

COMPLEXITY THEORY

Lecture 6: Nondeterministic Polynomial Time

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Knowledge-Based Systems

TU Dresden, 30th Oct 2023

More recent versions of this slide deck might be available.
For the most current version of this course, see
https://iccl.inf.tu-dresden.de/web/Complexity_Theory/en

The Class NP

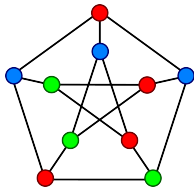
Beyond PTime

- We have seen that the class PTime provides a useful model of “tractable” problems
- This includes 2-Sat and 2-Colourability
- But what about 3-Sat and 3-Colourability?
- No polynomial time algorithms for these problems are known
- On the other hand . . .

Verifying Solutions

For many seemingly difficult problems, it is easy to **verify** the correctness of a “solution” if given.

p	q	r	$p \rightarrow q$
f	f	f	w
f	w	f	w
w	f	f	f
w	w	f	w
f	f	w	w
f	w	w	w
w	f	w	f
w	w	w	w



5	3			7	
		8			6
7		6		4	
4	1				
7	8	5	3	9	
			9	6	
5		1		7	
6			4		
	2			5	3

- **Satisfiability** – a satisfying assignment
- **k -Colourability** – a k -colouring
- **Sudoku** – a completed puzzle

Verifiers

Definition 6.1: A Turing machine \mathcal{M} which halts on all inputs is called a **verifier** for a language L if

$$L = \{w \mid \mathcal{M} \text{ accepts } (w\#c) \text{ for some string } c\}$$

The string c is called a **certificate** (or **witness**) for w .

Notation: # is a new separator symbol not used in words or certificates.

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Definition 6.2: A Turing machine \mathcal{M} is a **polynomial-time verifier** for L if \mathcal{M} is polynomially time bounded and

$$L = \{w \mid \mathcal{M} \text{ accepts } (w\#c) \text{ for some string } c \text{ with } |c| \leq p(|w|)\}$$

for some fixed polynomial p .

The Class NP

NP: “The class of dashed hopes and idle dreams.”¹

¹https://complexityzoo.net/Complexity_Zoo:N#np

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More formally:

the class of problems for which a possible solution can be verified in P

Definition 6.3: The class of languages that have polynomial-time verifiers is called NP.

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Definition 6.3: The class of languages that have polynomial-time verifiers is called **NP**.

In other words: NP is the class of all languages **L** such that:

- for every $w \in \mathbf{L}$, there is a **certificate** $c_w \in \Sigma^*$, where
- the length of c_w is polynomial in the length of w , and
- the language $\{(w\#c_w) \mid w \in \mathbf{L}\}$ is in P

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More Examples of Problems in NP

HAMILTONIAN PATH

Input: An undirected graph G

Problem: Is there a path in G that contains each vertex exactly once?

k -CLIQUE

Input: An undirected graph G

Problem: Does G contain a fully connected graph (clique) with k vertices?

More Examples of Problems in NP

SUBSET SUM

Input: A collection of positive integers

$S = \{a_1, \dots, a_k\}$ and a target integer t .

Problem: Is there a subset $T \subseteq S$ such that $\sum_{a_i \in T} a_i = t$?

TRAVELLING SALESPERSON

Input: A weighted graph G and a target number t .

Problem: Is there a simple path in G with weight $\leq t$?

Complements of NP are often not known to be in NP

No HAMILTONIAN PATH

Input: An undirected graph G

Problem: Is there no path in G that contains each vertex exactly once?

Whereas it is easy to certify that a graph has a Hamiltonian path, there does not seem to be a polynomial certificate that it has not.

But we may just not be clever enough to find one.

More Examples

COMPOSITE (NON-PRIME) NUMBER

Input: A positive integer $n > 1$

Problem: Are there integers $u, v > 1$ such that $u \cdot v = n$?

PRIME NUMBER

Input: A positive integer $n > 1$

Problem: Is n a prime number?

More Examples

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In fact: Composite Number (and thus Prime Number) was shown to be in P

N is for Nondeterministic

Reprise: Nondeterministic Turing Machines

A **nondeterministic Turing Machine** (NTM) $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}})$ consists of

- a finite set Q of **states**,
- an **input alphabet** Σ not containing \sqcup ,
- a **tape alphabet** Γ such that $\Gamma \supseteq \Sigma \cup \{\sqcup\}$.
- a **transition function** $\delta: Q \times \Gamma \rightarrow 2^{Q \times \Gamma \times \{L, R\}}$
- an **initial state** $q_0 \in Q$,
- an **accepting state** $q_{\text{accept}} \in Q$.

Note

An NTM can halt in any state if there are no options to continue

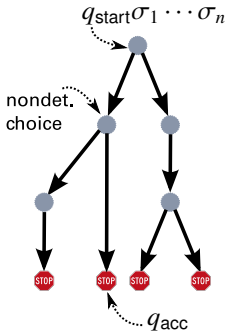
\leadsto no need for a special rejecting state

Reprise: Runs of NTMs

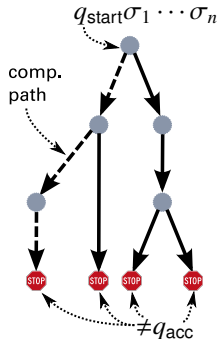
An (N)TM configuration can be written as a word uqv where $q \in Q$ is a state and $uv \in \Gamma^*$ is the current tape contents.

NTMs produce configuration trees that contain all possible runs:

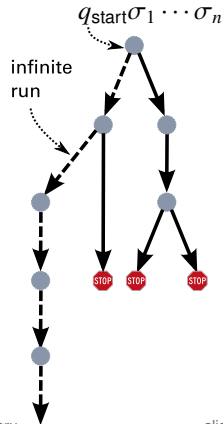
accept:



reject:



reject (not halting):

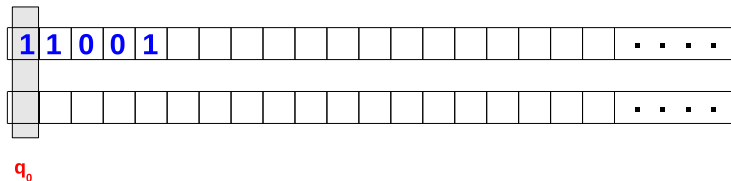


Example: Multi-Tape NTM

Consider the NTM $\mathcal{M} = (Q, \{0, 1\}, \{0, 1, \sqcup\}, q_0, \Delta, q_{\text{accept}})$ where

$$\Delta = \left\{ \begin{array}{l} (q_0, (-), q_0, (-), \binom{N}{R}) \\ (q_0, (-), q_0, (-), \binom{N}{R}) \\ (q_0, (-), q_{\text{check}}, (-), \binom{N}{R}) \\ \dots \\ \text{transition rules for } \mathcal{M}_{\text{check}} \end{array} \right\}$$

and where $\mathcal{M}_{\text{check}}$ is a deterministic TM deciding whether number on second tape is > 1 and divides the number on the first.

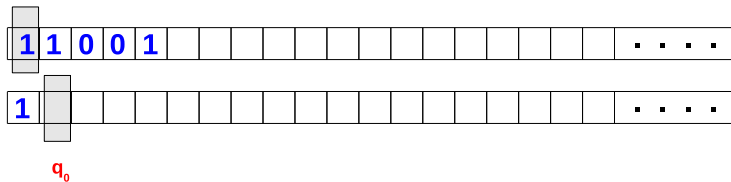


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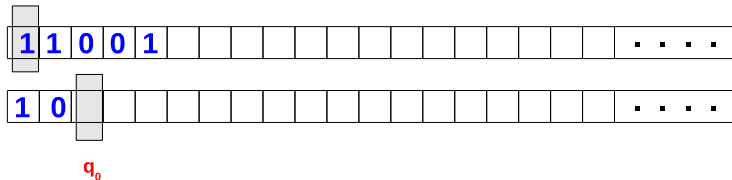


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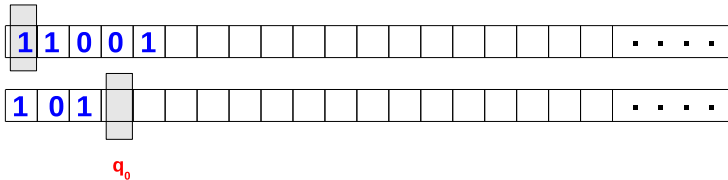


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and where $\mathcal{M}_{\text{check}}$ is a deterministic TM deciding whether number on second tape is > 1 and divides the number on the first.

The machine \mathcal{M} decides if the input is a composite number:

- guess a number on the second tape
- check if it divides the number on the first tape
- accept if a suitable number exists

Time and Space Bounded NTMs

Q: Which of the nondeterministic runs do time/space bounds apply to?

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A: To all of them!

Definition 6.4: Let \mathcal{M} be a nondeterministic Turing machine and let $f : \mathbb{N} \rightarrow \mathbb{R}^+$ be a function.

- (1) \mathcal{M} is **f -time bounded** if it halts on every input $w \in \Sigma^*$ and on every computation path after $\leq f(|w|)$ steps.
- (2) \mathcal{M} is **f -space bounded** if it halts on every input $w \in \Sigma^*$ and on every computation path using $\leq f(|w|)$ cells on its tapes.

(Here we typically assume that Turing machines have a separate input tape that we do not count in measuring space complexity.)

Nondeterministic Complexity Classes

Definition 6.5: Let $f : \mathbb{N} \rightarrow \mathbb{R}^+$ be a function.

- (1) $\mathbf{NTime}(f(n))$ is the class of all languages \mathbf{L} for which there is an $O(f(n))$ -time bounded nondeterministic Turing machine deciding \mathbf{L} .
- (2) $\mathbf{NSpace}(f(n))$ is the class of all languages \mathbf{L} for which there is an $O(f(n))$ -space bounded nondeterministic Turing machine deciding \mathbf{L} .

All Complexity Classes Have a Nondeterministic Variant

$$\text{NPTime} = \bigcup_{d \geq 1} \text{NTime}(n^d) \quad \text{nondet. polynomial time}$$

$$\text{NExp} = \text{NExpTime} = \bigcup_{d \geq 1} \text{NTime}(2^{n^d}) \quad \text{nondet. exponential time}$$

$$\text{N2Exp} = \text{N2ExpTime} = \bigcup_{d \geq 1} \text{NTime}(2^{2^{n^d}}) \quad \text{nond. double-exponential time}$$

$$\text{NL} = \text{NLogSpace} = \text{NSpace}(\log n) \quad \text{nondet. logarithmic space}$$

$$\text{NPSpace} = \bigcup_{d \geq 1} \text{NSpace}(n^d) \quad \text{nondet. polynomial space}$$

$$\text{NExpSpace} = \bigcup_{d \geq 1} \text{NSpace}(2^{n^d}) \quad \text{nondet. exponential space}$$

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Theorem 6.6: $NP = NPTime$.

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- Suppose $\mathbf{L} \in \text{NPTime}$.
- Then there is an NTM \mathcal{M} such that

$w \in \mathbf{L} \iff$ there is an accepting run of \mathcal{M} of length $O(n^d)$

for some d .

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for some d .

- This path can be used as a certificate for w .
- A DTM can check in polynomial time that a candidate certificate is a valid accepting run.

Therefore $\text{NP} \supseteq \text{NPTime}$.

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Equivalence of NP and NPTime

Theorem 6.6: $NP = NPTime$.

Proof: We now show $NP \subseteq NPTime$:

- Assume L has a polynomial-time verifier M with certificates of length at most $p(n)$ for a polynomial p .
- Then we can construct an NTM M^* deciding L as follows:
 - (1) M^* guesses a string of length $p(n)$
 - (2) M^* checks in deterministic polynomial time if this is a certificate.

Therefore $NP \subseteq NPTime$. □

NP and coNP

Note: the definition of NP is not symmetric

- there does not seem to be any polynomial certificate for Sudoku **unsolvability** or propositional logic **unsatisfiability** . . .
- converse of an NP problem is **coNP**
- similar for NExpTime and N2ExpTime

Other complexity classes are symmetric:

- Deterministic classes (coP = P etc.)
- Space classes mentioned above (esp. coNL = NL)

Deterministic vs. Nondeterministic Time

Theorem 6.7: $P \subseteq NP$, and also $P \subseteq coNP$.

(Clear since DTMs are a special case of NTMs)

It is not known to date if the converse is true or not.

- Put differently: “If it is easy to check a candidate solution to a problem, is it also easy to find one?”
- Exaggerated: “Can creativity be automated?” (Wigderson, 2006)
- Unresolved since over 35 years of effort
- One of the major problems in computer science and math of our time
- 1,000,000 USD prize for resolving it (“Millenium Problem”)
(might not be much money at the time it is actually solved)

Status of P vs. NP

Many people believe that $P \neq NP$

- Main argument: “If $NP = P$, someone ought to have found some polynomial algorithm for an NP-complete problem by now.”
- “This is, in my opinion, a very weak argument. The space of algorithms is very large and we are only at the beginning of its exploration.” (Moshe Vardi, 2002)
- Another source of intuition: Humans find it hard to solve NP-problems, and hard to imagine how to make them simpler – possibly “human chauvinistic bravado” (Zeilenberger, 2006)
- There are better arguments, but none more than an intuition

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- Even if $NP \neq P$, it is unclear if NP problems require exponential time in a strict sense – many super-polynomial functions exist . . .

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Many outcomes conceivable:

- $P = NP$ could be shown with a non-constructive proof
- The question might be independent of standard mathematics (ZFC)
- Even if $NP \neq P$, it is unclear if NP problems require exponential time in a strict sense – many super-polynomial functions exist . . .
- The problem might never be solved

Status of P vs. NP

Current status in research:

- Results of a poll among 152 experts [Gasarch 2012]:
 - $P \neq NP$: 126 (83%)
 - $P = NP$: 12 (9%)
 - Don't know or don't care: 7 (4%)
 - Independent: 5 (3%)
 - And 1 person (0.6%) answered: "I don't **want** it to be equal."
- Experts have guessed wrongly in other major questions before
- Over 100 "proofs" show $P = NP$ to be true/false/both/neither:
<https://www.win.tue.nl/~gwoegi/P-versus-NP.htm>

A Simple Proof for $P = NP$

Clearly
therefore
hence
that is
using $coP = P$
and hence
so by $P \subseteq NP$

$L \in P$ implies $L \in NP$
 $L \notin NP$ implies $L \notin P$
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 $NP = P$

q.e.d.?

Summary and Outlook

NP can be defined using polynomial-time verifiers or polynomial-time nondeterministic Turing machines

Many problems are easily seen to be in NP

NTM acceptance is not symmetric: coNP as complement class, which is assumed to be unequal to NP

What's next?

- NP hardness and completeness
- More examples of problems
- Space complexities