

Complexity Theory
Exercise 6: Diagonalization
6th December 2023

Exercise 6.1. Find the fault in the following proof of $P \neq NP$.

1. Assume that $P = NP$.
2. Then $SAT \in P$ and thus there exists a $k \in \mathbb{N}$ such that $SAT \in DTIME(n^k)$.
3. Because every language in NP is poly-time reducible to SAT , we have $NP \subseteq DTIME(n^k)$.
4. It follows that $P \subseteq DTIME(n^k)$.
5. By the Time Hierarchy Theorem there exist languages in $DTIME(n^{k+1})$ that are not in $DTIME(n^k)$, contradicting $P \subseteq DTIME(n^k)$.
6. Therefore, $P \neq NP$.

Exercise 6.2. Show the following.

1. $TIME(2^n) = TIME(2^{n+1})$
2. $TIME_*(2^n) \subset TIME_*(2^{2n})$
3. $NTIME(n) \subset PSPACE$

Exercise 6.3. Define a function that is computable but not time-constructible.

Exercise 6.4. Consider the function $pad: \Sigma^* \times \mathbb{N} \rightarrow \Sigma^* \#^*$ defined as $pad(s, \ell) = s \#^j$, where $j = \max(0, \ell - |s|)$. For some language $A \subseteq \Sigma^*$ and $f: \mathbb{N} \rightarrow \mathbb{N}$ define $pad(A, f) = \{ pad(s, f(|s|)) \mid s \in A \}$.

Show all of the following statements.

1. Show that, if $A \in DTIME(n^6)$, then $pad(A, n^2) \in DTIME(n^3)$.
2. Show that, if $NEXPTIME \neq EXPTIME$, then $P \neq NP$.
3. Show for every $A \subseteq \Sigma^*$ and $k \in \mathbb{N}$ that $A \in P$ if and only if $pad(A, n^k) \in P$.
4. Show $P \neq DSPACE(n)$.
5. Show $NP \neq DSPACE(n)$.

Exercise 6.5. You are given two oracles and one of them is the set **TRUE QBF**, but you do not know which one. Design a polynomial algorithm that decides **TRUE QBF** using these oracles.