

Decidable Gödel Description Logics without the Finitely-Valued Model Property*

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Abstract

In the last few years, there has been a large effort for analyzing the computational properties of reasoning in fuzzy description logics. This has led to a number of papers studying the complexity of these logics, depending on the chosen semantics. Surprisingly, despite being arguably the simplest form of fuzzy semantics, not much is known about the complexity of reasoning in fuzzy description logics w.r.t. witnessed models over the Gödel t-norm. We show that in the logic $G\mathcal{JALC}$, reasoning cannot be restricted to finitely-valued models in general. Despite this negative result, we also show that all the standard reasoning problems can be solved in exponential time, matching the complexity of reasoning in classical \mathcal{ALC} .

1 Introduction

Fuzzy Description Logics (DLs) have been studied as a means of representing vague or imprecise knowledge in a formal and well-understood manner. As for classical DLs (Baader et al. 2007), knowledge is expressed with the help of *concepts* and *roles*. What distinguishes fuzzy DLs from classical DLs are their semantics, which are based on *fuzzy sets*. Fuzzy sets associate every element of the domain of interest with a number from the interval $[0, 1]$, which intuitively represents the *degree* to which the element belongs to the fuzzy set.

When defining a fuzzy DL, one must also decide how to interpret the logical constructors, such as conjunction and implication, to handle the truth degrees. The simplest approach is to use the *minimum* operator to generalize intersection to fuzzy sets. Thus, the degree of membership of a conjunction is interpreted as the minimum of the membership degrees of the conjuncts. This operation, also known as the *Gödel t-norm*, can be used as a base to interpret all other logical constructors in a formally justified manner (Klement, Mesiar, and Pap 2000; Hájek 2001). The quantifiers \forall and \exists are interpreted as

infima and suprema of truth values, respectively. To avoid issues arising from having infinitely many truth values, reasoning in fuzzy DLs is usually restricted to so-called *witnessed models* (Hájek 2007).

The study of fuzzy DLs underwent a large change in recent years, after some relatively inexpressive fuzzy DLs were shown to be undecidable when reasoning w.r.t. general ontologies (Baader and Peñaloza 2011a; Baader and Peñaloza 2011b; Cerami and Straccia 2013). Since then, the limits of decidability have been explored, yielding very expressive decidable logics on the one hand (Borgwardt, Distel, and Peñaloza 2012), and inexpressive undecidable logics on the other (Borgwardt and Peñaloza 2012). Despite being widely regarded as the simplest t-norm, surprisingly little is known about fuzzy DLs based on Gödel semantics. It is generally believed that—at least w.r.t. witnessed models—these logics are decidable, but no proof exists to support this claim. The only results for similar logics restrict reasoning *a priori* to a finite subset of $[0, 1]$; in this case, a reduction to classical reasoning then yields decidability (Bobillo et al. 2009; Bobillo et al. 2012).

All existing approaches for reasoning in fuzzy DLs depend on limiting models to use only finitely many different truth degrees. Indeed, for these approaches to work, one must either (i) restrict the semantics to a finite set of truth degrees (Bobillo et al. 2009; Bobillo et al. 2012; Bobillo and Straccia 2011; Bobillo and Straccia 2013; Borgwardt and Peñaloza 2013a; Borgwardt and Peñaloza 2013b; Straccia 2006); (ii) prove that reasoning can be restricted to a finite set of degrees (Bobillo, Delgado, and Gómez-Romero 2008; Borgwardt, Distel, and Peñaloza 2012; Straccia 2001); or (iii) prove that models can be built from a finite pattern (Stoilos et al. 2007; Straccia and Bobillo 2007). In all three cases, the proofs of correctness of these algorithms imply the *finitely-valued model property*: an ontology has a model iff it has a model using only finitely many truth values. Conversely, the proofs of undecidability (Baader and Peñaloza 2011a; Baader and Peñaloza 2011b; Borgwardt and Peñaloza 2012; Cerami and Straccia 2013) construct a model that uses infinitely many truth degrees. Thus, this *finitely-valued model property* appears to be a good indicator of the decidability of a fuzzy DL.

In this paper we study the standard reasoning problems for the DL $G\mathcal{JALC}$, a fuzzy extension of \mathcal{ALC} based on the

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Gödel semantics w.r.t. witnessed models. First, we show that this logic does not have the finitely-valued model property. In fact, we provide very simple consistent ontologies that only have infinitely-valued models (see Section 3). The absence of the finitely-valued model property for these logics is a surprising result in itself, contradicting the common lore of the field. In contrast, we show in Sections 4 and 5 that consistency is decidable in exponential time for this logic. Our algorithm is based on the insight that under Gödel semantics, it is only necessary to know an ordering between the relevant truth degrees, rather the precise values they take. This idea has already been used for deciding validity of formulae in propositional Gödel logic (Guller 2012). We then extend our algorithm to also compute best subsumption degrees and best satisfiability degrees w.r.t. an ontology. The last section provides some pointers to future work.

2 Preliminaries

Before introducing fuzzy description logics, we briefly consider the operators of Gödel fuzzy logic and introduce auxiliary notions that will be useful for the reasoning procedures described in the following sections.

The two basic operators of Gödel fuzzy logic are conjunction and implication, interpreted by the *Gödel t-norm* and *residuum*, respectively. The Gödel t-norm of two fuzzy degrees $x, y \in [0, 1]$ is defined as minimum function $\min(x, y)$. The residuum \Rightarrow is uniquely defined by the equivalence $\min(x, y) \leq z$ iff $y \leq (x \Rightarrow z)$ for all $x, y, z \in [0, 1]$, and can be computed as

$$x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise.} \end{cases}$$

For a deeper introduction to t-norms and t-norm-based fuzzy logics, see (Cintula, Hájek, and Noguera 2011; Hájek 2001; Klement, Mesiar, and Pap 2000).

A *total preorder* over a set S is a transitive and total binary relation $\lesssim_* \subseteq S \times S$. For $x, y \in S$, we write $x \equiv_* y$ if $x \lesssim_* y$ and $y \lesssim_* x$. Notice that \equiv_* is an equivalence relation on S . Similarly, we write $x <_* y$ if $x \lesssim_* y$, but not $y \lesssim_* x$. By the symbol \bowtie we denote an arbitrary element of $\{=, \geq, >, \leq, <\}$, and by \bowtie_* the corresponding relation induced by the total preorder \lesssim_* , i.e. $\equiv_*, \gtrsim_*, >_*, \lesssim_*$, or $<_*$. Subscripts are used to distinguish these relations for different total preorders over the same carrier set S .

An *order structure* S is a finite set containing at least the numbers 0 and 1, together with an involutive unary operation $\text{inv}: S \rightarrow S$ such that $\text{inv}(x) = 1 - x$ for all $x \in S \cap [0, 1]$. For an order structure S , $\text{order}(S)$ denotes the set of all total preorders \lesssim_* over S that

- have 0 and 1 as least and greatest element, respectively,
- preserve the order of real numbers on $S \cap [0, 1]$, and
- satisfy $x \lesssim_* y$ iff $\text{inv}(y) \lesssim_* \text{inv}(x)$ for all $x, y \in S$.

Given $\lesssim_* \in \text{order}(S)$, the following functions on S that mimic the operators of Gödel fuzzy logic over $[0, 1]$ are well-defined since \lesssim_* is total:

$$\min_*(x, y) := \begin{cases} x & \text{if } x \lesssim_* y \\ y & \text{otherwise,} \end{cases}$$

Table 1: Semantics of G- \mathcal{JALC}

Constructor	Syntax	Semantics
top concept	\top	1
involutive negation	$\neg C$	$1 - C^{\mathcal{I}}(x)$
conjunction	$C \sqcap D$	$\min(C^{\mathcal{I}}(x), D^{\mathcal{I}}(x))$
implication	$C \rightarrow D$	$C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)$
existential restriction	$\exists r.C$	$\sup_{y \in \Delta^{\mathcal{I}}} \min(r^{\mathcal{I}}(x, y), C^{\mathcal{I}}(y))$
value restriction	$\forall r.C$	$\inf_{y \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x, y) \Rightarrow C^{\mathcal{I}}(y)$

$$\text{res}_*(x, y) := \begin{cases} 1 & \text{if } x \lesssim_* y \\ y & \text{otherwise.} \end{cases}$$

It is easy to see that these operators agree with \min and \Rightarrow on the set $S \cap [0, 1]$.

The fuzzy description logic G- \mathcal{JALC} is based on concepts and roles, which are interpreted as (fuzzy) unary and binary relations, respectively. Given the mutually disjoint sets N_I , N_R , and N_C of *individual*, *role*, and *concept names*, respectively, G- \mathcal{JALC} concepts are built through the rule

$$C ::= A \mid \top \mid \neg C \mid C \sqcap C \mid C \rightarrow C \mid \exists r.C \mid \forall r.C,$$

where $A \in N_C$ and $r \in N_R$. We call concepts of the form $\exists r.C$ or $\forall r.C$ *quantified concepts*. The semantics of this logic is given by means of interpretations. An *interpretation* is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where $\Delta^{\mathcal{I}}$ is a non-empty *domain*, and $\cdot^{\mathcal{I}}$ is a function that maps every $a \in N_I$ to an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$, every $A \in N_C$ to a fuzzy set $A^{\mathcal{I}}: \Delta^{\mathcal{I}} \rightarrow [0, 1]$, and every role name $r \in N_R$ to a fuzzy binary relation $r^{\mathcal{I}}: \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \rightarrow [0, 1]$. This function is extended to arbitrary concepts using the Gödel operators as shown in Table 1.

Notice that we have not introduced an explicit constructor for the *residual negation* $\ominus x := x \Rightarrow 0$ or disjunction, as they are expressible using \top , \neg , \sqcap , and \rightarrow . The residual negation is often used in fuzzy logics, but under Gödel semantics it is much less expressive than the involutive negation since we have $\ominus 0 = 1$ and $\ominus x = 0$ for all $x \in (0, 1]$.

In the literature on fuzzy DLs, interpretations are usually restricted to be witnessed (Hájek 2005), which means that existential and value restrictions must be interpreted as maxima and minima, respectively. More formally, an interpretation \mathcal{I} is *witnessed* if for every existential restriction $\exists r.C$ and every $x \in \Delta^{\mathcal{I}}$ there is a *witness* $y \in \Delta^{\mathcal{I}}$ such that $(\exists r.C)^{\mathcal{I}}(x) = \min(r^{\mathcal{I}}(x, y), C^{\mathcal{I}}(y))$, and similarly for value restrictions. We also adopt this restriction here, and for the rest of this paper consider only witnessed interpretations. For brevity, we call them simply *interpretations*.

The knowledge of a domain is represented using axioms that restrict the class of interpretations that are relevant for the different reasoning tasks.

Definition 1 (axioms). A *crisp assertion* is either a *concept assertion* of the form $a:C$ or a *role assertion* of the form $(a, b):r$ for a concept C , $r \in N_R$, and $a, b \in N_I$. An *(order) assertion* is of the form $\langle \alpha \bowtie \beta \rangle$, where α is a crisp assertion and β is either a crisp assertion or a value from $[0, 1]$. An interpretation \mathcal{I} *satisfies* an order assertion $\langle \alpha \bowtie \beta \rangle$ if $\alpha^{\mathcal{I}} \bowtie \beta^{\mathcal{I}}$, where $(a:C)^{\mathcal{I}} := C^{\mathcal{I}}(a^{\mathcal{I}})$,

$((a, b):r)^{\mathcal{I}} := r^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}})$, and $q^{\mathcal{I}} := q$ for all $q \in [0, 1]$. An *ordered ABox* \mathcal{A} is a finite set of order assertions. An interpretation is a *model* of \mathcal{A} if it satisfies all order assertions in \mathcal{A} .

A *general concept inclusion (GCI)* is an expression of the form $\langle C \sqsubseteq D \geq q \rangle$ for concepts C, D , and $q \in [0, 1]$. An interpretation \mathcal{I} satisfies this GCI if $C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \geq q$ holds for all $x \in \Delta^{\mathcal{I}}$. A *TBox* is a finite set of GCIs. An *ontology* is a pair $\mathcal{O} = (\mathcal{A}, \mathcal{T})$, where \mathcal{A} is an ordered ABox and \mathcal{T} is a TBox. An interpretation is a *model* of a TBox \mathcal{T} if it satisfies all GCIs in \mathcal{T} , and it is a *model* of an ontology $\mathcal{O} = (\mathcal{A}, \mathcal{T})$ if it is a model of both \mathcal{A} and \mathcal{T} .

We will usually abbreviate $\langle C \sqsubseteq D \geq 1 \rangle$ as $\langle C \sqsubseteq D \rangle$. Ordered ABoxes are more expressive than ABoxes usually considered for fuzzy DLs (Straccia 2001) since they allow to state order relations between concepts. This more general kind of ABox is better suited for our algorithms.

We denote by $\text{sub}(\mathcal{O})$ the closure under negation of the set of all subconcepts appearing in an ontology \mathcal{O} . The concepts $\neg C$ and C are equivalent, and we regard them here as equal, which means that $\text{sub}(\mathcal{O})$ is always finite. We further denote by $\mathcal{V}_{\mathcal{O}}$ the closure of the set of all truth degrees appearing in \mathcal{O} , together with 0, 0.5, and 1, under the operator $x \mapsto 1 - x$. Since this operator is involutive, $\mathcal{V}_{\mathcal{O}}$ is also always finite. We often denote the elements of $\mathcal{V}_{\mathcal{O}} \subseteq [0, 1]$ as $0 = q_0 < q_1 < \dots < q_k = 1$.

As with classical DLs, the most basic reasoning task in G- \mathcal{JALC} is to decide whether a given ontology has a (witnessed) model. However, one might also be interested in computing the degree to which an entailment holds.

Definition 2 (reasoning). An ontology \mathcal{O} is *consistent* if it has a model. Given $p \in [0, 1]$, a concept C is *p-satisfiable* w.r.t. \mathcal{O} if there is a model \mathcal{I} of \mathcal{O} and an $x \in \Delta^{\mathcal{I}}$ with $C^{\mathcal{I}}(x) \geq p$. The *best satisfiability degree* of C w.r.t. \mathcal{O} is the supremum over all p such that C is p -satisfiable w.r.t. \mathcal{O} . Furthermore, C is *p-subsumed* by a concept D w.r.t. \mathcal{O} if all models of \mathcal{O} satisfy the GCI $\langle C \sqsubseteq D \geq p \rangle$. The *best subsumption degree* of C and D w.r.t. \mathcal{O} is the supremum over all p such that C is p -subsumed by D w.r.t. \mathcal{O} .

If consistency is decidable, then satisfiability and subsumption can be restricted without loss of generality to ontologies containing an empty ABox: if \mathcal{O} is inconsistent, then these two problems are trivial, and if \mathcal{O} is consistent, then the ABox assertions cannot contradict the p -satisfiability of C , and therefore C is p -satisfiable w.r.t. $\mathcal{O} = (\mathcal{A}, \mathcal{T})$ iff it is p -satisfiable w.r.t. (\emptyset, \mathcal{T}) . A similar argument can be made for subsumptions.

We show in Section 5 that ontology consistency has the same complexity in G- \mathcal{JALC} as in classical \mathcal{ALC} : it is EXPTIME-complete. As a first step, we establish the complexity of consistency for the special case of ontologies with so-called *local* ordered ABoxes in Section 4, adapting an automata-based technique known from classical and finitely-valued DLs (Baader, Hladik, and Peñaloza 2008; Borgwardt and Peñaloza 2013a). We later lift these results to the satisfiability and subsumption problems. But first, we illustrate why the naïve approach of restricting reasoning to finitely-valued reasoning cannot work in this logic.

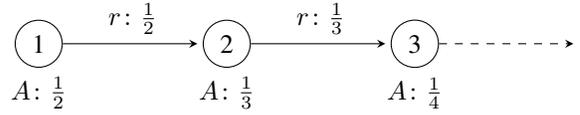


Figure 1: The model \mathcal{I}_1 from Example 3

3 Restricting to Finitely Many Values

It is a simple observation that any set of truth values that contains 0 and 1 is closed w.r.t. the Gödel connectives. Owing to this observation, it is common to restrict reasoning in fuzzy DLs with Gödel semantics to the finitely many truth values occurring in the ontology (Bobillo et al. 2012; Bobillo and Straccia 2013). This restriction is also sometimes justified by the “limited precision of computers” (Bobillo et al. 2009).

Earlier works have, however, neglected to study whether the restriction to a fixed finite set of values preserves the semantics of the logic. We now show that this is not the case, even for the simple description logics G- \mathcal{AL} , which allows only \exists , \forall , \sqcap , and \sqcup . and G- \mathcal{JEL} , where concepts are built using only \exists , \sqcap , \rightarrow , and \sqcup . We show even stronger results: reasoning in these logics cannot, without loss of generality, be restricted to *finitely-valued* models, i.e. models that only use values from an arbitrary finite subset of $[0, 1]$. We note that for the related *Zadeh semantics*, which differ from the Gödel semantics only in the operator used for implications, reasoning can be restricted to finitely-valued models without loss of generality (Straccia 2001; Bobillo, Delgado, and Gómez-Romero 2008).

Example 3. Let \mathcal{T}_1 be the G- \mathcal{AL} TBox

$$\mathcal{T}_1 = \{ \langle \forall r. A \sqsubseteq A \geq 1 \rangle, \langle \exists r. \top \sqsubseteq A \geq 1 \rangle \}.$$

We show that \top is *not* 1-subsumed by A w.r.t. the ontology $\mathcal{O} = (\emptyset, \mathcal{T}_1)$, but every finitely-valued model of this ontology also satisfies $\langle \top \sqsubseteq A \geq 1 \rangle$.

For the former, we construct a model \mathcal{I}_1 of \mathcal{T}_1 as follows (see Figure 1). Let $\Delta^{\mathcal{I}_1}$ be the set of all natural numbers. We define $A^{\mathcal{I}_1}(n) := r^{\mathcal{I}_1}(n, n+1) := \frac{1}{n+1}$ for all $n \in \mathbb{N}$ and $r^{\mathcal{I}_1}(n, m) := 0$ if $m \neq n+1$. It is straightforward to check that this is a witnessed model of \mathcal{T}_1 that violates $\langle \top \sqsubseteq A \geq 1 \rangle$. Thus, \top is not 1-subsumed by A w.r.t. \mathcal{O} . In fact, the best subsumption degree of \top and A w.r.t. \mathcal{O} is 0.

Assume now that there is a witnessed model \mathcal{I} of \mathcal{T}_1 using only finitely many truth values that violates $\langle \top \sqsubseteq A \geq 1 \rangle$. Since \mathcal{I} uses only finitely many truth values, there exists an element $y \in \Delta^{\mathcal{I}}$ for which $A^{\mathcal{I}}(y)$ is minimal, that is, $A^{\mathcal{I}}(y) \leq A^{\mathcal{I}}(x)$ holds for all $x \in \Delta^{\mathcal{I}}$. Furthermore, since \mathcal{I} violates $\langle \top \sqsubseteq A \geq 1 \rangle$ there must be some $x_0 \in \Delta^{\mathcal{I}}$ satisfying $A^{\mathcal{I}}(x_0) < 1$. In particular, this yields $A^{\mathcal{I}}(y) < 1$.

As \mathcal{I} is witnessed, there must exist a $z \in \Delta^{\mathcal{I}}$ such that $(\forall r. A)^{\mathcal{I}}(y) = r^{\mathcal{I}}(y, z) \Rightarrow A^{\mathcal{I}}(z)$. The first axiom of \mathcal{T}_1 entails $r^{\mathcal{I}}(y, z) \Rightarrow A^{\mathcal{I}}(z) \leq A^{\mathcal{I}}(y) < 1$, and in particular

$$r^{\mathcal{I}}(y, z) > A^{\mathcal{I}}(z). \quad (1)$$

The second axiom from \mathcal{T}_1 yields

$$r^{\mathcal{I}}(y, z) = \min(r^{\mathcal{I}}(y, z), 1) \leq (\exists r. \top)^{\mathcal{I}}(y) \leq A^{\mathcal{I}}(y). \quad (2)$$

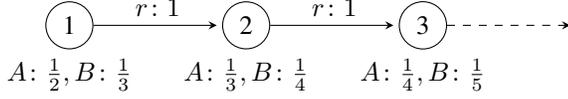


Figure 2: The model \mathcal{I}_2 from Example 4

From (1) and (2) we obtain $A^{\mathcal{I}}(y) > A^{\mathcal{I}}(z)$, contradicting the minimality of $A^{\mathcal{I}}(y)$. We have thus shown that a witnessed model of \mathcal{T}_1 with only finitely many truth values cannot violate $\langle \top \sqsubseteq A \geq 1 \rangle$. That is, \mathcal{T}_1 entails $\langle \top \sqsubseteq A \geq 1 \rangle$ when reasoning is restricted to finite sets of values.

It is thus not possible to restrict reasoning in $G\text{-}\mathcal{AL}$ to only finitely-valued models without changing the consequences. A similar example shows that this also holds for $G\text{-}\mathcal{JEL}$.

Example 4. Consider the TBox

$$\mathcal{T}_2 = \{ \langle B \sqsubseteq A \rangle, \langle A \rightarrow B \sqsubseteq B \rangle, \langle \top \sqsubseteq \exists r. \top \rangle, \langle \exists r. A \sqsubseteq B \rangle \}.$$

As in Example 3, one can show that \top is not 1-subsumed by A w.r.t. $\mathcal{O} := (\emptyset, \mathcal{T}_2)$, but every finitely-valued model of \mathcal{O} satisfies $\langle \top \sqsubseteq A \geq 1 \rangle$. A witnessed model \mathcal{I}_2 of \mathcal{T}_2 can be built as follows (see Figure 2). Let $\Delta^{\mathcal{I}_2}$ be the set of all natural numbers. Set $A^{\mathcal{I}_2}(n) := \frac{1}{n+1}$, $B^{\mathcal{I}_2}(n) := \frac{1}{n+2}$, $r^{\mathcal{I}_2}(n, n+1) := 1$ for all $n \in \mathbb{N}$, and $r^{\mathcal{I}_2}(n, m) := 0$ if $m \neq n+1$. It is straightforward to check that this is indeed a witnessed model of \mathcal{T}_2 that violates $\langle \top \sqsubseteq A \geq p \rangle$ for every $p > 0$; in particular for $p = 1$. Using a proof by contradiction similar to Example 3 it can be shown that no witnessed model of \mathcal{T}_2 with only finitely many truth values can violate $\langle \top \sqsubseteq A \geq 1 \rangle$. All details can be found in (Borgwardt, Distel, and Peñaloza 2013).

Recall that a (fuzzy) DL has the *finite model property* if every consistent ontology has a model with finite domain. A simple consequence of the last two examples is that $G\text{-}\mathcal{AL}$ and $G\text{-}\mathcal{JEL}$ do not have the finite model property. Indeed, each \mathcal{I}_i is a model of the ontology $(\{ \langle a: A = 0.5 \rangle \}, \mathcal{T}_i)$ if we interpret the individual name a as $a^{\mathcal{I}_i} := 1$. This shows that these ontologies are consistent. However, any finite model \mathcal{I} of \mathcal{T}_i uses only finitely many truth degrees. As shown in the examples, such an interpretation must satisfy $A^{\mathcal{I}}(x) = 1$ for all $x \in \Delta^{\mathcal{I}}$, and hence violate the assertion $\langle a: A = 0.5 \rangle$. We thus obtain the following result.

Theorem 5. $G\text{-}\mathcal{AL}$ and $G\text{-}\mathcal{JEL}$ do not have the finite model property or the finitely-valued model property.

The lack of the finitely-valued model property implies that some of the standard techniques used for reasoning in fuzzy DLs cannot be directly applied to any logic that contains $G\text{-}\mathcal{AL}$ or $G\text{-}\mathcal{JEL}$. For example, termination of the tableaux-based approach (Stoilos et al. 2007; Straccia and Bobillo 2007) relies on the existence of finitely many *types* that can describe domain elements by specifying the membership degrees for all relevant concepts, while any sound and complete reduction to crisp reasoning (Bobillo, Delgado, and Gómez-Romero 2008; Bobillo et al. 2012) implies the finitely-valued model property.

Moreover, all known undecidability proofs for fuzzy DLs (Baader and Peñaloza 2011a; Baader and Peñaloza 2011b;

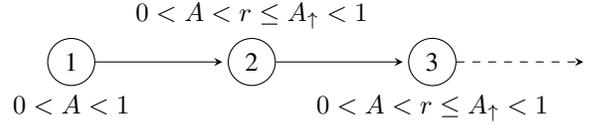


Figure 3: An abstract description of \mathcal{I}_1 from Example 3

Borgwardt and Peñaloza 2012; Cerami and Straccia 2013) are based on the fact that one can enforce models to have infinitely many values. One could thus be inclined to believe that consistency in $G\text{-}\mathcal{JALC}$ is also undecidable. In the rest of this paper, we show that this is not the case, providing EXPTIME automata-based algorithms that decide consistency, subsumption, and satisfiability.

4 Deciding Local Consistency

In this section, we consider only the special case where the ontology $\mathcal{O} = (\mathcal{A}, \mathcal{T})$ is such that \mathcal{A} is a *local* ordered ABox, which means that it contains no role assertions and uses only a single individual name a . In Section 5, we extend the approach to handle arbitrary ontologies.

The algorithm is based on the observation that the axioms and the semantics of the constructors only introduce restrictions on the *order* of the values that models can assign to concepts, not on the values themselves. For example, an interpretation \mathcal{I} satisfies an assertion $\langle a: (A \rightarrow B) = p \rangle$ iff $A^{\mathcal{I}}(a^{\mathcal{I}}) > B^{\mathcal{I}}(a^{\mathcal{I}})$ and $B^{\mathcal{I}}(a^{\mathcal{I}}) = p$. Thus, rather than building a model directly, we first create an abstract representation of a model that encodes for each domain element only the order between concepts.

Example 6. Consider again the TBox \mathcal{T}_1 from Example 3. When trying to construct a model violating $\langle \top \sqsubseteq A \geq 1 \rangle$, we start with a domain element satisfying the restriction that the value of A is strictly smaller than 1 (see Figure 3). The second axiom implies that the degree of any outgoing r -connection is bounded by the value of A . Moreover, the first axiom states that the witness of $\forall r. A$ must satisfy A to a degree strictly smaller than the value of the r -connection, and thus strictly smaller than the original value of A .

This yields an abstract description of two domain elements in terms of order relations between values of concepts at the current node and the parent node (denoted by a subscript \uparrow). Applying the same argument to the new element yields another element with the same restrictions. In order to construct a model, it is easy to see that the value of A at all considered elements has to be strictly greater than 0—once the value of A is 0, there can be no successors with smaller values for A . Note that it suffices to consider order relations between concepts of neighboring elements, which are directly connected by some role to a degree greater than 0.

As in this example, we use the subscript \uparrow to refer to values of the parent node in the tree-like model that we will construct. We additionally use a new element λ to represent the degree of the role connection from the parent node.

Definition 7 (order structure \mathcal{U}). We define the set $\text{sub}_{\uparrow}(\mathcal{O}) := \{ C_{\uparrow} \mid C \in \text{sub}(\mathcal{O}) \}$ and the order structure

$\mathcal{U} := \mathcal{V}_{\mathcal{O}} \cup \text{sub}(\mathcal{O}) \cup \text{sub}_{\uparrow}(\mathcal{O}) \cup \{\lambda, \neg\lambda\}$ with the involutive operation inv given by $\text{inv}(\lambda) := \neg\lambda$, $\text{inv}(C) := \neg C$, and $\text{inv}(C_{\uparrow}) := (\neg C)_{\uparrow}$ for all $C \in \text{sub}(\mathcal{O})$.

For convenience, we extend the notation of $\text{sub}_{\uparrow}(\mathcal{O})$ to the elements of $\mathcal{V}_{\mathcal{O}}$ by setting $q_{\uparrow} := q$ for all $q \in \mathcal{V}_{\mathcal{O}}$.

Using total preorders from $\text{order}(\mathcal{U})$, we can now describe the relationships between all the subconcepts from \mathcal{O} and the truth degrees from $\mathcal{V}_{\mathcal{O}}$ at given domain elements. One can think of such a preorder as the *type* of a domain element, from which a tree-shaped interpretation can be built, represented by a *Hintikka tree*.

In the following, let n be the number of quantified concepts in $\text{sub}(\mathcal{O})$ and ϕ an arbitrary but fixed bijection between the set of all quantified concepts in $\text{sub}(\mathcal{O})$ and $\{1, \dots, n\}$. This bijection specifies which quantified concept is witnessed by which successor in the Hintikka tree. For a given role $r \in \mathbb{N}_{\mathbb{R}}$, we denote by Φ_r the set of all indices $\phi(E)$ where $E \in \text{sub}(\mathcal{O})$ is a quantified concept of the form $\exists r.C$ or $\forall r.C$. Our algorithm will try to decide the existence of an n -ary infinite tree whose nodes are labeled with preorders from $\text{order}(\mathcal{U})$, such that the semantics of the constructors and all the axioms in \mathcal{O} are preserved.

Definition 8 (Hintikka ordering). A *Hintikka ordering* is a total preorder $\lesssim_H \in \text{order}(\mathcal{U})$ that satisfies the following conditions for every $C \in \text{sub}(\mathcal{O})$:

- $C = \top$ implies $C \equiv_H 1$,
- if $C = D_1 \sqcap D_2$, then $C \equiv_H \min_H(D_1, D_2)$,
- if $C = D_1 \rightarrow D_2$, then $C \equiv_H \text{res}_H(D_1, D_2)$.

This preorder is *compatible* with the TBox \mathcal{T} if for every GCI $\langle C \sqsubseteq D \geq q \rangle \in \mathcal{T}$ we have $\text{res}_H(C, D) \gtrsim_H q$. It is *compatible* with \mathcal{A} if for every order assertion $\langle a:C \bowtie q \rangle$ or $\langle a:C \bowtie a:D \rangle$ in \mathcal{A} , we have $C \bowtie_H q$ or $C \bowtie_H D$, respectively.

The conditions imposed on Hintikka orderings ensure that they preserve the semantics of all the *propositional* constructors. For every *quantified* concept E , we still need to ensure the existence of a witness. This is achieved through ϕ and the following Hintikka condition.

Definition 9 (Hintikka condition). The *Hintikka condition* consists of the following requirements for an $(n+1)$ -tuple $(\lesssim_0, \lesssim_1, \dots, \lesssim_n)$ of Hintikka orderings:

- for every $1 \leq i \leq n$ and all $\alpha, \beta \in \mathcal{V}_{\mathcal{O}} \cup \text{sub}(\mathcal{O})$, we have $\alpha \lesssim_0 \beta$ iff $\alpha_{\uparrow} \lesssim_i \beta_{\uparrow}$, where $q_{\uparrow} := q$ for all $q \in \mathcal{V}_{\mathcal{O}}$;
- for every $\exists r.D \in \text{sub}(\mathcal{O})$, we have
 - $(\exists r.D)_{\uparrow} \equiv_i \min_i(\lambda, D)$ for $i = \phi(\exists r.D)$, and
 - $(\exists r.D)_{\uparrow} \gtrsim_i \min_i(\lambda, D)$ for all $i \in \Phi_r$; and
- for every $\forall r.D \in \text{sub}(\mathcal{O})$, we have
 - $(\forall r.D)_{\uparrow} \equiv_i \text{res}_i(\lambda, D)$ for $i = \phi(\forall r.D)$, and
 - $(\forall r.D)_{\uparrow} \lesssim_i \text{res}_i(\lambda, D)$ for all $i \in \Phi_r$.

A *Hintikka tree* for \mathcal{O} is an infinite n -ary tree,¹ where every node u is associated with a Hintikka ordering \lesssim_u compatible with \mathcal{T} , such that:

¹We will use words from $\{1, \dots, n\}^*$ to denote the nodes in an infinite n -ary tree.

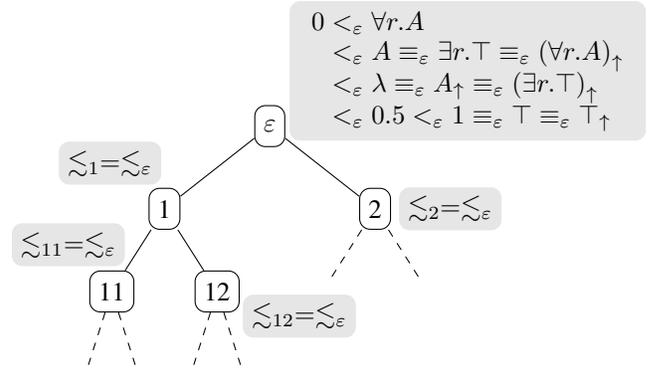


Figure 4: A Hintikka tree for Example 3

- every tuple $(\lesssim_u, \lesssim_{u_1}, \dots, \lesssim_{u_n})$ satisfies the Hintikka condition, and
- \lesssim_{ε} is compatible with \mathcal{A} .

For instance, Figure 4 shows a Hintikka tree for the TBox \mathcal{T}_1 from Example 3 and the ABox $\mathcal{A} = \{ \langle a:A < 1 \rangle \}$. Notice that every node is labeled with the same preorder and the tree is invariant w.r.t. the choice of ϕ . We now show that the existence of a Hintikka tree for an ontology \mathcal{O} characterizes the consistency of \mathcal{O} .

Proposition 10. *If there is a Hintikka tree for \mathcal{O} , then \mathcal{O} has a model.*

Proof. Given a Hintikka tree, we construct a model in two steps. In the first step, we recursively define a function $v: \mathcal{U} \times \{1, \dots, n\}^* \rightarrow [0, 1]$ satisfying the following conditions for all nodes u and all $\alpha, \beta \in \mathcal{U}$:

- (P1) for all values $q \in \mathcal{V}_{\mathcal{O}}$ we have $v(q, u) = q$,
- (P2) $v(\alpha, u) \leq v(\beta, u)$ iff $\alpha \lesssim_u \beta$,
- (P3) $v(\text{inv}(\alpha), u) = 1 - v(\alpha, u)$,
- (P4) for all $C \in \text{sub}(\mathcal{O})$ and all $i \in \{1, \dots, n\}$

$$v(C, u) = v(C_{\uparrow}, ui).$$

In the second step, we construct, with the help of this function v , an interpretation $\mathcal{I}_v = (\{1, \dots, n\}^*, \cdot^{\mathcal{I}_v})$ satisfying $C^{\mathcal{I}_v}(u) = v(C, u)$ for all concepts C and all nodes u , and show that \mathcal{I}_v is indeed a model of \mathcal{O} .

Step 1 The function v is defined recursively, starting from the root node ε . Let $\mathcal{U}/\equiv_{\varepsilon}$ be the set of all equivalence classes of \equiv_{ε} . Then \lesssim_{ε} yields a total order on $\mathcal{U}/\equiv_{\varepsilon}$. In particular, $[0]_{\varepsilon} <_{\varepsilon} [q_1]_{\varepsilon} <_{\varepsilon} [q_2]_{\varepsilon} <_{\varepsilon} \dots <_{\varepsilon} [q_{k-1}]_{\varepsilon} <_{\varepsilon} [1]_{\varepsilon}$ holds if we extend $<_{\varepsilon}$ to $\mathcal{U}/\equiv_{\varepsilon}$ in the obvious way. For an equivalence class $[\alpha]_{\varepsilon}$, we set $\text{inv}([\alpha]_{\varepsilon}) := [\text{inv}(\alpha)]_{\varepsilon}$, which is well-defined since \lesssim_{ε} is an element of $\text{order}(\mathcal{U})$.

We first define an auxiliary function $\tilde{v}_{\varepsilon}: \mathcal{U}/\equiv_{\varepsilon} \rightarrow [0, 1]$. For all $q \in \mathcal{V}_{\mathcal{O}}$ we define $\tilde{v}_{\varepsilon}([q]_{\varepsilon}) := q$. It remains to define a value for all equivalence classes that do not contain a value from $\mathcal{V}_{\mathcal{O}}$. Notice that due to the minimality of $[0]_{\varepsilon}$ and maximality of $[1]_{\varepsilon}$ every such class must be strictly between $[q_i]_{\varepsilon}$ and $[q_{i+1}]_{\varepsilon}$ for two adjacent truth degrees q_i, q_{i+1} . For

every $i \in \{0, \dots, k-1\}$, let ν_i be the number of equivalence classes that are strictly between $[q_i]_\varepsilon$ and $[q_{i+1}]_\varepsilon$. We assume that these classes are denoted by E_j^i such that $[q_i]_\varepsilon <_\varepsilon E_1^i <_\varepsilon E_2^i <_\varepsilon \dots <_\varepsilon E_{\nu_i}^i <_\varepsilon [q_{i+1}]_\varepsilon$. We then define values $q_i < s_1^i < s_2^i < \dots < s_{\nu_i}^i < q_{i+1}$ as

$$s_j^i := q_i + \frac{j}{\nu_i+1}(q_{i+1} - q_i) \quad (3)$$

and set $\tilde{v}_\varepsilon(E_j^i) := s_j^i$ for every $j, 1 \leq j \leq \nu_i$. Finally, we define $v(\alpha, \varepsilon) := \tilde{v}_\varepsilon([\alpha]_\varepsilon)$ for all $\alpha \in \mathcal{U}$. This construction ensures that (P1) and (P2) hold at the node ε . To see that (P3) is also satisfied, note that $1 - q_{i+1}$ and $1 - q_i$ are also adjacent in $\mathcal{V}_\mathcal{O}$ and have exactly the inverses $\text{inv}(E_j^i)$ between them in reversed order.

For the recursion step, assume that we have already defined v for a node u , such that (P1)–(P3) are satisfied at u and let $i \in \{1, \dots, n\}$. We initialize the auxiliary function $\tilde{v}_{ui}: \mathcal{U} \equiv_{ui} \rightarrow [0, 1]$ by setting $\tilde{v}_{ui}([q]_{ui}) := q$ for all $q \in \mathcal{V}_\mathcal{O}$ and $\tilde{v}_{ui}([C]_{ui}) := v(C, u)$ for all $C \in \text{sub}(\mathcal{O})$. To see that this is well-defined, consider $[C]_{ui} = [D]_{ui}$, i.e. $C \equiv_{ui} D$. From the Hintikka condition, it follows that $C \equiv_u D$, and from (P2) at u we obtain $v(C, u) = v(D, u)$. A similar argument can be used to show that $[q]_{ui} = [C]_{ui}$ implies $v(q, u) = v(C, u)$. For the remaining equivalence classes, we can use a construction analogous to the case for ε by considering the two unique neighboring equivalence classes that contain an element of $\mathcal{V}_\mathcal{O} \cup \text{sub}_\uparrow(\mathcal{O})$. We now define $v(\alpha, ui) := \tilde{v}_{ui}([\alpha]_{ui})$. This construction ensures that (P1)–(P3) hold at ui , and that (P4) holds for u .

Step 2 We define the interpretation \mathcal{I}_v over the domain $\{1, \dots, n\}^*$ as follows. For every concept name $A \in \mathbb{N}_C$ and all domain elements u , we set

$$A^{\mathcal{I}_v}(u) := \begin{cases} v(A, u) & \text{if } A \in \text{sub}(\mathcal{O}), \\ 0 & \text{otherwise.} \end{cases}$$

For every role name $r \in \mathbb{N}_R$ and all domain elements u , we likewise define

$$r^{\mathcal{I}_v}(u, w) := \begin{cases} v(\lambda, ui) & \text{if } w = ui \text{ with } i \in \Phi_r, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we define $a^{\mathcal{I}_v} := \varepsilon$ for the individual name a . It can be shown by induction on the structure of C that

$$C^{\mathcal{I}_v}(u) = v(C, u) \text{ for all } C \in \text{sub}(\mathcal{O}), u \in \{1, \dots, n\}^* \quad (4)$$

holds. In this proof by induction

- the base case follows trivially from the definition of \mathcal{I}_v ,
- the cases \top , $C \sqcap D$, and $C \rightarrow D$ follow from (P1), (P2), and Definition 8,
- the case $\neg C$ follows from (P3), and
- Definition 9 and (P4) entail the cases $\exists r.C$ and $\forall r.C$.

For details, we refer the reader to the technical report (Borgwardt, Distel, and Peñaloza 2013).

It remains to show that \mathcal{I}_v is indeed a model of \mathcal{O} . For every $\langle a: C \boxtimes q \rangle \in \mathcal{A}$, the Hintikka tree satisfies $C \boxtimes_\varepsilon q$, and thus we obtain from (4), (P1), and (P2):

$$C^{\mathcal{I}_v}(a^{\mathcal{I}_v}) = v(C, \varepsilon) \boxtimes v(q, \varepsilon) = q,$$

and similarly for assertions of the form $\langle a: C \boxtimes a: D \rangle$.

Now, let $u \in \{1, \dots, n\}^*$ be a domain element of \mathcal{I}_v and $\langle C \sqsubseteq D \geq q \rangle \in \mathcal{T}$. Since $p \in \mathcal{V}_\mathcal{O}$ and \lesssim_u is compatible with \mathcal{T} , it must hold that

$$q \lesssim_u \text{res}_u(C, D) = \begin{cases} 1 & \text{if } C \lesssim_u D \\ D & \text{if } D <_u C \end{cases} \\ \stackrel{\text{(P2)}}{=} \begin{cases} 1 & \text{if } v(C, u) \leq v(D, u) \\ D & \text{if } v(D, u) < v(C, u). \end{cases}$$

Thus, (P1) and (P2) yield

$$q = v(q, u) \leq \begin{cases} v(1, u) & \text{if } v(C, u) \leq v(D, u) \\ v(D, u) & \text{if } v(D, u) < v(C, u) \end{cases} \\ = v(C, u) \Rightarrow v(D, u) \\ = C^{\mathcal{I}_v}(u) \Rightarrow D^{\mathcal{I}_v}(u). \quad \square$$

Conversely, every model can be transformed into a Hintikka tree. The idea is to *unravel* the model into an infinite tree, and then abstract from the specific values by just considering the ordering between the elements of \mathcal{U} . This idea is formalized next.

Proposition 11. *If \mathcal{O} has a model, then there is a Hintikka tree for \mathcal{O} .*

Proof. Let \mathcal{I} be a model of \mathcal{O} . We use this model to guide the construction of a Hintikka tree for \mathcal{O} . During this construction, we will recursively generate a mapping $g: \{1, \dots, n\}^* \rightarrow \Delta^{\mathcal{I}}$ specifying which domain elements correspond to the nodes in the tree. This mapping will satisfy the following condition for all $\alpha, \beta \in \mathcal{V}_\mathcal{O} \cup \text{sub}(\mathcal{O})$ and all $u \in \{1, \dots, n\}^*$:

$$\text{(P5)} \quad \alpha \lesssim_u \beta \text{ iff } \alpha^{\mathcal{I}}(g(u)) \leq \beta^{\mathcal{I}}(g(u)),$$

where we define $q^{\mathcal{I}}(x) := q$ for all $q \in \mathcal{V}_\mathcal{O}$ and $x \in \Delta^{\mathcal{I}}$.

We first consider the root node ε of the tree. Recall that the ontology contains a local ordered ABox, using only the individual name a . We define $g(\varepsilon) := a^{\mathcal{I}}$ and the Hintikka ordering \lesssim_ε as follows for all $\alpha, \beta \in \mathcal{V}_\mathcal{O} \cup \text{sub}(\mathcal{O})$:

$$\alpha \lesssim_\varepsilon \beta \text{ iff } \alpha^{\mathcal{I}}(a^{\mathcal{I}}) \leq \beta^{\mathcal{I}}(a^{\mathcal{I}}).$$

We extend this order to the elements in $\text{sub}_\uparrow(\mathcal{O}) \cup \{\lambda, \neg\lambda\}$ arbitrarily, in such a way that for all $\alpha, \beta \in \mathcal{U}$ we have $\alpha \lesssim_\varepsilon \beta$ iff $\text{inv}(\beta) \lesssim_\varepsilon \text{inv}(\alpha)$. It is straightforward to show that \lesssim_ε is an element of order(\mathcal{U}) satisfying (P5) at ε , and that furthermore \lesssim_ε is a Hintikka ordering that is compatible with \mathcal{T} (cf. (Borgwardt, Distel, and Peñaloza 2013)).

Assume now that we have already defined $g(u)$ and \lesssim_u for a node $u \in \{1, \dots, n\}^*$ such that (P5) is satisfied. For all $i \in \{1, \dots, n\}$, we now construct \lesssim_{ui} in such a way that the tuple $(\lesssim_u, \lesssim_{u1}, \dots, \lesssim_{un})$ satisfies the Hintikka condition. For brevity, we consider only the case that $i = \phi(\exists r.D)$;

value restrictions can be handled using similar arguments. Since \mathcal{I} is witnessed, there must be a $y_i \in \Delta^{\mathcal{I}}$ such that $(\exists r.D)^{\mathcal{I}}(g(u)) = \min(r^{\mathcal{I}}(g(u), y_i), D^{\mathcal{I}}(y_i))$. We define $g(ui) := y_i$, and \lesssim_{ui} for all $\alpha, \beta \in \mathcal{U}$ by

$$\alpha \lesssim_{ui} \beta \text{ iff } \alpha^{\mathcal{I}}(g(ui)) \leq \beta^{\mathcal{I}}(g(ui)), \quad (5)$$

where we abbreviate $\lambda^{\mathcal{I}}(g(ui)) := r^{\mathcal{I}}(g(u), g(ui))$ and $(C_{\uparrow})^{\mathcal{I}}(g(ui)) := C^{\mathcal{I}}(g(u))$ for all concepts $C \in \text{sub}(\mathcal{O})$. It is clear that \lesssim_{ui} behaves on $\mathcal{V}_{\mathcal{O}} \cup \text{sub}_{\uparrow}(\mathcal{O})$ exactly as \lesssim_u does on $\mathcal{V}_{\mathcal{O}} \cup \text{sub}(\mathcal{O})$. Following the same arguments used for the root node, it is easy to show that \lesssim_{ui} is actually a Hintikka ordering compatible with \mathcal{T} .

We show the Hintikka condition for $(\lesssim_u, \lesssim_{u1}, \dots, \lesssim_{un})$. For the case $i = \phi(\exists r.D)$, the construction of g yields that $(\exists r.D)^{\mathcal{I}}(g(u)) = \min(r^{\mathcal{I}}(g(u), g(ui)), D^{\mathcal{I}}(g(ui)))$, and thus

$$((\exists r.D)_{\uparrow})^{\mathcal{I}}(g(ui)) = \min(\lambda^{\mathcal{I}}(g(ui)), D^{\mathcal{I}}(g(ui))).$$

Using (5), we obtain $(\exists r.D)_{\uparrow} \models_{ui} \min_{ui}(\lambda, D)$ as required. Furthermore, for all $i \in \Phi_r$, it holds that

$$\begin{aligned} (\exists r.D)^{\mathcal{I}}(g(u)) &= \sup_{y \in \Delta^{\mathcal{I}}} \min(r^{\mathcal{I}}(g(u), y), D^{\mathcal{I}}(y)) \\ &\geq \min(r^{\mathcal{I}}(g(u), g(ui)), D^{\mathcal{I}}(g(ui))), \end{aligned}$$

which similarly shows $(\exists r.D)_{\uparrow} \gtrsim_{ui} \min_{ui}(\lambda, D)$. Similar arguments apply to the value restrictions in $\text{sub}(\mathcal{O})$.

Finally, for every $\langle a : C \boxtimes q \rangle \in \mathcal{A}$, we have $C^{\mathcal{I}}(a^{\mathcal{I}}) \boxtimes q$, and thus $C \boxtimes_{\varepsilon} q$ by definition of \lesssim_{ε} , and similarly for assertions of the form $\langle a : C \boxtimes a : D \rangle$. Hence, the tree defined by \lesssim_u , for $u \in \{1, \dots, n\}^*$, is a Hintikka tree for \mathcal{O} . \square

Propositions 10 and 11 show that Hintikka trees characterize consistency of an ontology with a local ordered ABox. In other words, deciding the existence of a Hintikka tree for \mathcal{O} suffices for deciding consistency of \mathcal{O} . We now turn our attention to the former problem, and show that it can be solved in exponential time in the size of \mathcal{O} . For this, we construct a *looping tree automaton* whose runs correspond exactly to such Hintikka trees. Thus, the automaton accepts a non-empty language iff the ontology \mathcal{O} is consistent.

A *looping automaton* over n -ary (infinite) trees is a tuple $\mathbf{A} = (Q, I, \Delta)$, consisting of a non-empty set Q of *states*, a subset $I \subseteq Q$ of *initial states*, and a *transition relation* $\Delta \subseteq Q^{n+1}$. A *run* of this automaton is a mapping $\rho : \{1, \dots, n\}^* \rightarrow Q$ such that (i) $\rho(\varepsilon) \in I$, and (ii) for all $u \in \{1, \dots, n\}^*$, we have $(\rho(u), \rho(u1), \dots, \rho(un)) \in \Delta$. \mathbf{A} is *non-empty* iff it has a run.

Definition 12. The *Hintikka automaton* for an ontology \mathcal{O} is the looping tree automaton $\mathbf{A}_{\mathcal{O}} := (Q_{\mathcal{O}}, I_{\mathcal{O}}, \Delta_{\mathcal{O}})$, where

- $Q_{\mathcal{O}}$ is the set of all Hintikka orderings compatible with \mathcal{T} ,
- $I_{\mathcal{O}} := \{\lesssim_H \in Q_{\mathcal{O}} \mid \lesssim_H \text{ is compatible with } \mathcal{A}\}$, and
- $\Delta_{\mathcal{O}}$ contains all tuples from $Q_{\mathcal{O}}^{n+1}$ that satisfy the Hintikka condition.

It is easy to see that the runs of $\mathbf{A}_{\mathcal{O}}$ are exactly the Hintikka trees for \mathcal{O} . Observe that the number of Hintikka orderings for \mathcal{O} is bounded by $2^{|\mathcal{U}|^2}$ and the cardinality of

$\mathcal{U} = \mathcal{V}_{\mathcal{O}} \cup \text{sub}(\mathcal{O}) \cup \text{sub}_{\uparrow}(\mathcal{O}) \cup \{\lambda, \neg\lambda\}$ is linear in the size of \mathcal{O} . Likewise, the arity n of the automaton is bounded by $|\text{sub}(\mathcal{O})|$, which is linear in the size of \mathcal{O} . Thus, the size of the Hintikka automaton $\mathbf{A}_{\mathcal{O}}$ is exponential in the size of \mathcal{O} . Since (non-)emptiness of looping tree automata can be decided in polynomial time (Vardi and Wolper 1986), we obtain overall an EXPTIME-decision procedure for consistency of ontologies with local ordered ABoxes in G- \mathcal{JALC} . Note that concept satisfiability in classical \mathcal{ALC} is already EXPTIME-hard w.r.t. general TBoxes (Schild 1991), and hence our complexity bounds are tight.

Theorem 13. *Consistency in G- \mathcal{JALC} w.r.t. local ordered ABoxes and witnessed models is EXPTIME-complete.*

In the following section, we remove the restriction to local ordered ABoxes and show that consistency remains EXPTIME-complete in the general case.

5 Reducing Consistency to Local Consistency

To decide consistency of G- \mathcal{JALC} -ontologies containing more than one individual name, we adapt a technique from classical DLs known as *pre-completion* (Hollunder 1996). Intuitively, we try to build a forest-shaped model that satisfies the ontology. This model is composed of a finite set of trees, one for each individual name appearing in the ABox, whose roots can be arbitrarily interconnected due to the presence of role assertions. As before, rather than explicitly building such models, we use total preorders to represent them in an abstract manner.

The idea of pre-completion is to extend the input ABox to a full specification of each individual, and then decide consistency w.r.t. the local ABoxes associated with each individual name. In our setting, this amounts to extending the input ABox to a total preorder $\lesssim_{\mathcal{A}}$. This preorder represents the nucleus of a model of the ontology. To extend this to a full model, we check an (ordered) local consistency condition for each of the individual names, and use $\lesssim_{\mathcal{A}}$ to combine the resulting tree-shaped interpretations.

More formally, let $\mathcal{O} = (\mathcal{A}, \mathcal{T})$ be an ontology, and let $\text{Ind}(\mathcal{A})$ denote the set of individual names occurring in \mathcal{A} . We define the order structure

$$\begin{aligned} \mathcal{W} := & \mathcal{V}_{\mathcal{O}} \cup \{a : C \mid a \in \text{Ind}(\mathcal{A}), C \in \text{sub}(\mathcal{O})\} \\ & \cup \{(a, b) : r \mid a, b \in \text{Ind}(\mathcal{A}), r \text{ occurs in } \mathcal{O}\} \\ & \cup \{(a, b) : \neg r \mid a, b \in \text{Ind}(\mathcal{A}), r \text{ occurs in } \mathcal{O}\} \end{aligned}$$

with $\text{inv}(a : C) := a : \neg C$ and $\text{inv}((a, b) : r) := (a, b) : \neg r$.

Definition 14 (pre-completion). A *pre-completion* of \mathcal{A} w.r.t. \mathcal{T} is a total preorder $\lesssim_{\mathcal{A}} \in \text{order}(\mathcal{W})$ such that:

- for every $a \in \text{Ind}(\mathcal{A})$ and all $C \in \text{sub}(\mathcal{O})$,
 - if $C = \top$, then $a : C \equiv_{\mathcal{A}} 1$,
 - if $C = D_1 \sqcap D_2$, then $a : C \equiv_{\mathcal{A}} \min_{\mathcal{A}}(a : D_1, a : D_2)$,
 - if $C = D_1 \rightarrow D_2$, then $a : C \equiv_{\mathcal{A}} \text{res}_{\mathcal{A}}(a : D_1, a : D_2)$;
- for every $\exists r.C \in \text{sub}(\mathcal{O})$ and $a, b \in \text{Ind}(\mathcal{A})$, we have $a : \exists r.C \gtrsim_{\mathcal{A}} \min_{\mathcal{A}}((a, b) : r, b : C)$;
- for every $\forall r.C \in \text{sub}(\mathcal{O})$ and $a, b \in \text{Ind}(\mathcal{A})$, we have $a : \forall r.C \lesssim_{\mathcal{A}} \text{res}_{\mathcal{A}}((a, b) : r, b : C)$;

- d) for all $a \in \text{Ind}(\mathcal{A})$ and every GCI $\langle C \sqsubseteq D \geq q \rangle \in \mathcal{T}$, we have $\text{res}_{\mathcal{A}}(a:C, a:D) \succeq_{\mathcal{A}} q$; and
- e) for every assertion $\langle \alpha \bowtie \beta \rangle \in \mathcal{A}$, we have $\alpha \bowtie_{\mathcal{A}} \beta$.

This definition generalizes the local conditions of Definitions 8 and 9 to handle several named individuals simultaneously. One difference is that we do not create witnesses for the quantified concepts here. This will be taken care of by testing the following local ordered ABoxes for consistency. For a pre-completion $\lesssim_{\mathcal{A}}$ and $a \in \text{Ind}(\mathcal{A})$, we define the local ordered ABox \mathcal{A}_a as the set of all order assertions $\langle \alpha \bowtie \beta \rangle$ over a and $\text{sub}(\mathcal{O})$ satisfying $\alpha \bowtie_{\mathcal{A}} \beta$.² That is,

$$\mathcal{A}_a := \{ \langle a:C \bowtie q \rangle \mid C \in \text{sub}(\mathcal{O}), q \in \mathcal{V}_{\mathcal{O}}, a:C \bowtie_{\mathcal{A}} q \} \\ \cup \{ \langle a:C \bowtie a:D \rangle \mid C, D \in \text{sub}(\mathcal{O}), a:C \bowtie_{\mathcal{A}} a:D \}.$$

Lemma 15. *An ontology $\mathcal{O} = (\mathcal{A}, \mathcal{T})$ is consistent iff there is a pre-completion $\lesssim_{\mathcal{A}}$ of \mathcal{A} w.r.t. \mathcal{T} such that, for every $a \in \text{Ind}(\mathcal{A})$, the ontology $\mathcal{O}_a := (\mathcal{A}_a, \mathcal{T})$ is consistent.*

Proof. Let \mathcal{I} be a model of \mathcal{O} . We define the total pre-order $\lesssim_{\mathcal{A}}$ by setting $\alpha \lesssim_{\mathcal{A}} \beta$ if and only if $\alpha^{\mathcal{I}} \leq \beta^{\mathcal{I}}$, where we set $((a, b):\neg r)^{\mathcal{I}} := 1 - r^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}})$. A straightforward argument that $\lesssim_{\mathcal{A}}$ is a pre-completion of \mathcal{A} w.r.t. \mathcal{T} and that \mathcal{I} is a model of $(\mathcal{A}_a, \mathcal{T})$ for each $a \in \text{Ind}(\mathcal{A})$ can be found in (Borgwardt, Distel, and Peñaloza 2013).

Conversely, let $\lesssim_{\mathcal{A}}$ be a pre-completion of \mathcal{A} w.r.t. \mathcal{T} and each $(\mathcal{A}_a, \mathcal{T})$ be consistent. By Proposition 11, there are Hintikka trees for $(\mathcal{A}_a, \mathcal{T})$ that consist of Hintikka orderings \lesssim_u^a for all $u \in \{1, \dots, n\}^*$, where n is the number of existential and value restrictions in $\text{sub}(\mathcal{O})$. Similar to the proof of Proposition 10, we first construct a function $v: \mathcal{W} \cup (\text{Ind}(\mathcal{A}) \times \mathcal{U} \times \{1, \dots, n\}^*) \rightarrow [0, 1]$ such that

- for all values $q \in \mathcal{V}_{\mathcal{O}}$, we have $v(q) = q$,
- for all $\alpha, \beta \in \mathcal{W}$, we have $v(\alpha) \leq v(\beta)$ iff $\alpha \lesssim_{\mathcal{A}} \beta$,
- for all $\alpha \in \mathcal{W}$, we have $v(\text{inv}(\alpha)) = 1 - v(\alpha)$,
- for every $C \in \text{sub}(\mathcal{O})$ and all $a \in \text{Ind}(\mathcal{A})$, we have $v(a:C) = v(a, C, \varepsilon)$,
- for all $u \in \{1, \dots, n\}^*$ and all $a \in \text{Ind}(\mathcal{A})$,
 - for all values $q \in \mathcal{V}_{\mathcal{O}}$, we have $v(a, q, u) = q$,
 - for all $\alpha, \beta \in \mathcal{U}$, we have $v(a, \alpha, u) \leq v(a, \beta, u)$ iff $\alpha \lesssim_u^a \beta$,
 - for all $\alpha \in \mathcal{U}$, we have $v(a, \text{inv}(\alpha), u) = 1 - v(a, \alpha, u)$, and
 - for all concepts $C \in \text{sub}(\mathcal{O})$ and all $i \in \{1, \dots, n\}$, we have that $v(a, C, u) = v(a, C_{\uparrow}, ui)$.

We will then use this function to define a model of \mathcal{O} .

Using the technique from the proof of Proposition 10, we first define v on \mathcal{W} . On the set $\mathcal{W}/\equiv_{\mathcal{A}}$ of all equivalence classes of $\equiv_{\mathcal{A}}$, we define an auxiliary function $\tilde{v}_{\mathcal{A}}: \mathcal{W}/\equiv_{\mathcal{A}} \rightarrow [0, 1]$, by setting $\tilde{v}_{\mathcal{A}}([q]_{\mathcal{A}}) := q$ for each $q \in \mathcal{V}_{\mathcal{O}}$ and treating the remaining equivalence classes as in (3). We then define $v(\alpha) := \tilde{v}_{\mathcal{A}}([\alpha]_{\mathcal{A}})$ for all $\alpha \in \mathcal{W}$.

For each $a \in \text{Ind}(\mathcal{A})$, $C \in \text{sub}(\mathcal{O})$, and $q \in \mathcal{V}_{\mathcal{O}}$, we now set $v(a, C, \varepsilon) := v(a:C)$ and $v(a, q, \varepsilon) := q$. The values of

²It actually suffices to consider only $\bowtie \in \{>, =, <\}$.

$v(a, \alpha, \varepsilon)$ for elements $\alpha \in \text{sub}_{\uparrow}(\mathcal{O}) \cup \{\lambda, \neg\lambda\}$ are irrelevant for the desired properties and can be fixed arbitrarily, as long as we have $v(a, \alpha, \varepsilon) \leq v(a, \beta, \varepsilon)$ iff $\alpha \lesssim_{\varepsilon}^a \beta$ and $v(a, \text{inv}(\alpha), \varepsilon) = 1 - v(a, \alpha, \varepsilon)$ for all $\alpha, \beta \in \mathcal{U}$, e.g. using the technique in (3). The definition of $v(a, \alpha, u)$ can now proceed as in the proof of Proposition 10 based on the Hintikka trees for $(\mathcal{A}_a, \mathcal{T})$. This construction ensures that v has the desired properties.

We now define the interpretation \mathcal{I} as follows:

- $\Delta^{\mathcal{I}} := \text{Ind}(\mathcal{A}) \times \{1, \dots, n\}^*$,
- $a^{\mathcal{I}} := (a, \varepsilon)$ for each $a \in \text{Ind}(\mathcal{A})$,
- $A^{\mathcal{I}}(a, u) := v(a, A, u)$ for all $a \in \text{Ind}(\mathcal{A})$, concept names $A \in \text{sub}(\mathcal{O})$, and $u \in \{1, \dots, n\}^*$, and
- $r^{\mathcal{I}}((a, u), (b, u')) := \begin{cases} v(a, \lambda, ui) & \text{if } a = b \text{ and } u' = ui \text{ with } i \in \Phi_r, \\ v((a, b):r) & \text{if } u = u' = \varepsilon \text{ and } r \text{ occurs in } \mathcal{O}, \\ 0 & \text{otherwise.} \end{cases}$

The interpretation of the remaining individual and concept names is irrelevant and can be fixed arbitrarily. As in Proposition 10, we can show by induction on the structure of C that $C^{\mathcal{I}}(a, u) = v(a, C, u)$ holds for all $C \in \text{sub}(\mathcal{O})$, $a \in \text{Ind}(\mathcal{A})$, and $u \in \{1, \dots, n\}^*$ by induction on the structure of C . The claim for \top , $\neg C$, $C \sqcap D$, and $C \rightarrow D$ follows as before from Condition a) of Definition 14 and the fact that each \lesssim_u^a is a Hintikka ordering.

Consider now an existential restriction $\exists r.C \in \text{sub}(\mathcal{O})$ and the domain element (a, ε) for some $a \in \text{Ind}(\mathcal{A})$. By the Hintikka condition and the induction hypothesis, we have $v(a, \exists r.C, u) = \min(r^{\mathcal{I}}((a, \varepsilon), (a, i_0)), C^{\mathcal{I}}(a, i_0))$, where $i_0 = \phi(\exists r.C)$, as in the proof of Proposition 10. Likewise, $v(a, \exists r.C, u) \geq \min(r^{\mathcal{I}}((a, \varepsilon), (a, i)), C^{\mathcal{I}}(a, i))$ holds for all $i \in \Phi_r$. Finally, for each $b \in \text{Ind}(\mathcal{A})$, we have $v(a, \exists r.C, u) \geq \min(r^{\mathcal{I}}((a, \varepsilon), (b, \varepsilon)), C^{\mathcal{I}}(b, \varepsilon))$ by Condition b) of Definition 14. Since (a, ε) does not have any other relevant r -successors, this shows the claim for $\exists r.C$ at (a, ε) . At the other domain elements, it can be shown as for Proposition 10. Similar arguments apply to all $\forall r.C \in \text{sub}(\mathcal{O})$.

Finally, the fact that \mathcal{I} is actually a model of \mathcal{O} is ensured by compatibility of all Hintikka orderings with \mathcal{T} and Conditions e) and d) of Definition 14. \square

Note that the cardinality of $\text{order}(\mathcal{W})$ is exponential in the size of \mathcal{O} , and all elements of $\text{order}(\mathcal{W})$ are of polynomial size. We can thus enumerate $\text{order}(\mathcal{W})$, check for each element whether it satisfies Definition 14 in polynomial time, and then execute the polynomially many local consistency tests as described by Lemma 15. This yields the following complexity result.

Corollary 16. *Consistency in G-JALC w.r.t. witnessed models is EXPTIME-complete.*

6 Satisfiability and Subsumption

We have described an exponential-time algorithm for deciding consistency of G-JALC ontologies. We now direct our attention at other standard reasoning problems in fuzzy

DLs; namely, deciding concept satisfiability and subsumption, and computing the best truth degrees to which these hold. Recall from Section 2 that for these reasoning problems we can restrict our attention to ontologies with an empty ABox.

Let now $\mathcal{O} = (\emptyset, \mathcal{T})$ be an ontology. It is easy to see that p -subsumption and p -satisfiability w.r.t. \mathcal{O} can be reduced in polynomial time to consistency w.r.t. local ordered ABoxes. More precisely, for any two concepts C, D and $p \in [0, 1]$,

- C is p -satisfiable w.r.t. \mathcal{O} iff $(\{\langle a : C \geq p \rangle\}, \mathcal{T})$ is consistent, and
- C is p -subsumed by D w.r.t. \mathcal{O} iff $(\{\langle a : C \rightarrow D < p \rangle\}, \mathcal{T})$ is inconsistent,

where a is an arbitrary individual name. We thus obtain the following result from Theorem 13.

Theorem 17. *Satisfiability and subsumption in $G\text{-}\mathcal{JALC}$ w.r.t. witnessed models are EXPTIME-complete.*

We now shift our attention to the problems of computing the *best* satisfiability and subsumption degrees. We first show that the local consistency checks required for deciding p -satisfiability and p -subsumption only depend on the position of p relative to the values occurring in \mathcal{T} , but not on the precise value of p . To prove this, we again use the preorders of the previous sections, and in particular Hintikka trees.

Lemma 18. *Let $p, p' \in (q_i, q_{i+1})$ for two adjacent values $q_i, q_{i+1} \in \mathcal{V}_{\mathcal{O}}$, and C be a concept. Then $(\{\langle a : C \bowtie p \rangle\}, \mathcal{T})$ is consistent iff $(\{\langle a : C \bowtie p' \rangle\}, \mathcal{T})$ is consistent.*

Proof. By Propositions 10 and 11, both consistency conditions are equivalent to the existence of Hintikka trees, albeit over different order structures. We denote by \mathcal{U}_p the order structure from Definition 7 over $\mathcal{V}_p := \mathcal{V}_{\mathcal{O}} \cup \{p, 1-p\}$, and by $\mathcal{U}_{p'}$ the one over $\mathcal{V}_{p'} := \mathcal{V}_{\mathcal{O}} \cup \{p', 1-p'\}$. Observe that the bijection $\iota : \mathcal{V}_p \rightarrow \mathcal{V}_{p'}$ that simply maps p to p' and $1-p$ to $1-p'$ and leaves the other values as they are, can be extended to a bijection between \mathcal{U}_p and $\mathcal{U}_{p'}$ by defining it as the identity on all elements outside of \mathcal{V}_p . Furthermore, it is compatible with the involutive operator inv , i.e. we have $\iota(\text{inv}(\alpha)) = \text{inv}(\iota(\alpha))$ for all $\alpha \in \mathcal{U}_p$.

It is straightforward to extend this bijection to Hintikka orderings and Hintikka trees (see (Borgwardt, Distel, and Peñaloza 2013) for details). Then there is a Hintikka tree for $(\{\langle a : C \bowtie p \rangle\}, \mathcal{T})$ iff there is one for $(\{\langle a : C \bowtie p' \rangle\}, \mathcal{T})$, which concludes the proof. \square

This shows that subsumption between C and D or satisfiability of C either holds for all values in an interval (q_i, q_{i+1}) , or for none of them.

Corollary 19. *For any two concepts C and D , the best subsumption degree of C and D w.r.t. \mathcal{O} and the best satisfiability degree of C w.r.t. \mathcal{O} are always in $\mathcal{V}_{\mathcal{O}}$.*

Since the best subsumption degree p of C and D is always a subsumption degree, i.e. C is p -subsumed by D , it suffices to check subsumption w.r.t. the values from $\mathcal{V}_{\mathcal{O}}$ in order to determine the best subsumption degree. Thus, we only have to execute linearly many (in-)consistency checks to compute the best subsumption degree.

However, it is possible that C is p -satisfiable for every $p \in (q_i, q_{i+1})$, but not q_{i+1} -satisfiable. Therefore, we check satisfiability for all values $\frac{q_i + q_{i+1}}{2}$. The best satisfiability degree is then the largest q_{i+1} for which this check succeeds (or 0 if it never succeeds). Again, this means that we have to execute linearly many consistency checks to compute the best satisfiability degree.

By combining these reductions with Theorem 13, we obtain the following results.

Corollary 20. *In $G\text{-}\mathcal{JALC}$ w.r.t. witnessed models, best subsumption and satisfiability degrees can be computed in exponential time.*

7 Conclusions

We have studied the standard reasoning problems for the fuzzy DL $G\text{-}\mathcal{JALC}$ w.r.t. witnessed model semantics. The contributions of the paper are twofold. First, we have shown that, contrary to popular belief, reasoning in this logic cannot be restricted to reasoning over finitely-valued models without affecting its consequences. In particular, this implies that the algorithms based on maintaining only a finite set of truth degrees (Bobillo et al. 2009; Bobillo et al. 2012) are incomplete for the general semantics. Moreover, this also implies that the logic does not have the finite model property, and hence standard tableau-based approaches cannot terminate (Bobillo and Straccia 2007; Straccia and Bobillo 2007; Bobillo, Bou, and Straccia 2011).

As the second contribution of the paper, we showed that all standard reasoning problems can be solved in exponential time. To achieve this, we developed an automaton that decides the existence of a Hintikka tree, which is an abstract representation of a model of a given ontology. The main insight needed for this approach is that we can abstract from the precise truth degrees assigned by an interpretation, and focus only on their ordering.

As an added benefit, in our formalism we can express order assertions like $\langle \text{ana} : \text{Tall} > \text{bob} : \text{Tall} \rangle$, intuitively stating that Ana is taller than Bob, without needing to specify the precise degrees to which ana and bob belong to the concept Tall. This is similar to concrete domains (Lutz 2003), which can even compare values at unnamed domain elements. But concrete domains allow only for atomic attributes, whereas order assertions can also contain complex concepts.

As we have developed an automata-based algorithm, it is natural to ask whether previous automata-based approaches (Baader, Hladik, and Peñaloza 2008; Borgwardt and Peñaloza 2013a) can be adapted to this setting in order to handle the expressivity up to $G\text{-}\mathcal{JSHI}$, or provide better upper-bounds for reasoning w.r.t. *acyclic* TBoxes. We will study this problem in future work. We also plan to adapt these ideas into a tableau-based algorithm which is more suitable for implementation.

Recall that we have restricted our framework to reasoning w.r.t. *witnessed* models only. Indeed, this restriction is fundamental for our proof of Proposition 11. One open question is whether consistency of $G\text{-}\mathcal{JALC}$ ontologies w.r.t. *general* models is still decidable. We conjecture that it is, and in fact remains in EXPTIME.

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