Technische Universität Dresden

International Center for Computational Logic

COMPLEXITY THEORY

Lecture 8: NP-Complete Problems

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Knowledge-Based Systems

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More recent versions of this slide deck might be available.
For the most current version of this course, see
`https://iccl.inf.tu-dresden.de/web/Complexity_Theory/en`

The First NP-Complete Problem

- The problem of deciding if a propositional formula is satisfiable is NP-complete.
  - See the Cook–Levin theorem from the previous lecture.

The First NP-Complete Problem

- The problem of deciding if a propositional formula is satisfiable is NP-complete.
  - See the Cook–Levin theorem from the previous lecture.
- **Sat** is a special case: the formula must be in CNF.
  - It is also NP-complete, but to show that, we need to modify our proof of the theorem.

Towards More NP-Complete Problems

Starting with **Sat**, one can show other problems **P** to be NP-complete, each time performing two steps:

(1) Show that **P** $\in$ NP

(2) Find a known NP-complete problem **P′** and reduce **P′** $\leq_p$ **P**

Thousands of problems have now been shown to be NP-complete.
(See Garey and Johnson for an early survey)

In this course:

	$\leq_p$	**Clique**	$\leq_p$	**Independent Set**
Sat	$\leq_p$	**3-Sat**	$\leq_p$	**Dir. Hamiltonian Path**
	$\leq_p$	**Subset Sum**	$\leq_p$	**Knapsack**

NP-Completeness of **Clique**

Theorem 8.1: Clique is NP-complete.

Clique: Given G, k, does G contain a clique of size k?

Proof:

(1) **Clique** $\in$ NP

Take the vertex set of a clique of size k as a certificate.

(2) **Clique** is NP-hard

We show **Sat** $\leq_p$ **Clique**

To every CNF-formula φ, assign a graph G_φ and a number k_φ such that

$$\varphi \text{ satisfiable} \iff G_\varphi \text{ contains a clique of size } k_\varphi$$

SAT $\leq_p$ CLIQUE

To every CNF-formula φ assign a graph G_φ and a number k_φ such that

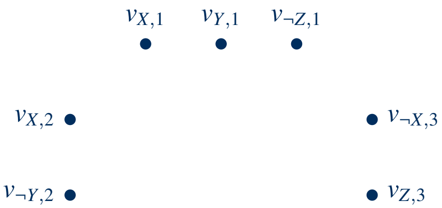
$$\varphi \text{ satisfiable if and only if } G_\varphi \text{ contains a clique of size } k_\varphi$$

Given $\varphi = C_1 \wedge \cdots \wedge C_k$,

- Set $k_\varphi := k$
- For each clause C_j and literal $L \in C_j$, add a vertex $v_{L,j}$
- Add edge $\{v_{L,j}, v_{K,i}\}$ if $i \neq j$ and $L \wedge K$ is satisfiable (that is: if $L \neq \neg K$ and $\neg L \neq K$)

Example 8.2:

$$\underbrace{(X \vee Y \vee \neg Z)}_{C_1} \wedge \underbrace{(X \vee \neg Y)}_{C_2} \wedge \underbrace{(\neg X \vee Z)}_{C_3}$$



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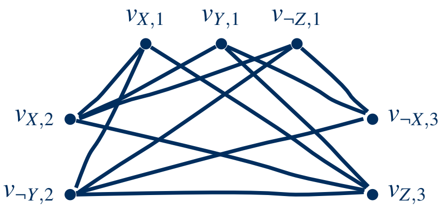

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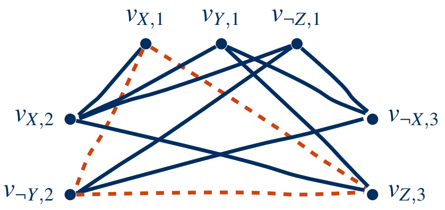

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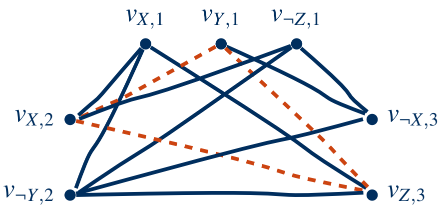

SAT $\leq_p$ CLIQUE

To every CNF-formula φ assign a graph G_φ and a number k_φ such that

$$\varphi \text{ satisfiable if and only if } G_\varphi \text{ contains clique of order } k_\varphi$$

Given $\varphi = C_1 \wedge \cdots \wedge C_k$:

- Set $k_\varphi := k$
- For each clause C_j and literal $L \in C_j$ add a vertex $v_{L,j}$
- Add edge $\{v_{L,j}, v_{K,i}\}$ if $i \neq j$ and $L \wedge K$ is satisfiable (that is: if $L \neq \neg K$ and $\neg L \neq K$)

Correctness:

G_φ has a clique of order k iff φ is satisfiable.

Complexity:

The reduction is clearly computable in polynomial time.

NP-Completeness of INDEPENDENT SET

INDEPENDENT SET

Input:	An undirected graph G and a natural number k
Problem:	Does G contain k vertices that share no edges (independent set)?

Theorem 8.3: INDEPENDENT SET is NP-complete.

NP-Completeness of **Independent Set**

Independent Set

Input:	An undirected graph G and a natural number k
Problem:	Does G contain k vertices that share no edges (independent set)?

Theorem 8.3: **Independent Set** is NP-complete.

Proof: Hardness by reduction **Clique** $\leq_p$ **Independent Set**:

- Given $G := (V, E)$ construct $\overline{G} := (V, \{\{u, v\} \mid \{u, v\} \notin E \text{ and } u \neq v\})$

NP-Completeness of Independent Set

Independent Set

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Problem:	Does G contain k vertices that share no edges (independent set)?

Theorem 8.3: **Independent Set** is NP-complete.

Proof: Hardness by reduction **Clique** $\leq_p$ **Independent Set**:

- Given $G := (V, E)$ construct $\overline{G} := (V, \{\{u, v\} \mid \{u, v\} \notin E \text{ and } u \neq v\})$
- A set $X \subseteq V$ induces a clique in G iff X induces an independent set in $\overline{G}$.
- Reduction: G has a clique of order k iff $\overline{G}$ has an independent set of order k.

□

NP-Completeness of **3-Sat**

3-Sat: Satisfiability of formulae in CNF with ≤ 3 literals per clause

Theorem 8.4: 3-Sat is NP-complete.

Proof: Hardness by reduction **Sat** $\leq_p$ **3-Sat**:

- Given: φ in CNF
- Construct φ' by replacing clauses $C_i = (L_1 \vee \cdots \vee L_k)$ with $k > 3$ by

$$C_i' := (L_1 \vee Y_1) \;\wedge\; (\neg Y_1 \vee L_2 \vee Y_2) \;\wedge \ldots \;\wedge\; (\neg Y_{k-1} \vee L_k)$$

  Here, the Y_j are fresh variables for each clause.
- Claim: φ is satisfiable iff φ' is satisfiable.

Example

Let $\varphi := (X_1 \lor X_2 \lor \neg X_3 \lor X_4) \quad \land \quad (\neg X_4 \lor \neg X_2 \lor X_5 \lor \neg X_1)$

Then $\varphi' :=$

$$
\begin{aligned}
&(X_1 \lor Y_1) \land \\
&(\neg Y_1 \lor X_2 \lor Y_2) \land \\
&(\neg Y_2 \lor \neg X_3 \lor Y_3) \land \\
&(\neg Y_3 \lor X_4) \land \\
&(\neg X_4 \lor Z_1) \land \\
&(\neg Z_1 \lor \neg X_2 \lor Z_2) \land \\
&(\neg Z_2 \lor X_5 \lor Z_3) \land \\
&(\neg Z_3 \lor \neg X_1)
\end{aligned}
$$

Proving NP-Completeness of 3-Sat

"⇒" Given $\varphi := \bigwedge_{i=1}^{m} C_i$ with clauses C_i, show that, if φ is satisfiable, then φ' is satisfiable.

For a satisfying assignment β for φ, define an assignment β' for φ' as follows.

For each $C := (L_1 \vee \cdots \vee L_k)$ with $k > 3$, the formula φ contains

$$C' = (L_1 \vee Y_1) \;\wedge\; (\neg Y_1 \vee L_2 \vee Y_2) \;\wedge \ldots \wedge\; (\neg Y_{k-1} \vee L_k) \text{ in } \varphi'$$

As β satisfies φ, there is $i \leq k$ such that $\beta(L_i) = 1$, i.e., $\begin{array}{l} \beta(X) = 1 \text{ if } L_i = X \\ \beta(X) = 0 \text{ if } L_i = \neg X \end{array}$

Set $\begin{array}{ll} \beta'(Y_j) = 1 & \text{for } j < i \\ \beta'(Y_j) = 0 & \text{for } j \geq i \\ \beta'(X) = \beta(X) & \text{for all variables in } \varphi \end{array}$

This is a satisfying assignment for φ'.

Proving NP-Completeness of 3-Sat

"$\Leftarrow$" Show that, if φ' is satisfiable, then so is φ.

Suppose β is a satisfying assignment for φ'—then β satisfies φ:

Let $C := (L_1 \vee \cdots \vee L_k)$ be a clause of φ

(1) If $k \leq 3$, then C is a clause of φ'.

(2) If $k > 3$, then

$$C' = (L_1 \vee Y_1) \ \wedge \ (\neg Y_1 \vee L_2 \vee Y_2) \ \wedge \ldots \ \wedge \ (\neg Y_{k-1} \vee L_k) \text{ in } \varphi'$$

β must satisfy at least one L_i, $1 \leq i \leq k$

Case (2) follows since, if $\beta(L_i) = 0$ for all $i \leq k$, then C' can be reduced to

$$\begin{aligned} C' &= (Y_1) \ \wedge \ (\neg Y_1 \vee Y_2) \ \wedge \ldots \ \wedge \ (\neg Y_{k-1}) \\ &\equiv Y_1 \ \wedge \ (Y_1 \to Y_2) \ \wedge \ldots \ \wedge \ (Y_{k-2} \to Y_{k-1}) \ \wedge \neg Y_{k-1} \end{aligned}$$

which is not satisfiable. $\square$

NP-Completeness of Directed Hamiltonian Path

Directed Hamiltonian Path

Input:	A directed graph G.
Problem:	Is there a directed path in G containing every vertex exactly once?

Theorem 8.5: **Directed Hamiltonian Path** is NP-complete.

NP-Completeness of **Directed Hamiltonian Path**

Directed Hamiltonian Path

Input: A directed graph G.

Problem: Is there a directed path in G containing every vertex exactly once?

Theorem 8.5: **Directed Hamiltonian Path** is NP-complete.

Proof:

(1) **Directed Hamiltonian Path** $\in$ NP:
Take the path to be the certificate.

Digression: How to design reductions

Task: Show that problem **P** (**Directed Hamiltonian Path**) is NP-hard.

- Arguably, the most important part is to decide where to start from.

  That is, which problem to reduce to **Directed Hamiltonian Path**?

Digression: How to design reductions

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- Arguably, the most important part is to decide where to start from.

  That is, which problem to reduce to **Directed Hamiltonian Path**?

- Considerations:
  - Is there an NP-complete problem similar to **P**?
    (for example, **Clique** and **Independent Set**)
  - It is not always beneficial to choose a problem of the same type
    (for example, reducing a graph problem to a graph problem)
    - For instance, **Clique** and **Independent Set** are “local” problems
      (is there a set of vertices inducing some structure?)
    - Hamiltonian Path is a global problem
      (find a structure—the Hamiltonian path—containing all vertices)

Digression: How to design reductions

Task: Show that problem **P** (**Directed Hamiltonian Path**) is NP-hard.

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  That is, which problem to reduce to **Directed Hamiltonian Path**?
- Considerations:
  - Is there an NP-complete problem similar to **P**?
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  - It is not always beneficial to choose a problem of the same type
    (for example, reducing a graph problem to a graph problem)
    - For instance, **Clique** and **Independent Set** are “local” problems
      (is there a set of vertices inducing some structure?)
    - Hamiltonian Path is a global problem
      (find a structure—the Hamiltonian path—containing all vertices)
- How to design the reduction:
  - Does your problem come from an optimisation problem?
    If so: a maximisation problem? a minimisation problem?
  - Learn from examples, have good ideas.

Directed Hamiltonian Path

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Theorem 8.5: **Directed Hamiltonian Path** is NP-complete.

Proof:

(1) **Directed Hamiltonian Path** $\in$ NP:
Take the path to be the certificate.

(2) **Directed Hamiltonian Path** is NP-hard:
3-Sat $\leq_p$ **Directed Hamiltonian Path**

NP-Completeness of Directed Hamiltonian Path

Proof (Proof idea): (see blackboard for details)
Let $\varphi := \bigwedge_{i=1}^{k} C_i$ and $C_i := (L_{i,1} \vee L_{i,2} \vee L_{i,3})$

- For each variable X occurring in φ, we construct a directed graph ("gadget") that allows only two Hamiltonian paths: "true" and "false"
- Gadgets for each variable are "chained" in a directed fashion, so that all variables must be assigned one value
- Clauses are represented by vertices that are connected to the gadgets in such a way that they can only be visited on a Hamiltonian path that corresponds to an assignment where they are true

Details are also given in [Sipser, Theorem 7.46].

Example 8.6: $\varphi := C_1 \wedge C_2$ where $C_1 := (X \vee \neg Y \vee Z)$ and $C_2 := (\neg X \vee Y \vee \neg Z)$ (see blackboard)

Towards More NP-Complete Problems

Starting with **Sat**, one can show other problems **P** to be NP-complete, each time performing two steps:

(1) Show that **P** $\in$ NP

(2) Find a known NP-complete problem **P′** and reduce **P′** $\leq_p$ **P**

Thousands of problems have now been shown to be NP-complete.
(See Garey and Johnson for an early survey)

In this course:

	$\leq_p$	**Clique**	$\leq_p$	**Independent Set**
Sat	$\leq_p$	**3-Sat**	$\leq_p$	**Dir. Hamiltonian Path**
	$\leq_p$	**Subset Sum**	$\leq_p$	**Knapsack**

NP-Completeness of **Subset Sum**

Subset Sum

Input: A collection[1] of positive integers $S = \{a_1, \ldots, a_k\}$ and a target integer t.

Problem: Is there a subset $T \subseteq S$ such that $\sum_{a_i \in T} a_i = t$?

Theorem 8.7: **Subset Sum** is NP-complete.

Proof:

(1) **Subset Sum** $\in$ NP: Take T to be the certificate.

(2) **Subset Sum** is NP-hard: **3-Sat** $\leq_p$ **Subset Sum**

[1]) This "collection" is supposed to be a multi-set, i.e., we allow the same number to occur several times. The solution "subset" can likewise use numbers multiple times, but not more often than they occured in the given collection.

Example

$$(X_1 \vee X_2 \vee X_3) \wedge (\neg X_1 \vee \neg X_4 \vee X_5) \wedge (X_4 \vee \neg X_2 \vee \neg X_3)$$

		X_1	X_2	X_3	X_4	X_5	C_1	C_2	C_3
t_1	=	1	0	0	0	0	1	0	0
f_1	=	1	0	0	0	0	0	1	0
t_2	=		1	0	0	0	1	0	0
f_2	=		1	0	0	0	0	0	1
t_3	=			1	0	0	1	0	0
f_3	=			1	0	0	0	0	1
t_4	=				1	0	0	0	1
f_4	=				1	0	0	1	0
t_5	=					1	0	1	0
f_5	=					1	0	0	0
$m_{1,1}$	=						1	0	0
$m_{1,2}$	=						1	0	0
$m_{2,1}$	=							1	0
$m_{2,2}$	=							1	0
$m_{3,1}$	=								1
$m_{3,2}$	=								1
t	=	1	1	1	1	1	3	3	3

$\textsc{Sat} \leq_p \textsc{Subset Sum}$

Given: $\varphi := C_1 \wedge \cdots \wedge C_k$ in 3-CNF

Let $X_1, \ldots, X_n$ be the variables in φ. For each X_i let

$t_i := a_1 \ldots a_n c_1 \ldots c_k$, where $a_j := \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ and $c_j := \begin{cases} 1 & X_i \text{ occurs in } C_j \\ 0 & \text{otherwise} \end{cases}$

$f_i := a_1 \ldots a_n c_1 \ldots c_k$, where $a_j := \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ and $c_j := \begin{cases} 1 & \neg X_i \text{ occurs in } C_j \\ 0 & \text{otherwise} \end{cases}$

Example

$$(X_1 \vee X_2 \vee X_3) \wedge (\neg X_1 \vee \neg X_4 \vee X_5) \wedge (X_4 \vee \neg X_2 \vee \neg X_3)$$

		X_1	X_2	X_3	X_4	X_5	C_1	C_2	C_3
t_1	=	1	0	0	0	0	1	0	0
f_1	=	1	0	0	0	0	0	1	0
t_2	=		1	0	0	0	1	0	0
f_2	=		1	0	0	0	0	0	1
t_3	=			1	0	0	1	0	0
f_3	=			1	0	0	0	0	1
t_4	=				1	0	0	0	1
f_4	=				1	0	0	1	0
t_5	=					1	0	1	0
f_5	=					1	0	0	0
$m_{1,1}$	=						1	0	0
$m_{1,2}$	=						1	0	0
$m_{2,1}$	=							1	0
$m_{2,2}$	=							1	0
$m_{3,1}$	=								1
$m_{3,2}$	=								1
t	=	1	1	1	1	1	3	3	3

SAT $\leq_p$ SUBSET SUM

Further, for each clause C_i, take two integers $m_{i,1} = m_{i,2} = 10^{k-i}$.

Definition of S: Let

$$S := \{t_i, f_i \mid 1 \leq i \leq n\} \cup \{m_{i,1}, m_{i,2} \mid 1 \leq i \leq k\}$$

Target: Finally, choose as target

$$t := a_1 \ldots a_n c_1 \ldots c_k \text{ where } a_i := 1 \text{ and } c_i := 3$$

Claim: There is $T \subseteq S$ with $\sum_{a_i \in T} a_i = t$ iff φ is satisfiable.

Example

$$(X_1 \vee X_2 \vee X_3) \wedge (\neg X_1 \vee \neg X_4 \vee X_5) \wedge (X_4 \vee \neg X_2 \vee \neg X_3)$$

		X_1	X_2	X_3	X_4	X_5	C_1	C_2	C_3
t_1	=	1	0	0	0	0	1	0	0
f_1	=	1	0	0	0	0	0	1	0
t_2	=		1	0	0	0	1	0	0
f_2	=		1	0	0	0	0	0	1
t_3	=			1	0	0	1	0	0
f_3	=			1	0	0	0	0	1
t_4	=				1	0	0	0	1
f_4	=				1	0	0	1	0
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$m_{3,1}$	=								1
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t	=	1	1	1	1	1	3	3	3

NP-Completeness of Subset Sum

Let $\varphi := \bigwedge C_i$ $\qquad$ C_i: clauses

Show: If φ is satisfiable, then there is $T \subseteq S$ with $\sum_{s \in T} s = t$.

Let β be a satisfying assignment for φ.

$$\text{Set} \quad T_1 := \quad \{t_i \mid \beta(X_i) = 1,\ 1 \leq i \leq m\} \cup \\ \{f_i \mid \beta(X_i) = 0,\ 1 \leq i \leq m\}$$

Further, for each clause C_i let r_i be the number of satisfied literals in C_i (w.r.t. β).

Set $T_2 := \{m_{i,j} \mid 1 \leq i \leq k, \quad 1 \leq j \leq 3 - r_i\}$,

and define $T := T_1 \cup T_2$.

It follows that $\sum_{s \in T} s = t$.

NP-Completeness of Subset Sum

Show: If there is $T \subseteq S$ with $\sum_{s \in T} s = t$, then φ is satisfiable.

Suppose $\sum_{s \in T} s = t$ for some $T \subseteq S$.

Define $\beta(X_i) = \begin{cases} 1 & \text{if } t_i \in T \\ 0 & \text{if } f_i \in T \end{cases}$

This is well defined as, for all i, we have $t_i \in T$ or $f_i \in T$ but not both.

Further, for each clause, there must be one literal set to 1, as, for all i,

$m_{i,1} + m_{i,2} = 2 \neq 3$. □

Towards More NP-Complete Problems

Starting with **Sat**, one can show other problems **P** to be NP-complete, each time performing two steps:

(1) Show that **P** $\in$ NP

(2) Find a known NP-complete problem **P′** and reduce **P′** $\leq_p$ **P**

Thousands of problems have now been shown to be NP-complete.
(See Garey and Johnson for an early survey)

In this course:

$$
\begin{array}{llll}
 & \leq_p \textbf{Clique} & \leq_p \textbf{Independent Set} \\
\textbf{Sat} & \leq_p \textbf{3-Sat} & \leq_p \textbf{Dir. Hamiltonian Path} \\
 & \leq_p \textbf{Subset Sum} & \leq_p \textbf{Knapsack}
\end{array}
$$

NP-completeness of Knapsack

Knapsack

Input: A set $I := \{1, \ldots, n\}$ of items
each of value v_i and weight w_i for $1 \leq i \leq n$,
target value t and weight limit ℓ

Problem: Is there $T \subseteq I$ such that
$\sum_{i \in T} v_i \geq t$ and $\sum_{i \in T} w_i \leq \ell$?

Theorem 8.8: **Knapsack** is NP-complete.

NP-completeness of KNAPSACK

KNAPSACK

Input: A set $I := \{1, \ldots, n\}$ of items
each of value v_i and weight w_i for $1 \leq i \leq n$,
target value t and weight limit ℓ

Problem: Is there $T \subseteq I$ such that
$\sum_{i \in T} v_i \geq t$ and $\sum_{i \in T} w_i \leq \ell$?

Theorem 8.8: **KNAPSACK** is NP-complete.

Proof:

(1) **KNAPSACK** $\in$ NP: Take T to be the certificate.

(2) **KNAPSACK** is NP-hard: **SUBSET SUM** $\leq_p$ **KNAPSACK**

Subset Sum $\leq_p$ Knapsack

	Given:	$S := \{a_1, \ldots, a_n\}$	collection of positive integers
Subset Sum:		t	target integer
	Problem:	Is there a subset $T \subseteq S$ such that $\sum_{a_i \in T} a_i = t$?	

SUBSET SUM $\leq_p$ KNAPSACK

	Given:	$S := \{a_1, \ldots, a_n\}$	collection of positive integers
Subset Sum:		t	target integer
	Problem:	Is there a subset $T \subseteq S$ such that $\sum_{a_i \in T} a_i = t$?	

Reduction: From this input to **SUBSET SUM** construct

- set of items $I := \{1, \ldots, n\}$
- weights and values $v_i = w_i = a_i$ for all $1 \leq i \leq n$
- target value $t' := t$ and weight limit $\ell := t$

Subset Sum $\leq_p$ Knapsack

	Given:	$S := \{a_1, \ldots, a_n\}$	collection of positive integers
Subset Sum:		t	target integer
	Problem:	Is there a subset $T \subseteq S$ such that $\sum_{a_i \in T} a_i = t$?	

Reduction: From this input to **Subset Sum** construct

- set of items $I := \{1, \ldots, n\}$
- weights and values $v_i = w_i = a_i$ for all $1 \leq i \leq n$
- target value $t' := t$ and weight limit $\ell := t$

Clearly: For every $T \subseteq S$

$$\sum_{a_i \in T} a_i = t \qquad \text{iff} \qquad \begin{array}{ll} \sum_{a_i \in T} v_i \geq t' & = t \\ \sum_{a_i \in T} w_i \leq \ell & = t \end{array}$$

Hence: The reduction is correct and polynomial-time.

A Polynomial-Time Algorithm for **Knapsack**

Knapsack can be solved in time $O(n\ell)$ using dynamic programming

Initialisation:

- Create an $(\ell + 1) \times (n + 1)$ matrix M
- Set $M(w, 0) := 0$ for all $1 \leq w \leq \ell$ and $M(0, i) := 0$ for all $1 \leq i \leq n$

Example

Input $I = \{1, 2, 3, 4\}$ with

Values: $v_1 = 1$ $v_2 = 3$ $v_3 = 4$ $v_4 = 2$

Weight: $w_1 = 1$ $w_2 = 1$ $w_3 = 3$ $w_4 = 2$

Weight limit: $\ell = 5$ Target value: $t = 7$

weight limit w	max. total value from first i items				
	$i = 0$	$i = 1$	$i = 2$	$i = 3$	$i = 4$
0					
1					
2					
3					
4					
5					

Set $M(w, 0) := 0$ for all $1 \leq w \leq \ell$ and $M(0, i) := 0$ for all $1 \leq i \leq n$

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Weight limit: $\ell = 5$ Target value: $t = 7$

weight limit w	max. total value from first i items				
	$i = 0$	$i = 1$	$i = 2$	$i = 3$	$i = 4$
0	0	0	0	0	0
1	0				
2	0				
3	0				
4	0				
5	0				

Set $M(w, 0) := 0$ for all $1 \le w \le \ell$ and $M(0, i) := 0$ for all $1 \le i \le n$

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A Polynomial-Time Algorithm for KNAPSACK

KNAPSACK can be solved in time $O(n\ell)$ using dynamic programming

Initialisation:

- Create an $(\ell + 1) \times (n + 1)$ matrix M
- Set $M(w, 0) := 0$ for all $1 \leq w \leq \ell$ and $M(0, i) := 0$ for all $1 \leq i \leq n$

Computation: Assign further $M(w, i)$ to be the largest total value obtainable by selecting from the first i items with weight limit w:

For $i = 0, 1, \ldots, n - 1$ set $M(w, i + 1)$ as

$$M(w, i + 1) := \max\left\{M(w, i),\ M(w - w_{i+1}, i) + v_{i+1}\right\}$$

Here, if $w - w_{i+1} < 0$ we always take $M(w, i)$.

Acceptance: If M contains an entry $\geq t$, accept. Otherwise reject.

Example

Input $I = \{1, 2, 3, 4\}$ with

Values: $v_1 = 1$ $v_2 = 3$ $v_3 = 4$ $v_4 = 2$

Weight: $w_1 = 1$ $w_2 = 1$ $w_3 = 3$ $w_4 = 2$

Weight limit: $\ell = 5$ Target value: $t = 7$

weight limit w	max. total value from first i items				
	$i = 0$	$i = 1$	$i = 2$	$i = 3$	$i = 4$
0	0	0	0	0	0
1	0				
2	0				
3	0				
4	0				
5	0				

For $i = 0, 1, \ldots, n-1$ set $M(w, i+1) := \max\{M(w, i),\ M(w - w_{i+1}, i) + v_{i+1}\}$

Example

Input $I = \{1, 2, 3, 4\}$ with

Values: $v_1 = 1$ $v_2 = 3$ $v_3 = 4$ $v_4 = 2$

Weight: $w_1 = 1$ $w_2 = 1$ $w_3 = 3$ $w_4 = 2$

Weight limit: $\ell = 5$ Target value: $t = 7$

weight limit w	max. total value from first i items				
	$i = 0$	$i = 1$	$i = 2$	$i = 3$	$i = 4$
0	0	0	0	0	0
1	0	1			
2	0	1			
3	0	1			
4	0	1			
5	0	1			

For $i = 0, 1, \ldots, n-1$ set $M(w, i+1) := \max\{M(w, i),\ M(w - w_{i+1}, i) + v_{i+1}\}$

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Weight limit: $\ell = 5$ Target value: $t = 7$

weight limit w	max. total value from first i items				
	$i = 0$	$i = 1$	$i = 2$	$i = 3$	$i = 4$
0	0	0	0	0	0
1	0	1	3		
2	0	1			
3	0	1			
4	0	1			
5	0	1			

For $i = 0, 1, \ldots, n-1$ set $M(w, i+1) := \max\{M(w, i),\ M(w - w_{i+1}, i) + v_{i+1}\}$

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	$i = 0$	$i = 1$	$i = 2$	$i = 3$	$i = 4$
0	0	0	0	0	0
1	0	1	3		
2	0	1	4		
3	0	1			
4	0	1			
5	0	1			

For $i = 0, 1, \ldots, n-1$ set $M(w, i+1) := \max\{M(w, i),\ M(w - w_{i+1}, i) + v_{i+1}\}$

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	$i = 0$	$i = 1$	$i = 2$	$i = 3$	$i = 4$
0	0	0	0	0	0
1	0	1	3		
2	0	1	4		
3	0	1	4		
4	0	1			
5	0	1			

For $i = 0, 1, \ldots, n-1$ set $M(w, i+1) := \max\{M(w, i),\ M(w - w_{i+1}, i) + v_{i+1}\}$

Example

Input $I = \{1, 2, 3, 4\}$ with

Values: $v_1 = 1 \quad v_2 = 3 \quad v_3 = 4 \quad v_4 = 2$

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weight limit w	max. total value from first i items				
	$i = 0$	$i = 1$	$i = 2$	$i = 3$	$i = 4$
0	0	0	0	0	0
1	0	1	3		
2	0	1	4		
3	0	1	4		
4	0	1	4		
5	0	1			

For $i = 0, 1, \ldots, n-1$ set $M(w, i+1) := \max\left\{M(w, i),\ M(w - w_{i+1}, i) + v_{i+1}\right\}$

Example

Input $I = \{1, 2, 3, 4\}$ with

Values: $v_1 = 1$ $v_2 = 3$ $v_3 = 4$ $v_4 = 2$

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	$i = 0$	$i = 1$	$i = 2$	$i = 3$	$i = 4$
0	0	0	0	0	0
1	0	1	3		
2	0	1	4		
3	0	1	4		
4	0	1	4		
5	0	1	4		

For $i = 0, 1, \ldots, n-1$ set $M(w, i+1) := \max\{M(w, i),\ M(w - w_{i+1}, i) + v_{i+1}\}$

Example

Input $I = \{1, 2, 3, 4\}$ with

Values: $v_1 = 1$ $v_2 = 3$ $v_3 = 4$ $v_4 = 2$

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Weight limit: $\ell = 5$ Target value: $t = 7$

weight limit w	max. total value from first i items				
	$i = 0$	$i = 1$	$i = 2$	$i = 3$	$i = 4$
0	0	0	0	0	0
1	0	1	3	3	3
2	0	1	4	4	4
3	0	1	4	4	5
4	0	1	4	7	7
5	0	1	4	8	8

For $i = 0, 1, \ldots, n-1$ set $M(w, i+1) := \max\{M(w, i),\ M(w - w_{i+1}, i) + v_{i+1}\}$

Did we prove P = NP?

Summary:

- Theorem 8.8: **Knapsack** is NP-complete
- **Knapsack** can be solved in time $O(n\ell)$ using dynamic programming

What went wrong?

Knapsack

Input: A set $I := \{1, \ldots, n\}$ of items
each of value v_i and weight w_i for $1 \leq i \leq n$,
target value t and weight limit ℓ

Problem: Is there $T \subseteq I$ such that
$\sum_{i \in T} v_i \geq t$ and $\sum_{i \in T} w_i \leq \ell$?

Pseudo-Polynomial Time

The previous algorithm is not sufficient to show that **Knapsack** is in P

- The algorithm fills a $(\ell + 1) \times (n + 1)$ matrix M
- The size of the input to **Knapsack** is $O(n \log \ell)$

$\leadsto$ the size of M is not bounded by a polynomial in the length of the input!

Pseudo-Polynomial Time

The previous algorithm is not sufficient to show that **Knapsack** is in P

- The algorithm fills a $(\ell + 1) \times (n + 1)$ matrix M
- The size of the input to **Knapsack** is $O(n \log \ell)$

$\leadsto$ the size of M is not bounded by a polynomial in the length of the input!

Definition 8.9 (Pseudo-Polynomial Time): Problems decidable in time polynomial in the sum of the input length and the value of numbers occurring in the input.

Equivalently: Problems decidable in polynomial time when using unary encoding for all numbers in the input.

- If **Knapsack** is restricted to instances with $\ell \leq p(n)$ for a polynomial p, then we obtain a problem in P.
- **Knapsack** is in polynomial time for unary encoding of numbers.

Strong NP-completeness

Pseudo-Polynomial Time: Algorithms polynomial in the maximum of the input length and the value of numbers occurring in the input.

Examples:

- **Knapsack**
- **Subset Sum**

Strong NP-completeness: Problems which remain NP-complete even if all numbers are bounded by a polynomial in the input length (equivalently: even for unary encoding of numbers).

Examples:

- **Clique**
- **Sat**
- **Hamiltonian Cycle**
- ...

Note: Showing **Sat** $\leq_p$ **Subset Sum** required exponentially large numbers.

Beyond NP

The Class coNP

Recall that coNP is the complement class of NP.

Definition 8.10:

- For a language $\mathbf{L} \subseteq \Sigma^*$ let $\overline{\mathbf{L}} := \Sigma^* \setminus \mathbf{L}$ be its complement
- For a complexity class C, we define $\mathrm{coC} := \{\mathbf{L} \mid \overline{\mathbf{L}} \in \mathrm{C}\}$
- In particular $\mathrm{coNP} = \{\mathbf{L} \mid \overline{\mathbf{L}} \in \mathrm{NP}\}$

A problem belongs to coNP, if no-instances have short certificates.

Examples:

- **No Hamiltonian Path**: Does the graph G not have a Hamiltonian path?
- **Tautology**: Is the propositional logic formula φ a tautology (true under all assignments)?
- ...

coNP-completeness

Definition 8.11: A language **C** $\in$ coNP is coNP-complete, if **L** $\leq_p$ **C** for all **L** $\in$ coNP.

Theorem 8.12:

(1) P = coP

(2) Hence, P $\subseteq$ NP $\cap$ coNP

Open questions:

- NP = coNP?

  Most people do not think so.

- P = NP $\cap$ coNP?

  Again, most people do not think so.

Summary and Outlook

CLIQUE, **INDEPENDENT SET**, **3-SAT**, and **HAMILTONIAN PATH** are also NP-complete.

So are **SUBSET SUM** and **KNAPSACK**, but only if numbers are encoded effiently (pseudo-polynomial time).

There do not seem to be polynomial certificates for coNP instances; and for some problems there seem to be certificates neither for instances nor for non-instances.

What's next?

- Space
- Games
- Relating complexity classes