Extracting Confident General Concept Inclusions from Finite Interpretations

Daniel Borchmann

Automated is a Good Idea

Goal (of AI)
Let computers do what they can do (and leave all the rest to the humans.)
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Requirement
Computers must know about the real world.
Motivation and Introduction

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Goal (of Knowledge Representation)
Represent knowledge in a way suitable for computers,
Motivation and Introduction

Automation is a Good Idea

Goal (of AI)
Let computers do what they can do (and leave all the rest to the humans.)

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Computers must know about the real world.

Goal (of Knowledge Representation)
Represent knowledge in a way suitable for computers, i.e. as a description logics ontology.
Ontologies contain assertional knowledge and terminological knowledge.
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Example ($\mathcal{EL}$-Ontology)

$(\mathcal{T}, \mathcal{A})$ is an ontology, where

\[
\mathcal{T} = \{ \text{Cat} \sqsubseteq \text{Animal} \cap \exists \text{hunts}.\text{Mouse}, \\
\qquad \text{Cat} \cap \text{Mouse} \sqsubseteq \bot \} \\
\mathcal{A} = \{ \text{Cat}(\text{Tom}), \text{Mouse}(\text{Jerry}), \text{hunts}(\text{Tom}, \text{Jerry}) \}
\]
Motivation and Introduction

Description Logics Ontologies

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$(\mathcal{T}, \mathcal{A})$ is an ontology, where

$$
\mathcal{T} = \{ \text{Cat} \sqsubseteq \text{Animal} \land \exists \text{hunts} \cdot \text{Mouse}, \\
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\mathcal{A} = \{ \text{Cat}(\text{Tom}), \text{Mouse}(\text{Jerry}), \text{hunts}(\text{Tom}, \text{Jerry}) \}
$$

Definition

Terminological axioms of the form $C \sqsubseteq D$ are called general concept inclusions (GCIs.)
Problem

Construction of real world ontologies is a difficult task
Motivation and Introduction

Description Logics Ontologies

Problem

*Construction of real world ontologies is a difficult task*

But unstructured information is often already available (i.e. as textual publication)
Motivation and Introduction

Description Logics Ontologies

Problem

Construction of real world ontologies is a difficult task

But unstructured information is often already available (i.e. as textual publication)

Goal

Automatically construct ontologies from unstructured data
Motivation and Introduction

Description Logics Ontologies

Problem

*Construction of real world ontologies is a difficult task*

But unstructured information is often already available (i.e. as textual publication)

Goal

*Semi-Automatically construct the terminological part of ontologies from unstructured data*
Unstructured Data

Question

What is unstructured data?
Unstructured Data

Question

What is unstructured data?

Example (RDF Triples)

<http://dbpedia.org/resource/Autism>
<http://www.w3.org/1999/02/22-rdf-syntax-ns#type>

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Approach

Unstructured data is given as a *finite interpretation* (finite vertex- and edge-labeled graphs)
Example (Interpretation $\mathcal{I}_{\text{pets}}$)

- **Node 1**: Cat, Mammal
- **Node 2**: Dog, Mammal
- **Node 3**: Mouse, Mammal
- **Node 4**: Cheese

**Edges**:
- **1 to 3**: hunts (from Cat to Mouse)
- **2 to 3**: fights (from Dog to Mouse)
- **3 to 4**: eats (from Mouse to Cheese)
- **2 to 1**: fights (from Dog to Cat)

**Explanation**

The elements (vertices) satisfying $C = \text{Mammal}[\text{hunts}.\text{Mouse}]$ are $C[I] = t_1u$. $C[I]$ is called the extension of $C$. 

Daniel Borchmann  
Extracting Confident GCIs  
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A Simple Example

Example (Interpretation $\mathcal{I}_{\text{pets}}$)

The elements (vertices) satisfying $C = \text{Mammal} \cap \exists \text{hunts}. \text{Mouse}$ are

$$C_\mathcal{I} = \{1\}.$$ 

$C_\mathcal{I}$ is called the extension of $C$. 
Goal

Extract all *terminological knowledge*, i.e. all valid GCIs, from $\mathcal{I}$. 
Definition

Let $C, D$ be $\mathcal{EL}^\perp$-concept descriptions. Then the GCI $C \subseteq D$ holds in $\mathcal{I}$ if and only if $C^\mathcal{I} \subseteq D^\mathcal{I}$. 

Problem

The number of valid GCIs of $\mathcal{I}$ is (normally) infinite.

Example

$\text{Cat} \subseteq \text{Mammal}$ holds in $\mathcal{I}$, and so do $\text{Dhunts}$.

$\text{Cat} \subseteq \text{Dhunts}$.

$\text{Mammal} \subseteq \text{Dhunts}$.

$\text{Dhunts}$. . .
Terminological Knowledge of Interpretations

Definition

Let $C, D$ be $\mathcal{EL}^\bot$-concept descriptions. Then the GCI $C \sqsubseteq D$ holds in $I$ if and only if $C^I \subseteq D^I$.

Problem

The number of valid GCIs of $I$ is (normally) infinite.
### Terminological Knowledge of Interpretations

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Cat $\sqsubseteq$ Mammal holds in $\mathcal{I}_{\text{pets}}$. 
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Cat $\subseteq$ Mammal holds in $\mathcal{I}_{\text{pets}}$, and so do

\[ \exists\text{hunts.Cat} \subseteq \exists\text{hunts.Mammal}, \]
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Bases of Valid GCIs

Approach

Consider bases of valid GCIs of $\mathcal{I}$, i.e. sets $\mathcal{B}$ of GCIs such that every valid GCI of $\mathcal{I}$ already follows from $\mathcal{B}$ (i.e., $\mathcal{B}$ is sound) and every valid GCI of $\mathcal{I}$ already follows from $\mathcal{B}$ (i.e., $\mathcal{B}$ is complete).

Goal

Find a finite base of all valid GCIs of $\mathcal{I}$.

Theorem (Baader, Distel 2008)

Finite bases of all valid ELK-GCIs of $\mathcal{I}$ always exist. One can be constructed effectively.
Approach

Consider *bases* of valid GCIs of $\mathcal{I}$, i.e. sets $\mathcal{B}$ of GCIs such that

- $\mathcal{B}$ contains only valid GCIs of $\mathcal{I}$ ($\mathcal{B}$ is *sound*).
Bases of Valid GCIs

Approach

Consider *bases* of valid GCIs of $\mathcal{I}$, i.e. sets $\mathcal{B}$ of GCIs such that

- $\mathcal{B}$ contains only valid GCIs of $\mathcal{I}$ (*$\mathcal{B}$ is sound*)
- every valid GCI of $\mathcal{I}$ already *follows* from $\mathcal{B}$ (*$\mathcal{B}$ is complete.*)
Bases of Valid GCIs

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Consider bases of valid GCIs of $\mathcal{I}$, i.e. sets $\mathcal{B}$ of GCIs such that
- $\mathcal{B}$ contains only valid GCIs of $\mathcal{I}$ ($\mathcal{B}$ is sound)
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Problem: Errors in DBpedia

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Approach assumes data set $\mathcal{I}$ to be complete and free of errors.
Problem: Errors in DBpedia

**Problem**

*Approach assumes data set $\mathcal{I}$ to be complete and free of errors.*

**Example**

The GCI

$$\exists \text{child. } \top \subseteq \text{Person}$$

does not hold in $\mathcal{I}_{\text{DBpedia}}$. 
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Example

The GCI

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does not hold in $\mathcal{I}_{\text{DBpedia}}$, but there are only four erroneous counterexamples (in 5262 individuals.)

Idea

Consider confident GCIs, i.e. GCIs that allow some few “exceptions.”
Formal Concept Analysis
From Mathematical Order Theory to a «Theory of Data»
Formal Concept Analysis

What is FCA?

Formal Concept Analysis is a restructuring attempt to modern lattice theory.
What is FCA?

Formal Concept Analysis is a restructuring attempt to modern lattice theory.
Motivation for FCA (back in the 1980s)

Claim: lattice theory has turned into a meaningless manipulation of symbols

Goal: (re)introduce meaning into this theory

Use a theory of concepts for this

Literature
Restructuring Lattice Theory: An Approach Based on Hierarchies of Concepts; R. Wille 1982
Formal Concept Analysis Mathematical Foundations; R. Wille and B. Ganter; 1999
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The fundamental notion of FCA is the one of a formal context.
The fundamental notion of FCA is the one of a *formal context*.

**Definition**

Let $G, M$ be sets and let $I \subseteq G \times M$. Then the triple $\mathbf{K} = (G, M, I)$ is called a *formal context*. 
The fundamental notion of FCA is the one of a *formal context*.

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**Example**

$$(\{1, \ldots, 5\}, \{1, \ldots, 5\}, \{(x, y) \mid x \leq y\})$$
The fundamental notion of FCA is the one of a *formal context*.

**Definition**

Let $G, M$ be sets and let $I \subseteq G \times M$. Then the triple $\mathbb{K} = (G, M, I)$ is called a *formal context*.

**Example**

$$\left( \{1, \ldots, 5\}, \{1, \ldots, 5\}, \{(x, y) \mid x \leq y\} \right)$$

**Uhm…**

**Meaning?**
Let $K = (G, M, I)$ be a formal context. We then introduce the following interpretation:
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- Elements of $G$ are called *objects* (Gegenstände)
Let $\mathcal{K} = (G, M, I)$ be a formal context.

We then introduce the following interpretation:

- Elements of $G$ are called *objects* (Gegenstände)
- Elements of $M$ are called *attributes* (Merkmale)
Formal Contexts – Basic Interpretation

Let $\mathcal{K} = (G, M, I)$ be a formal context. We then introduce the following interpretation:

- Elements of $G$ are called *objects* (Gegenstände)
- Elements of $M$ are called *attributes* (Merkmale)
- We say that the object $g$ has the attribute $m$ if and only if $(g, m) \in I$
Formal Contexts – Graphical Representation

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Definition (Derivation Operators)
Let $A \subseteq G$, $B \subseteq M$. Then we define

\[
A' := \{ m \in M \mid \forall g \in A : g \downarrow m \},
\]

\[
B' := \{ g \in G \mid \forall m \in B : g \downarrow m \}.
\]
Formal Concepts

Definition (Derivation Operators)
Let $A \subseteq G$, $B \subseteq M$. Then we define

$$A' := \{ m \in M \mid \forall g \in A : g \perp m \},$$
$$B' := \{ g \in G \mid \forall m \in B : g \perp m \}.$$

Definition (Formal Concepts)
The pair $(A, B)$ is called a formal concept of $\mathbb{K}$ if and only if $A \subseteq G$, $B \subseteq M$ and

$$A' = B \quad \text{and} \quad B' = A.$$
Formal Concepts

Definition (Derivation Operators)
Let \( A \subseteq G, B \subseteq M \). Then we define

\[
A' := \{ m \in M \mid \forall g \in A : g \mid m \},
\]
\[
B' := \{ g \in G \mid \forall m \in B : g \mid m \}.
\]

Definition (Formal Concepts)
The pair \((A, B)\) is called a formal concept of \( \mathbf{K} \) if and only if \( A \subseteq G, B \subseteq M \) and

\[ A' = B \quad \text{and} \quad B' = A. \]

The set of all formal contexts of \( \mathbf{K} \) is denoted by \( \mathcal{B}(\mathbf{K}) \).
### Formal Concepts – Example

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**Example (Formal Concepts)**
### Example (Formal Concepts)

- \((\{ \text{Mercury, Venus, Earth, Mars, Pluto} \}, \{ \text{small} \})\)
### Formal Concepts – Example

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**Example (Formal Concepts)**

\[
\{ \{ \text{Mercury, Venus, Earth, Mars, Pluto} \}, \{ \text{small} \} \} \models \text{small planets}
\]
### Example (Formal Concepts)

- $\left( \{ \text{Mercury, Venus, Earth, Mars, Pluto} \}, \{ \text{small} \} \right) \models \text{small planets}$
- $\left( \{ \text{Pluto} \}, \{ \text{small, far, moon} \} \right)$
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**Example (Formal Concepts)**

- \( \langle \{ \text{Mercury, Venus, Earth, Mars, Pluto} \}, \{ \text{small} \} \rangle \uparrow \equiv \text{small planets} \)
- \( \langle \{ \text{Pluto} \}, \{ \text{small, far, moon} \} \rangle \uparrow \equiv \text{small planets far away from sun} \)
Concept Lattices

Observation

Concepts can be ordered by *generality*.
Concept Lattices

Observation
Concepts can be ordered by *generality*.

Example (Formal Concepts)

- \(
\{ \text{Mercury, Venus, Earth, Mars, Pluto} \}, \{ \text{small} \} \) \supseteq \text{small planets}
- \(
\{ \text{Pluto} \}, \{ \text{small, far, moon} \} \) \supseteq \text{small planets far away from sun}
Concept Lattices

Observation
Concepts can be ordered by *generality*.

Example (Formal Concepts)
- \((\{ \text{Mercury, Venus, Earth, Mars, Pluto} \}, \{ \text{small} \}) \models \text{small planets}\)
- \((\{ \text{Pluto} \}, \{ \text{small, far, moon} \}) \models \text{small planets far away from sun}\)

Definition
Let \((A_1, B_1), (A_2, B_2) \in \mathcal{B}(\mathcal{I}K)\). Then define

\[(A_1, B_1) \leq (A_2, B_2) \iff A_1 \subseteq A_2.\]
## Concept Lattices

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Concept Lattices

Mercury, Venus, no-moon
Mars, Earth
Jupiter, Saturn, large
Uranus, Neptune, medium
Pluto

small
moon
near
far

no-moon
Mercury, Venus
Mars, Earth
Uranus, Neptune
Pluto
Implications

FCA can also be used to examine *dependencies* between attributes of $\mathcal{K}$.
FCA can also be used to examine dependencies between attributes of \( \mathbb{K} \).

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FCA can also be used to examine *dependencies* between attributes of \( K \).

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**Observation**

Every planet, that is far away from sun has a moon.
Implications

FCA can also be used to examine *dependencies* between attributes of \( \mathbb{I} \).

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**Observation**

Every planet, that is far away from sun has a moon.

far planet \( \Rightarrow \)
Implications

FCA can also be used to examine *dependencies* between attributes of $\mathbb{I}$. 

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Observation

Every planet, that is far away from sun has a moon.

$\text{far planet} \models \{ \{ \text{Jupiter, Saturn, Uranus, Neptune, Pluto} \},$
Implications

FCA can also be used to examine *dependencies* between attributes of $\mathbb{K}$.

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**Observation**

Every planet, that is far away from sun has a moon.

$$\text{far planet} \implies (\{ \text{Jupiter, Saturn, Uranus, Neptune, Pluto} \}, \{ \text{far, moon} \}).$$
Implications

Definition (Implication (Syntax))

Let $M$ be a set, $A, B \subseteq M$. Then the pair $(A, B)$ may be called an *implication* on $M$ and is written as $A \rightarrow B$. 

Remark

$A \rightarrow B$ holds in $K$ if and only if all objects that have all attributes from $A$ also have all attributes from $B$.

This is a model-based semantics!
Implications

Definition (Implication (Syntax))
Let $M$ be a set, $A, B \subseteq M$. Then the pair $(A, B)$ may be called an *implication* on $M$ and is written as $A \rightarrow B$.

Definition (Implication (Semantics))
Let $\mathbf{K} = (G, M, I)$ be a formal context and let $A \rightarrow B$ be an implication $M$. Then $A \rightarrow B$ holds if and only if $A_1 \subseteq B_1$.

Remark
$A \rightarrow B$ holds in $\mathbf{K}$ if and only if all objects that have all attributes from $A$ also have all attributes from $B$. This is a model-based semantics!
Implications

Definition (Implication (Syntax))
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Let $\mathcal{K} = (G, M, I)$ be a formal context and let $A \rightarrow B$ be an implication $M$. Then $A \rightarrow b$ holds in $\mathcal{K}$ if and only if

$$A' \subseteq B'.$$
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$A \rightarrow B$ holds in $\mathcal{K}$ if and only if all objects that have all attributes from $A$ also have all attributes from $B$. 
Implications

Definition (Implication (Syntax))
Let $M$ be a set, $A, B \subseteq M$. Then the pair $(A, B)$ may be called an *implication* on $M$ and is written as $A \longrightarrow B$.

Definition (Implication (Semantics))
Let $\mathcal{K} = (G, M, I)$ be a formal context and let $A \longrightarrow B$ be an implication $M$. Then $A \longrightarrow b$ holds in $\mathcal{K}$ if and only if

$$A' \subseteq B'.$$

Remark
$A \longrightarrow B$ holds in $\mathcal{K}$ if and only if all objects that have all attributes from $A$ also have all attributes from $B$.

This is a *model-based* semantics!
Bases of Implications

Recall
Want to find a finite base of all GCIs of a finite interpretation
Bases of Implications

Recall

Want to find a finite base of all GCIs of a finite interpretation

In terms of FCA

Find all valid implications of $\mathcal{I}K$
Bases of Implications

Recall
Want to find a finite base of all GCIs of a finite interpretation

In terms of FCA
Find a good representation of all valid implications of \( K \)
Bases of Implications

Recall
Want to find a finite base of all GCIs of a finite interpretation

In terms of FCA
Find a *good representation* of all valid implications of $\mathcal{K}$

Definition
Let $\mathcal{B}$ be a set of implications of $\mathcal{K}$. 
Bases of Implications

Recall
Want to find a finite base of all GCIs of a finite interpretation

In terms of FCA
Find a *good representation* of all valid implications of $\mathbb{K}$

Definition
Let $\mathcal{B}$ be a set of implications of $\mathbb{K}$.
- $\mathcal{B}$ is called *sound*, if all implications in $\mathcal{B}$ hold in $\mathbb{K}$;
Bases of Implications

Recall
Want to find a finite base of all GCIs of a finite interpretation

In terms of FCA
Find a *good representation* of all valid implications of $\mathbb{K}$

Definition
Let $\mathcal{B}$ be a set of implications of $\mathbb{K}$.
- $\mathcal{B}$ is called *sound*, if all implications in $\mathcal{B}$ hold in $\mathbb{K}$;
- $\mathcal{B}$ is called *complete*, if all implications valid in $\mathbb{K}$ follow from $\mathcal{B}$. 
Bases of Implications

Recall
Want to find a finite base of all GCIs of a finite interpretation

In terms of FCA
Find a good representation of all valid implications of $\mathcal{I}_K$

Definition
Let $\mathcal{B}$ be a set of implications of $\mathcal{I}_K$.
- $\mathcal{B}$ is called sound, if all implications in $\mathcal{B}$ hold in $\mathcal{I}_K$;
- $\mathcal{B}$ is called complete, if all implications valid in $\mathcal{I}_K$ follow from $\mathcal{B}$.
$\mathcal{B}$ is called a base if it is sound and complete.
Recall
Want to find a finite base of all GCIs of a finite interpretation

In terms of FCA
Find a *good representation* of all valid implications of $\mathbb{K}$

Definition
Let $\mathcal{B}$ be a set of implications of $\mathbb{K}$.

- $\mathcal{B}$ is called *sound*, if all implications in $\mathcal{B}$ hold in $\mathbb{K}$;
- $\mathcal{B}$ is called *complete*, if all implications valid in $\mathbb{K}$ follow from $\mathcal{B}$.

$\mathcal{B}$ is called a *base* if it is sound and complete. $\mathcal{B}$ is called an *irredundant base* if $\mathcal{B}$ is a base and every proper subset $\mathcal{B}' \subsetneq \mathcal{B}$ is not a base.
Canonical Base

One can explicitly describe some bases of $\mathcal{K}$.
One can explicitly describe some bases of $\mathcal{I}K$

**Theorem**

*The set*

$$\{ A \rightarrow A'' \mid A \subseteq M \}$$

*is a base of $\mathcal{I}K$.***
Canonical Base

One can explicitly describe some bases of $\mathcal{I}K$

**Theorem**

The set

$$\{ A \rightarrow A'' \mid A \subseteq M \}$$

is a base of $\mathcal{I}K$.

This base is in general not irredundant.
Canonical Base

One can explicitly describe some bases of $\mathbb{K}$

**Theorem**

The set

$$\{ A \rightarrow A'' \mid A \subseteq M \}$$

*is a base of $\mathbb{K}$.*

This base is in general not irredundant.

**Remark**

One can explicitly describe a base of $\mathbb{K}$ *with minimal cardinality*, the so-called *canonical base* of $\mathbb{K}$. 
Description Logics
Formalizing Knowledge The Right Way
What are Description Logics about?

In a Nutshell

Description Logics are formal languages to represent knowledge that provide methods to reason about this knowledge.
What are Description Logics about?

In a Nutshell

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What are Description Logics about?

In a Nutshell

Description Logics are formal languages to represent knowledge that provide methods to reason about this knowledge.
The Plan

- Syntax of $\mathcal{ALC}$
- Semantics of $\mathcal{ALC}$
- TBoxes, ABoxes and Ontologies
- Standard Reasoning Tasks

Literature

Syntax of $\mathcal{ALC}$

Fix the following sets:

- $N_C$ of concept names
- $N_R$ of role names
Syntax of $\mathcal{ALC}$

Fix the following sets:

- $N_C$ of concept names
- $N_R$ of role names

Example

\begin{align*}
N_C &= \{ \text{Person, Male, Female} \} \\
N_R &= \{ \text{hasChild} \}
\end{align*}
Syntax of $\mathcal{AL}$

Definition (Syntax of $\mathcal{ALC}$)

The following terms form the set $\mathcal{C}$ of all $\mathcal{ALC}$-concept descriptions.

Example:

Person $\{\text{Female}\}$ $\left[\text{hasChild}\right]$ $\left[\text{Male}\right]$ $\left[\text{hasChild}\right]$ $\left[\text{Male}\right]$.

A mother which has only sons.
Syntax of \( \mathcal{AL} \)

**Definition (Syntax of \( \mathcal{ALC} \))**

The following terms form the set \( C \) of all \( \mathcal{ALC} \)-concept descriptions:

- \( \top, \bot \) (universal and bottom concept)
- \( A \) for \( A \in N_C \) (atomic concepts)
- \( \lnot C \) for \( C \in C \) (negation)
- \( C \sqcap D \) for \( C, D \in C \) (conjunction)
- \( C \sqcup D \) for \( C, D \in C \) (disjunction)
- \( \forall r.C \) for \( r \in N_R, C \in C \) (value restriction)
- \( \exists r.C \) for \( r \in N_R, C \in C \) (existential restriction)
Syntax of $\mathcal{AL}$

Definition (Syntax of $\mathcal{ALC}$)

The following terms form the set $C$ of all $\mathcal{ALC}$-concept descriptions:

- $\top, \bot$ (universal and bottom concept)
- $A$ for $A \in N_C$ (atomic concepts)
- $\neg C$ for $C \in C$ (negation)
- $C \sqcap D$ for $C, D \in C$ (conjunction)
- $C \sqcup D$ for $C, D \in C$ (disjunction)
- $\forall r.C$ for $r \in N_R, C \in C$ (value restriction)
- $\exists r.C$ for $r \in N_R, C \in C$ (existential restriction)

Example

Person $\sqcap$ Female $\sqcap$ $\exists$hasChild.$\top$ $\sqcap$ $\forall$hasChild.Male
Definition (Syntax of $\mathcal{ALC}$)

The following terms form the set $C$ of all $\mathcal{ALC}$-concept descriptions:

- $\top, \bot$ (universal and bottom concept)
- $A$ for $A \in N_C$ (atomic concepts)
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- $\forall r.C$ for $r \in N_R, C \in C$ (value restriction)
- $\exists r.C$ for $r \in N_R, C \in C$ (existential restriction)

Example

$$\text{Person} \cap \text{Female} \cap \exists \text{hasChild}. \top \cap \forall \text{hasChild}. \text{Male}$$

A mother which has only sons.
Interpretations

Semantics of description logics are defined using interpretations.
Interpretations

Semantics of description logics are defined using interpretations.

Definition

An interpretation $\mathcal{I}$ is a pair $(\Delta_{\mathcal{I}}, \cdot_{\mathcal{I}})$ where $\Delta_{\mathcal{I}}$ is a set and $\cdot_{\mathcal{I}}$ is mapping such that
Interpretations

Semantics of description logics are defined using *interpretations*.

**Definition**

An *interpretation* \( \mathcal{I} \) is a pair \((\Delta_\mathcal{I}, \cdot^\mathcal{I})\) where \(\Delta_\mathcal{I}\) is a set and \(\cdot^\mathcal{I}\) is mapping such that

- \(A^\mathcal{I} \subseteq \Delta_\mathcal{I}\) for each \(A \in N_C\)
Interpretations

Semantics of description logics are defined using interpretations.

Definition

An interpretation $\mathcal{I}$ is a pair $(\Delta_{\mathcal{I}}, \cdot_{\mathcal{I}})$ where $\Delta_{\mathcal{I}}$ is a set and $\cdot_{\mathcal{I}}$ is mapping such that

- $A^\mathcal{I} \subseteq \Delta_{\mathcal{I}}$ for each $A \in N_C$
- $r^\mathcal{I} \subseteq \Delta_{\mathcal{I}} \times \Delta_{\mathcal{I}}$ for each $r \in N_R$
Example

Consider $\Delta_I = \{1, 2, 3, 4\}$ and

\[
\begin{align*}
\text{Person}_I &= \{1, 2, 3, 4\} \\
\text{Male}_I &= \{2, 3\} \\
\text{Female}_I &= \{1, 4\} \\
\text{hasChild}_I &= \{(1, 3), (2, 3), (3, 4)\}.
\end{align*}
\]
Consider $\Delta_I = \{1, 2, 3, 4\}$ and

\[
\begin{align*}
\text{Person}_I &= \{1, 2, 3, 4\} \\
\text{Male}_I &= \{2, 3\} \\
\text{Female}_I &= \{1, 4\} \\
\text{hasChild}_I &= \{(1, 3), (2, 3), (3, 4)\}.
\end{align*}
\]
Semantics of $\mathcal{ALC}$

Definition

Let $C, D$ be $\mathcal{ALC}$-concept descriptions, $r \in N_R$.

- $\top^\mathcal{I} = \Delta^\mathcal{I}$
- $\bot^\mathcal{I} = \emptyset$
- $(C \sqcap D)^\mathcal{I} = C^\mathcal{I} \cap D^\mathcal{I}$
- $(C \sqcup D)^\mathcal{I} = C^\mathcal{I} \cup D^\mathcal{I}$
- $(\forall r. C)^\mathcal{I} = \{ x \in \Delta^\mathcal{I} \mid \forall y \in \Delta^\mathcal{I} : (x, y) \in r^\mathcal{I} \implies y \in C^\mathcal{I} \}$
- $(\exists r. C)^\mathcal{I} = \{ x \in \Delta^\mathcal{I} \mid \exists y \in \Delta^\mathcal{I} : (x, y) \in r^\mathcal{I} \land y \in C^\mathcal{I} \}$
Semantics of \textit{ALC}

Example

![Diagram showing the relationships between Person, Male, Person, Female, and their hasChild connections to Person, Male and Person, Female, with an equality assertion in the center.]
Semantics of $\mathcal{ALC}$

Example

\[
\begin{align*}
(\text{Person} \cap \text{Female} \cap \exists \text{hasChild.} \top \sqcap \forall \text{hasChild.Male})^\mathcal{I} &=
\end{align*}
\]
Example

\[(\text{Person} \sqcap \text{Female} \sqcap \exists \text{hasChild. } \top \sqcap \forall \text{hasChild. Male})^{I} = \{ 1 \}\]
Goal

Use Description Logics to represent knowledge
Goal

Use Description Logics to represent knowledge

Different forms of knowledge:
Goal

Use Description Logics to represent knowledge

Different forms of knowledge:

- *terminological knowledge*, i.e. “a cat is a mammal which hunts mice”
Goal

Use Description Logics to represent knowledge

Different forms of knowledge:

- *terminological knowledge*, i.e. “a cat is a mammal which hunts mice”
  \(\rightsquigarrow\) TBoxes \(\mathcal{T}\)
Goal

Use Description Logics to represent knowledge

Different forms of knowledge:

- *terminological knowledge*, i.e. “a cat is a mammal which hunts mice”
  \[ \leadsto \ TBoxes \ T \]
- *assertional knowledge*, i.e. “Tom is a cat”
Goal

Use Description Logics to represent knowledge

Different forms of knowledge:

- *terminological knowledge*, i.e. “a cat is a mammal which hunts mice”
  - $\leadsto$ TBoxes $\mathcal{T}$
- *assertional knowledge*, i.e. “Tom is a cat”
  - $\leadsto$ ABoxes $\mathcal{A}$
Goal
Use Description Logics to represent knowledge

Different forms of knowledge:
- *terminological knowledge*, i.e. “a cat is a mammal which hunts mice”
  \[\rightsquigarrow\] TBoxes \( \mathcal{T} \)
- *assertional knowledge*, i.e. “Tom is a cat”
  \[\rightsquigarrow\] ABoxes \( \mathcal{A} \)

Definition (Ontology)
An *ontology* is a pair \((\mathcal{T}, \mathcal{A})\), where \(\mathcal{T}\) is a TBox and \(\mathcal{A}\) is an ABox.
Example (\(\mathcal{EL}^\perp\)-Ontology)

\((\mathcal{T}, \mathcal{A})\) is an ontology, where

\[
\mathcal{T} = \{ \text{Cat} \sqsubseteq \text{Animal} \land \exists \text{hunts} . \text{Mouse}, \\
\text{Cat} \sqcap \text{Mouse} \sqsubseteq \bot \} \\
\mathcal{A} = \{ \text{Cat(Tom)}, \text{Mouse(Jerry)}, \text{hunts(Tom, Jerry)} \}
\]
Terminological Knowledge and TBoxes

Definition (Terminological Axioms)

Terminological Axioms are of the form

\[ A \sqsubseteq C \]

where \( C \) is a concept description and \( A \) is a defined concept name (concept definition).

\[ C \sqsubseteq D \]

where \( C, D \) are concept descriptions (general concept inclusion).

A TBox \( T \) is a finite set of terminological axioms, where each defined concept name appears at most once.

Example

\( T = \text{Cat} \sqsubseteq \text{Animal} \)

\[ \text{Mouse}, \text{Cat} \sqsubseteq \text{Ku} \]
Terminological Knowledge and TBoxes

**Definition (Terminological Axioms)**

*Terminological Axioms* are of the form

- $A \equiv C$, where $C$ is a concept description and $A \not\equiv N_C$ is a *defined concept name* (concept definition)
Terminological Knowledge and TBoxes

Definition (Terminological Axioms)

Terminological Axioms are of the form

- \( A \equiv C \), where \( C \) is a concept description and \( A \not\equiv N_C \) is a defined concept name (concept definition)
- \( C \sqsubseteq D \), where \( C, D \) are concept descriptions (general concept inclusion)
Terminological Knowledge and TBoxes

Definition (Terminological Axioms)

Terminological Axioms are of the form

- \[ A \equiv C, \text{ where } C \text{ is a concept description and } A \not\equiv N_C \text{ is a defined concept name (concept definition)} \]
- \[ C \sqsubseteq D, \text{ where } C, D \text{ are concept descriptions (general concept inclusion)} \]

A TBox \( \mathcal{T} \) is a finite set of terminological axioms, where each defined concept name appears at most once.
Terminological Knowledge and TBoxes

Definition (Terminological Axioms)

Terminological Axioms are of the form

- \( A \equiv C \), where \( C \) is a concept description and \( A \not\equiv N_C \) is a defined concept name (concept definition)
- \( C \sqsubseteq D \), where \( C, D \) are concept descriptions (general concept inclusion)

A TBox \( \mathcal{T} \) is a finite set of terminological axioms, where each defined concept name appears at most once.

Example

\[ \mathcal{T} = \{ \text{Cat} \sqsubseteq \text{Animal} \land \exists \text{hunts.Mouse}, \text{Cat} \sqcap \text{Mouse} \sqsubseteq \bot \} \]
TBox Semantics

Definition (Descriptive Semantics)
An interpretation $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ is a model of a TBox $\mathcal{T}$ if and only if

$$A^\mathcal{I} = C^\mathcal{I} \quad \text{and} \quad C^\mathcal{I} \subseteq D^\mathcal{I}$$

for all $(A \equiv C), (C \sqsubseteq D) \in \mathcal{T}$. 
Definition (Descriptive Semantics)

An interpretation $\mathcal{I} = (\Delta_\mathcal{I}, \cdot_\mathcal{I})$ is a model of a TBox $\mathcal{T}$ if and only if

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for all $(A \equiv C), (C \sqsubseteq D) \in \mathcal{T}$.

Extend the interpretation function $\cdot_\mathcal{I}$ to all defined concept names such that

$$A^\mathcal{I} \subseteq \Delta_\mathcal{I}.$$
Definition (Descriptive Semantics)

An interpretation $\mathcal{I} = (\Delta_\mathcal{I}, \cdot_\mathcal{I})$ is a model of a TBox $\mathcal{T}$ if and only if

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for all $(A \equiv C), (C \subseteq D) \in \mathcal{T}$.

Extend the interpretation function $\cdot_\mathcal{I}$ to all defined concept names such that

$$A^\mathcal{I} \subseteq \Delta_\mathcal{I}.$$

Other semantics:

- greatest fixpoint semantics
- least fixpoint semantics
Confident GCIs of Finite Interpretations
Handling Errors in Knowledge
Work by Baader and Distel

Theorem (Baader, Distel 2008)

Finite bases of all valid $\mathcal{EL}^\perp$-GCIs of $\mathcal{I}$ always exists. One can be constructed effectively.
Work by Baader and Distel

Theorem (Baader, Distel 2008)

*Finite bases of all valid $\mathcal{EL}^\perp$-GCIs of $\mathcal{I}$ always exists. One can be constructed effectively.*

Goal

Extend approach to also handle errors.
Work by Baader and Distel

Theorem (Baader, Distel 2008)

Finite bases of all valid $\mathcal{EL}_{\perp}\!\!\!\downarrow$-GCIs of $\mathcal{I}$ always exists. One can be constructed effectively.

Goal

Extend approach to also handle errors.

Plan

- Introduce necessary terminology
- Define confident GCIs as an approach to handle errors
- Discuss some relevant ideas from FCA
- Present first results
Theorem

The set

\[ B_2 := \{ \bigcap U \subseteq ((\bigcap U)^\mathcal{I})^\mathcal{I} \mid U \subseteq M_\mathcal{I} \} \]

is a finite base of \( \mathcal{I} \).
The set 

$$B_2 := \{ \bigcap U \subseteq (\bigcap U)^I \mid U \subseteq M_I \}$$

is a finite base of $I$.

Questions:
In More Detail

Theorem

The set

\[ \mathcal{B}_2 := \{ \bigcap U \subseteq ((\bigcap U)^I)^I \mid U \subseteq M_I \} \]

is a finite base of \( I \).

Questions:
- What is \( M_I \)?
In More Detail

Theorem

The set

\[ B_2 := \{ \bigcap U \subseteq (\bigcap U)^I | U \subseteq M_I \} \]

is a finite base of \( I \).

Questions:

- What is \( M_I \)? \( \sim \) set of concept descriptions (no more details here)
Theorem

The set

$$\mathcal{B}_2 := \{ \bigcap U \subseteq ((\bigcap U)^\mathcal{I})^\mathcal{I} \mid U \subseteq M_\mathcal{I} \}$$

is a finite base of $\mathcal{I}$.

Questions:

- What is $M_\mathcal{I}$? \(\sim\) set of concept descriptions (no more details here)
- What is $\bigcap U$?
In More Detail

Theorem

The set

\[ B_2 := \{ \bigcap U \subseteq ((\bigcap U)^\mathcal{I}^\mathcal{I})^\mathcal{I} \mid U \subseteq M_\mathcal{I} \} \]

is a finite base of \( \mathcal{I} \).

Questions:

- What is \( M_\mathcal{I} \)? \( \sim \) set of concept descriptions (no more details here)
- What is \( \bigcap U \)?
- What is \( ((\bigcap U)^\mathcal{I})^\mathcal{I} \)?
In More Detail

Theorem

The set

\[ \mathcal{B}_2 := \{ \prod U \subseteq ((\prod U^\mathcal{I})^\mathcal{I} \mid U \subseteq M^\mathcal{I} \} \]

is a finite base of \( \mathcal{I} \).

Questions:

- What is \( M^\mathcal{I} \)? \( \sim \) set of concept descriptions (no more details here)
- What is \( \prod U \)?
- What is \( ((\prod U^\mathcal{I})^\mathcal{I})^\mathcal{I} \)?

Definition

\[ \prod U := \begin{cases} \top & U = \emptyset \\ \prod_{V \in U} V & \text{otherwise.} \end{cases} \]
Let $X \subseteq \Delta_\mathcal{I}$. Then $X^\mathcal{I}$ denotes the *model-based most-specific concept description* of $X$ in $\mathcal{I}$.
Let $X \subseteq \Delta_{\mathcal{I}}$. Then $X^\mathcal{I}$ denotes the \textit{model-based most-specific concept description} of $X$ in $\mathcal{I}$.

\textbf{Definition}

A concept description $C$ is a \textit{model-based most-specific concept description} of $X$ in $\mathcal{I}$ iff
Model-Based Most-Specific Concept Descriptions

Let $X \subseteq \Delta_I$. Then $X^I$ denotes the *model-based most-specific concept description* of $X$ in $I$.

**Definition**

A concept description $C$ is a *model-based most-specific concept description* of $X$ in $I$ iff

- $C^I \supseteq X$,
Model-Based Most-Specific Concept Descriptions

Let $X \subseteq \Delta_I$. Then $X^I$ denotes the *model-based most-specific concept description* of $X$ in $I$.

**Definition**

A concept description $C$ is a *model-based most-specific concept description* of $X$ in $I$ iff

- $C^I \supseteq X$,
- if $D$ is a concept description such that $D^I \supseteq X$, then $C \subseteq D$. 


Model-Based Most-Specific Concept Descriptions

Let $X \subseteq \Delta_\mathcal{I}$. Then $X^\mathcal{I}$ denotes the *model-based most-specific concept description* of $X$ in $\mathcal{I}$.

**Definition**

A concept description $C$ is a *model-based most-specific concept description* of $X$ in $\mathcal{I}$ iff

- $C^\mathcal{I} \supseteq X$,
- if $D$ is a concept description such that $D^\mathcal{I} \supseteq X$, then $C \subseteq D$.

**Observation**
Model-Based Most-Specific Concept Descriptions

Let $X \subseteq \Delta_I$. Then $X^I$ denotes the model-based most-specific concept description of $X$ in $I$.

**Definition**

A concept description $C$ is a model-based most-specific concept description of $X$ in $I$ iff

- $C^I \supseteq X$,
- if $D$ is a concept description such that $D^I \supseteq X$, then $C \subseteq D$.

**Observation**

- $C$ (as above) is a most specific concept description that describes $X$. 
Let $X \subseteq \Delta_{\mathcal{I}}$. Then $X^\mathcal{I}$ denotes the \textit{model-based most-specific concept description} of $X$ in $\mathcal{I}$.

\textbf{Definition}

A concept description $C$ is a \textit{model-based most-specific concept description} of $X$ in $\mathcal{I}$ iff

\begin{itemize}
  \item $C^\mathcal{I} \supseteq X$,
  \item if $D$ is a concept description such that $D^\mathcal{I} \supseteq X$, then $C \subseteq D$.
\end{itemize}

\textbf{Observation}

\begin{itemize}
  \item $C$ (as above) is a \textit{most specific concept description that describes} $X$.
  \item $C$ is unique up to equivalence, denoted by $X^\mathcal{I}$.
Model-Based Most-Specific Concept Descriptions

Problem

*Model-based most-specific concept descriptions do not need to exist in $\mathcal{EL}^\bot$.*

\[ X \xrightarrow{r} X \]
Problem

*Model-based most-specific concept descriptions do not need to exist in $\mathcal{EL}^\perp$.*

Solution: Consider $\mathcal{EL}^\perp_{gfp}$ concept descriptions.
Model-Based Most-Specific Concept Descriptions

Problem

Model-based most-specific concept descriptions do not need to exist in $\mathcal{EL}^\perp$.

Solution: Consider $\mathcal{EL}_{gfp}$ concept descriptions.

Lemma

In $\mathcal{EL}_{gfp}$ model-based most-specific concept descriptions always exist.
Model-Based Most-Specific Concept Descriptions

Problem

*Model-based most-specific concept descriptions do not need to exist in $\mathcal{EL}$.*

Solution: Consider $\mathcal{EL}_{gfp}$ concept descriptions.

Lemma

In $\mathcal{EL}_{gfp}$ *model-based most-specific concept descriptions always exist.*

Lemma

If $\mathcal{B}$ is an $\mathcal{EL}_{gfp}$-base of $\mathcal{I}$, then one can effectively compute an $\mathcal{EL}$-base $\mathcal{B}'$ from $\mathcal{B}$. 
Ontologies from Data: an Example

Experiment (B. 2010)

DBpedia: automatically extracted RDF triples from Wikipedia Infoboxes
Ontologies from Data: an Example

Experiment (B. 2010)

DBpedia: automatically extracted RDF triples from Wikipedia Infoboxes
Take relation hasChild $\leadsto$ interpretation $\mathcal{I}_{DBpedia}$
Ontologies from Data: an Example

Experiment (B. 2010)

DBpedia: automatically extracted RDF triples from Wikipedia Infoboxes
Take relation hasChild $\sim$ interpretation $\mathcal{I}_{\text{DBpedia}}$
$|\Delta \mathcal{I}_{\text{DBpedia}}| = 5626$, Base of GCIs of size 1252.
Ontologies from Data: an Example

Experiment (B. 2010)

DBpedia: automatically extracted RDF triples from Wikipedia Infoboxes

Take relation hasChild $\sim$ interpretation $\mathcal{I}_{DBpedia}$

$|\Delta \mathcal{I}_{DBpedia}| = 5626$, Base of GCIs of size 1252.

Observation

$\exists \text{hasChild}. \top \sqsubseteq \text{Person}$

*does not hold* in $\mathcal{I}_{DBpedia}$
Ontologies from Data: an Example

Experiment (B. 2010)

DBpedia: automatically extracted RDF triples from Wikipedia Infoboxes
Take relation hasChild \(\sim\) interpretation \(\mathcal{I}_{\text{DBpedia}}\)
\[|\Delta \mathcal{I}_{\text{DBpedia}}| = 5626, \text{Base of GCIs of size 1252.}\]

Observation

\[\exists \text{hasChild}. \top \subseteq \text{Person}\]

*does not hold* in \(\mathcal{I}_{\text{DBpedia}}\), but there are only 4 *erroneous* counterexamples.
Ontologies from Data: an Example

Experiment (B. 2010)

DBpedia: automatically extracted RDF triples from Wikipedia Infoboxes
Take relation hasChild \sim interpretation \mathcal{I}_{DBpedia}
|\Delta\mathcal{I}_{DBpedia}| = 5626, Base of GCIs of size 1252.

Observation

\exists \text{hasChild}. \top \sqsubseteq \text{Person}

does not hold in \mathcal{I}_{DBpedia}, but there are only 4 erroneous counterexamples.

Idea

Also consider GCIs that “almost” hold in \mathcal{I}_{DBpedia}. 
Confidence of GCI

Definition

The confidence of $C \sqsubseteq D$ in $\mathcal{I}$ is defined as

$$
\text{conf}_{\mathcal{I}}(C \sqsubseteq D) := \begin{cases} 
1 & \text{if } C^\mathcal{I} = \emptyset, \\
\frac{|(C \cap D)^\mathcal{I}|}{|C^\mathcal{I}|} & \text{otherwise}.
\end{cases}
$$
Confidence of GCIs

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The *confidence* of $C \sqsubseteq D$ in $\mathcal{I}$ is defined as

$$\text{conf}_\mathcal{I}(C \sqsubseteq D) := \begin{cases} 1 & \text{if } C^\mathcal{I} = \emptyset, \\ \frac{|(C \cap D)^\mathcal{I}|}{|C^\mathcal{I}|} & \text{otherwise}. \end{cases}$$

Let $c \in [0, 1]$. Define $\text{Th}_c(\mathcal{I})$ as the set of all GCIs having confidence of at least $c$ in $\mathcal{I}$. 

Confidence of GCIs

Definition

The *confidence* of $C \subseteq D$ in $\mathcal{I}$ is defined as

$$\text{conf}_{\mathcal{I}}(C \subseteq D) := \begin{cases} 1 & \text{if } C^\mathcal{I} = \emptyset, \\ \frac{|C \cap D|^\mathcal{I}}{|C|^\mathcal{I}} & \text{otherwise.} \end{cases}$$

Let $c \in [0, 1]$. Define $\text{Th}_c(\mathcal{I})$ as the set of all GCIs having confidence of at least $c$ in $\mathcal{I}$.

Approach

Consider $\text{Th}_c(\mathcal{I})$ as set of “almost” valid GCIs of $\mathcal{I}$.
Confidence of GCIs

Definition

The confidence of \( C \sqsubseteq D \) in \( \mathcal{I} \) is defined as

\[
\text{conf}_\mathcal{I}(C \sqsubseteq D) := \begin{cases} 
1 & \text{if } C^\mathcal{I} = \emptyset, \\
\frac{|(C \cap D)^\mathcal{I}|}{|C^\mathcal{I}|} & \text{otherwise}.
\end{cases}
\]

Let \( c \in [0, 1] \). Define \( \text{Th}_c(\mathcal{I}) \) as the set of all GCIs having confidence of at least \( c \) in \( \mathcal{I} \).

Approach

Consider \( \text{Th}_c(\mathcal{I}) \) as set of “almost” valid GCIs of \( \mathcal{I} \).

Question

Can we find a finite base for \( \text{Th}_c(\mathcal{I}) \)?
A Base for Confident GCIs

Answer

There exist finite bases of $\text{Th}_c(\mathcal{I})$. 
A Base for Confident GCIs

Answer

There exist finite bases of $\text{Th}_c(I)$.

Use ideas from Formal Concept Analysis for this!
Implications with Confidence

Definition

For an implication $A \rightarrow B$ of a formal context $\mathbf{K}$ define its confidence to be

$$conf_{\mathbf{K}}(A \rightarrow B) := \begin{cases} 1 & A' = \emptyset \\ \frac{|(A \cup B)'|}{|A'|} & \text{otherwise.} \end{cases}$$
Implications with Confidence

Definition
For an implication $A \rightarrow B$ of a formal context $\mathcal{K}$ define its confidence to be

$$\text{conf}_{\mathcal{K}}(A \rightarrow B) := \begin{cases} 1 & A' = \emptyset \\ \frac{|(A \cup B)'|}{|A'|} & \text{otherwise.} \end{cases}$$

Goal
Find “small” representation of all implications with confidence at least $c \in [0, 1]$. 
Implications with Confidence

Definition

For an implication $A \rightarrow B$ of a formal context $IK$ define its confidence to be

$$\text{conf}_{IK}(A \rightarrow B) := \begin{cases} 1 & A' = \emptyset \\ \frac{|(A \cup B)'|}{|A'|} & \text{otherwise}. \end{cases}$$

Goal

Find “small” representation of all implications with confidence at least $c \in [0, 1]$. More precisely, let

$$\text{Th}_c(IK) := \{ A \rightarrow B \mid \text{conf}_{IK}(A \rightarrow B) \geq c \}.$$
Implications with Confidence

Definition
For an implication $A \rightarrow B$ of a formal context $\mathbb{K}$ define its confidence to be

$$\text{conf}_{\mathbb{K}}(A \rightarrow B) := \begin{cases} 1 & \text{if } A' = \emptyset \\ \frac{|(A \cup B)'|}{|A'|} & \text{otherwise} \end{cases}$$

Goal
Find “small” representation of all implications with confidence at least $c \in [0, 1]$. More precisely, let

$$\text{Th}_c(\mathbb{K}) := \{ A \rightarrow B \mid \text{conf}_{\mathbb{K}}(A \rightarrow B) \geq c \} ,$$

Then: find a set $\mathcal{B} \subseteq \text{Th}_c(\mathbb{K})$ that is complete for $\text{Th}_c(\mathbb{K})$, i.e. that entails all implications from $\text{Th}_c(\mathbb{K})$. 
Implications with Confidence

Observation

Plan (Luxenburger)
Implications with Confidence

Observation

Plan (Luxenburger)

- Restrict attention to implications with confidence $< 1$
Implications with Confidence

Observation

Plan (Luxenburger)

- Restrict attention to implications with confidence $< 1$
- Consider only implications of the form $A'' \rightarrow B''$, where $B'' \supseteq A''$
Implications with Confidence

Observation

Plan (Luxenburger)

- Restrict attention to implications with confidence < 1
- Consider only implications of the form $A'' \rightarrow B''$, where $B'' \supseteq A''$
- Consider only implications $A'' \rightarrow B''$ where $A''$ and $B''$ are directly neighbored

Lemma

For $A \subseteq B \subseteq C \subseteq M$ it is true that $\text{conf}(A \rightarrow C) = \text{conf}(A \rightarrow B) \cdot \text{conf}(B \rightarrow C)$. 
Implications with Confidence

Observation

Plan (Luxenburger)

- Restrict attention to implications with confidence < 1
- Consider only implications of the form $A'' \rightarrow B''$, where $B'' \supseteq A''$
- Consider only implications $A'' \rightarrow B''$ where $A''$ and $B''$ are directly neighbored

Lemma

For $A \subseteq B \subseteq C \subseteq M$ it is true that

$$\text{conf}_K(A \rightarrow C) = \text{conf}_K(A \rightarrow B) \cdot \text{conf}_K(B \rightarrow C).$$
Implications with Confidence

Theorem

Let $\mathcal{K} = (G, M, I)$ be a finite non-empty formal context and $c \in [0, 1]$. Let $\mathcal{B}$ be a base of $\mathcal{K}$ and define

$$C := \{ A'' \rightarrow C'' \mid A \subseteq C \subseteq M, \text{conf}_\mathcal{K}(A'' \rightarrow C'') \in [c, 1), \exists B' : A'' \not\subseteq B'' \not\subseteq C'' \}.$$ 

Then $\mathcal{B} \cup C$ is a base of $\text{Th}_c(\mathcal{K})$. 
An Order Isomorphism
An Order Isomorphism

$(\text{Int}(K_\mathcal{I}), \subseteq) \cong \emptyset''$

$(\text{mmsc}(\mathcal{I}), \equiv) \cong \bigcap \emptyset''$

$\text{pr}_{M_\mathcal{I}}$
An Order Isomorphism

\[(\text{Int}(K_I), \subseteq), \varnothing'' \]  
\[\text{conf}_{K_I}(P'' \rightarrow Q'') \in [c, 1)\]

\[(\text{mmsc}(I), \equiv), \prod \varnothing''\]
An Order Isomorphism

\[
\begin{align*}
\text{(Int}(K_I), \subseteq) & \quad \varnothing'' \\
\text{conf}_{K_I}(P'' \rightarrow Q'') & \in [c, 1)
\end{align*}
\]

\[
\begin{align*}
\text{(mmsc}(I), \equiv) & \quad \bigcap \varnothing'' \\
\text{pr}_{M_I} & \\
\text{pr}_{M_I}
\end{align*}
\]
An Order Isomorphism

\[ (\text{Int}(K_I), \subseteq) \overset{\varnothing''}{\triangleright} \text{conf}_{K_I}(P'' \rightarrow Q'') \in [c, 1) \]

\[ (\text{mmsc}(I), \equiv) \overset{\prod \varnothing''}{\triangleright} \prod P'' \]

\[ \text{pr}_{M_I} \]
An Order Isomorphism

$M_\mathcal{I}$

$\text{pr}_{M_\mathcal{I}}((Y^\mathcal{I})^\mathcal{I})$

$\text{pr}_{M_\mathcal{I}}((X^\mathcal{I})^\mathcal{I})$

$(\text{Int}(\mathbb{K}_\mathcal{I}), \subseteq) \quad \emptyset''$

$(\text{mmsc}(\mathcal{I}), \sqsupseteq) \prod \emptyset''$

$\text{conf}_\mathcal{I}((X^\mathcal{I})^\mathcal{I} \sqsubseteq (Y^\mathcal{I})^\mathcal{I}) \in [c, 1)$
A Base for Confident GCIs

Theorem (B. 2012)

Let $\mathcal{B}$ be a finite base of $\mathcal{I}$, $c \in [0, 1]$ and

$$\text{Conf}(\mathcal{I}, c) := \{ X^\mathcal{I} \sqsubseteq Y^\mathcal{I} \mid Y \subseteq X \subseteq \Delta^\mathcal{I}, 1 > \text{conf}_\mathcal{I}(X^\mathcal{I} \sqsubseteq Y^\mathcal{I}) \geq c \}.$$ 

Then $\mathcal{B} \cup \mathcal{C}$ is a finite base of $\text{Th}_c(\mathcal{I})$. 
A Base for Confident GCIs

Theorem (B. 2012)

Let \( B \) be a finite base of \( \mathcal{I} \), \( c \in [0, 1] \) and

\[
\text{Conf}(\mathcal{I}, c) := \{ X^\mathcal{I} \subseteq Y^\mathcal{I} \mid Y \subseteq X \subseteq \Delta_\mathcal{I}, 1 > \text{conf}_\mathcal{I}(X^\mathcal{I} \subseteq Y^\mathcal{I}) \geq c \}.
\]

Then \( B \cup C \) is a finite base of \( \text{Th}_c(\mathcal{I}) \).

Theorem (B. 2012)

The set

\[
\mathcal{D} := \{ (X^\mathcal{I} \subseteq Y^\mathcal{I}) \in \text{Conf}(\mathcal{I}, c) \mid \exists Z \subseteq \Delta_\mathcal{I} : Y^\mathcal{I} \nsubseteq Z^\mathcal{I} \nsubseteq X^\mathcal{I} \}
\]

is complete for \( C \). In particular, \( B \cup \mathcal{D} \) is a finite base for \( \text{Th}_c(\mathcal{I}) \).
Thank You for Your Attention!