Mutual Irreducibility of Revision and Multiple Revision

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Abstract. We show that revision by single sentences and multiple revision are, in general, mutually irreducible processes. The result is given for revision in base logics, a framework for studying revision in arbitrary classical logics for various notions of bases. Within this framework, we demonstrate that revision by single sentences can not be represented as multiple revision. The result employs general relational semantics for base revision. In combination with the well-known result that multiple revision cannot be reduced to sentence revision, we obtain that sentence revision and multiple revision are mutually irreducible.

1 Introduction

Adaptation of an agent's beliefs according to new information is a central issue of artificial intelligence. One of the premier change processes is

revision: the integration of new information,
while resolving inconsistencies, whenever possible.

Revision is studied in several different flavours [8]; notably one distinguishes between the task of integrating one piece of information—which is here called *sentence revision*—and *multiple revision*, which is the task of revision by multiple pieces of information at the same time, such that all pieces are integrated.

It is known that, in general, multiple revision cannot be reduced to sentence revision [16]. Our main contribution is to complement this observation by showing that sentence revision cannot be reduced to multiple revision, which might seem counterintuitive at first sight. This shows that sentence revision and multiple revision are mutually different kinds of operations. Our example employs recent results on the semantics of revision within base logics[3] [5–7], a framework generalizing work on the semantics of revision [4] to arbitrary Tarskian logics, a broad class of logics including propositional logic and first-order predicate logic.

In the following sections, we consider the preliminaries of this paper, introduce base logics and change operators within this framework. We consider revision within base logics and the semantics of base revision. Finally, we show that sentence revision cannot be reduced to multiple revision.

[3] Disclaimer: Some texts here are from joint work with Faiq Miftakhul Falakh [7].

2 Preliminaries

In order to study (sentence and multiple) revision formally, one uses logics, which is convenient, as logics provide an intrinsic notion of consistency and consequence, which describes interrelations between beliefs. We will employ a generic view on logics with a classical model-theoretic semantics represented by satisfaction systems [20]. A logic's satisfaction system is a triple $(\mathcal{L}, \Omega, \models)$, consisting of a (possibly infinite) set of *sentences* $\mathcal{L}$, a (potentially infinite) class Ω of *interpretations* (also referred to as *worlds*) and a binary relation $\models$ between Ω and $\mathcal{L}$ where $\omega \models \varphi$ indicates that ω is a *model* of φ. For some $\varphi \in \mathcal{L}$, we use the shortcut $[\![\varphi]\!]$ to denote the set $\{\omega \mid \omega \models \varphi\}$ of models of φ. Logical entailment is defined as usual via models: for two sentences φ and ψ we say φ *entails* ψ (written $\varphi \models \psi$) if $[\![\varphi]\!] \subseteq [\![\psi]\!]$. Note that this means we overload the symbol "$\models$" in the usual way.

The notions of modelhood and entailment can be easily lifted from single sentences to sets. We obtain the models of a set $\mathcal{K} \subseteq \mathcal{L}$ of sentences via $[\![\mathcal{K}]\!] = \bigcap_{\varphi \in \mathcal{K}} [\![\varphi]\!]$. For $\mathcal{K} \subseteq \mathcal{L}$ and $\mathcal{K}' \subseteq \mathcal{L}$ we say $\mathcal{K}$ *entails* $\mathcal{K}'$ (written $\mathcal{K} \models \mathcal{K}'$) if $[\![\mathcal{K}]\!] \subseteq [\![\mathcal{K}']\!]$. We write $\mathcal{K} \equiv \mathcal{K}'$ to express $[\![\mathcal{K}]\!] = [\![\mathcal{K}']\!]$. A sentence, respectively, a set of sentences, is called *consistent* if it has models. A set of sentences is called *consistent with* another set of sentences if the two have models in common.

The existence of a classical model-theoretic semantics as above is equivalent to the logic being *Tarskian* [21], i.e., the following is satisfied[4]:

$$\text{If } \varphi \in \mathcal{K} \text{ then } \mathcal{K} \models \varphi. \qquad \text{(extensivity)}$$

$$\text{If } \mathcal{K} \models \varphi \text{ and } \mathcal{K} \subseteq \mathcal{K}', \text{ then } \mathcal{K}' \models \varphi. \qquad \text{(monotonicity)}$$

$$\text{If } \mathcal{K} \models \mathcal{K}' \text{ and } \mathcal{K}' \models \varphi, \text{ then } \mathcal{K} \models \varphi. \qquad \text{(idempotence)}$$

The notion of Tarskian logic captures many well-known classical logical formalisms, including propositional logic, first- and second-order predicate logic, and well-known fragments as Horn fragments and description logics. A great variety of formalisms can be conceived as Tarskian logics, even if they're typically not considered as logics in the stricter sense [7].

In knowledge representation and its applications, one often works with (knowledge) bases which are sets of formulas of some underlying logic. In the belief revision community, the term of *base* commonly denotes an arbitrary (possibly infinite) set of sentences. However, in certain scenarios, other assumptions might be more appropriate. Hence, for the sake of generality, we decided to define the notion of a base on an abstract level with minimal requirements (just as we introduced our notion of *logic*), allowing for its instantiation in many ways.

Definition 1 ([7]). *A* base logic *is a tuple* $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup)$*, where*

- $(\mathcal{L}, \Omega, \models)$ *is a logic (specified by a satisfaction system),*
- $\mathfrak{B} \subseteq \mathcal{P}(\mathcal{L})$ *is a family of sets of sentences, called* bases*, and*
- $\Cup : \mathfrak{B} \times \mathfrak{B} \to \mathfrak{B}$ *is a binary operator over bases, called the* abstract union*, satisfying* $[\![\mathcal{B}_1 \Cup \mathcal{B}_2]\!] = [\![\mathcal{B}_1]\!] \cap [\![\mathcal{B}_2]\!]$.

[4] See [7] for a proof of equivalence via the closure $\mathrm{Cn}(K) = \{\varphi \mid \mathcal{K} \models \varphi\}$.

Given a logic, typical choices of which kind of sets to pick as bases are *arbitrary sets* (as in the setting of base revision à la Hansson), *finite sets* which are a natural representation in the context of computation, e.g. when computational properties or implementations are to be investigated. Another common scenario are *belief sets*, i.e., the bases are deductively closed sets of sentences. One might also consider the setting where the bases are all *singleton sets*, which comes close to the setting investigated by Katsuno and Mendelzon [12].

3 AGM Revision and Multiple Revision in Base Logics

The work by Alchourrón, Gärdenfors, and Makinson [1] (AGM) answers the question of how to conduct sentence revision for many settings that satisfy the principle of minimal change. In AGM theory, an agent's beliefs are represented as set of sentences $\mathcal{K}$ and then, one formalizes revision of $\mathcal{K}$ by a sentence φ as the application of a binary operation $*$, resulting in $\mathcal{K} * \varphi$.

As one observe by inspecting Definition 1, in the setting of base logics, sentences are not first-level citizens of the formalism. To accommodate the setting of AGM theory with the the setting of base logics, we identify sentences with singleton bases and focus on base logics that contain such bases.

Definition 2. *A base logic* $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup)$ *is called* sentential *if*

- $\{\ \{\psi\}\ \mid \psi \in \mathcal{L}\} \subseteq \mathfrak{B}$ *holds and*
- *for all* $\psi_1, \psi_2 \in \mathcal{L}$ *there is* $\psi_1 \owedge \psi_2 \in \mathcal{L}$ *with* $[\![\psi_1 \owedge \psi_2]\!] = [\![\psi_1]\!] \cap [\![\psi_2]\!]$.

Here, by $\psi_1 \owedge \psi_2$ we refer to a formula that behaves semantically like the conjunction of ψ_1 and ψ_2. The logic $(\mathcal{L}, \Omega, \models)$ does not necessarily need to feature conjunction $\wedge$ as a syntactic construct. However, if this is the case, then we can just set $\psi_1 \owedge \psi_2 = \psi_1 \wedge \psi_2$. For a sentential base logic we write $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup, \owedge)$ to explicitly expose $\owedge$ (which formally is a function of type $\mathcal{L} \times \mathcal{L} \to \mathcal{L}$).

Belief change operators for single sentences are called *sentential base change operators* in our setting.

Definition 3. *Let* $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup, \owedge)$ *be a sentential base logic. A* sentential base change operator *for* $\mathbb{B}$ *is a function* $* : \mathfrak{B} \times \mathcal{L} \to \mathfrak{B}$.

Most famously, AGM provide an axiomatization of those revision operators that satisfy minimal change [1]. In the following, we use $\mathcal{K} + \psi$ as a shortcut for $\mathcal{K} \Cup \{\psi\}$. The AGM postulates for sentence revision are present in the following, accommodated to the setting of a sentential base logic $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup, \owedge)$:

(S1) $\mathcal{K} * \psi \models \psi$
(S2) If $[\![\mathcal{K} + \psi]\!] \neq \emptyset$ then $\mathcal{K} * \psi \equiv \mathcal{K} + \psi$
(S3) If $[\![\psi]\!] \neq \emptyset$ then $[\![\mathcal{K} * \psi]\!] \neq \emptyset$
(S4) If $\mathcal{K}_1 \equiv \mathcal{K}_2$ and $\psi_1 \equiv \psi_2$ then $\mathcal{K}_1 * \psi_1 \equiv \mathcal{K}_2 * \psi_2$
(S5) $(\mathcal{K} * \psi_1) + \psi_2 \models \mathcal{K} * (\psi_1 \owedge \psi_2)$
(S6) If $[\![(\mathcal{K} * \psi_1) + \psi_2]\!] \neq \emptyset$ then $\mathcal{K} * (\psi_1 \owedge \psi_2) \models (\mathcal{K} * \psi_1) + \psi_2$

For an explanation of these postulates, we recommend considering one of the books on the topic [8, 9, 11].

In the context of base logics, the default setting is to change a base based on information that is provided as a base. This corresponds to what is often called multiple revision, i.e., when multiple pieces of information are given at the same time to revise with. Formally, base change operators take two bases as inputs and output a base.

Definition 4. *Let* $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup)$ *be a base logic. A* base change operator *for* $\mathbb{B}$ *is a function* $\circ : \mathfrak{B} \times \mathfrak{B} \to \mathfrak{B}$.

Analogously, multiple revision of prior beliefs $\mathcal{K}$ by a set of sentences Γ is formalized as the application of a binary operation $\circ$, resulting in $\mathcal{K} \circ \Gamma$. The postulates (S1)–(S6) for revision were extended to sets of sentences [15, 16]. In the following, such postulates are given in the notation of this paper:

(G1) $\mathcal{K} \circ \Gamma \models \Gamma$
(G2) If $[\![\mathcal{K} \Cup \Gamma]\!] \neq \emptyset$ then $\mathcal{K} \circ \Gamma \equiv \mathcal{K} \Cup \Gamma$
(G3) If $[\![\Gamma]\!] \neq \emptyset$ then $[\![\mathcal{K} \circ \Gamma]\!] \neq \emptyset$
(G4) If $\mathcal{K}_1 \equiv \mathcal{K}_2$ and $\Gamma_1 \equiv \Gamma_2$ then $\mathcal{K}_1 \circ \Gamma_1 \equiv \mathcal{K}_2 \circ \Gamma_2$
(G5) $(\mathcal{K} \circ \Gamma_1) \Cup \Gamma_2 \models \mathcal{K} \circ (\Gamma_1 \Cup \Gamma_2)$
(G6) If $[\![(\mathcal{K} \circ \Gamma_1) \Cup \Gamma_2]\!] \neq \emptyset$ then $\mathcal{K} \circ (\Gamma_1 \Cup \Gamma_2) \models (\mathcal{K} \circ \Gamma_1) \Cup \Gamma_2$

Clearly, the postulates (G1)–(G6) follow the same intuition as (S1)–(S6).

As discussed, we consider "revision by bases", also referred to as *multiple revision* [8]. Thereby, we focus on *package semantics*, meaning that all given sentences have to be incorporated, that is, given a base $\mathcal{K}$ and new information Γ (also a base here), we demand – as per (G1) – success of revision for all elements from Γ, i.e., $\mathcal{K} \circ \Gamma \models \varphi$ for every $\varphi \in \Gamma$.

4 Multiple Revision to Sentence Revision

Pavlos Peppas studied the interrelation of sentence revision and multiple revision [16]. First, he observed that in logics, where everything can be represented with finite sets, e.g., propositional logic, sentence revision and multiple revision are interdefinable. One can mutually reduce sentence revision and multiple revision in the following way:

[S⇒M] For each sentence revision operator $*$, there is a multiple revision operator $\circ$ such that $\mathcal{K} * \varphi \equiv \mathcal{K} \circ \{\varphi\}$ for all $\mathcal{K}$ and φ.
[M⇒S] For each multiple revision operator $\circ$, there is a sentence revision operator $*$ such that $\mathcal{K} \circ \Gamma \equiv \mathcal{K} * \bigwedge_{\varphi \in \Gamma} \varphi$ for all $\mathcal{K}$ and finite Γ.

In a setting where infinite sets are permitted, one cannot employ the construction from [M⇒S], as in many logics conjunction is limited to finite conjunctions.

Observation 5 ([16])**.** *In general, multiple revision cannot be reduced to sentence revision.*

Interestingly, Peppas showed that, even in infinite settings, one can reduce multiple revision to sentence revision in some interesting cases by employing constructions different from [M⇒S] [16]. We do not consider this direction further, and will focus on whether sentence revision can be reduced to multiple revision.

5 Relational Semantics of Base Revision

Recently, the general relational semantics of AGM-style revision in the sense of (G1)–(G6) in base logics was studied [7]. In the following, we present the main definitions and representation result.

For describing belief revision on the semantic level, it is expedient to endow the interpretation space Ω with some structure [10, 12]. We will employ binary relations $\preceq$ over Ω (formally: $\preceq \subseteq \Omega \times \Omega$), where the intuitive meaning of $\omega_1 \preceq \omega_2$ is that ω_1 is "equally good or better" than ω_2 when it comes to serving as a model. We call $\preceq$ *total* if $\omega_1 \preceq \omega_2$ or $\omega_2 \preceq \omega_1$ for any $\omega_1, \omega_2 \in \Omega$ holds. We write $\omega_1 \prec \omega_2$ as a shorthand, whenever $\omega_1 \preceq \omega_2$ and $\omega_2 \not\preceq \omega_1$ (the intuition being that ω_1 is "strictly better" than ω_2). For a selection $\Omega' \subseteq \Omega$ of interpretations, an $\omega \in \Omega'$ is called $\preceq$*-minimal in* Ω' if $\omega \preceq \omega'$ for all $\omega' \in \Omega'$.[5] We let $\min(\Omega', \preceq)$ denote the set of $\preceq$-minimal interpretations in Ω'. We call $\preceq$ a *preorder* if it is transitive and reflexive. While in many classical logics (most notably propositional logic), revision operators are known to be representable by total-preorders over the interpretations [10, 12], generalizing the representation theorems to arbitrary Tarskian logics requires adjusting the formal properties of the preference relation.

Definition 6 (min-complete, min-retractive, min-friendly). *Given a base logic* $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup)$*, a binary relation* $\preceq$ *over* Ω *is called*

- min-complete *(for* $\mathbb{B}$*) if* $\min(\llbracket\Gamma\rrbracket, \preceq) \neq \emptyset$ *holds for every* $\Gamma \in \mathfrak{B}$ *with* $\llbracket\Gamma\rrbracket \neq \emptyset$*,*
- min-retractive *(for* $\mathbb{B}$*) if, for every* $\Gamma \in \mathfrak{B}$ *and* $\omega', \omega \in \llbracket\Gamma\rrbracket$ *with* $\omega' \preceq \omega$*,* $\omega \in \min(\llbracket\Gamma\rrbracket, \preceq)$ *implies* $\omega' \in \min(\llbracket\Gamma\rrbracket, \preceq)$*,*
- min-friendly *(for* $\mathbb{B}$*) if it is both min-retractive and min-complete.*

Min-completeness is close to the limit assumption by Lewis [14] or smoothness by Kraus, Lehmann and Magidor [13]. Specific for the general setting of base logics is min-retractivity, that guarantees that minimality of models with respect to a base carries over to equal elements. Note that every total preorder (over interpretations) is a min-retractive relation.

Definition 7 (assignment, faithful). *Let* $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup)$ *be a base logic. An* assignment *for* $\mathbb{B}$ *is a function* $\preceq_{(.)}\colon \mathfrak{B} \to \mathcal{P}(\Omega \times \Omega)$ *that assigns to each belief base* $\mathcal{K} \in \mathfrak{B}$ *a total binary relation* $\preceq_{\mathcal{K}}$ *over* Ω*. An assignment* $\preceq_{(.)}$ *for* $\mathbb{B}$ *is called* faithful *if it satisfies for all* $\omega, \omega' \in \Omega$ *and all* $\mathcal{K}, \mathcal{K}' \in \mathfrak{B}$*:*

(F1) If $\omega, \omega' \models \mathcal{K}$*, then* $\omega \prec_{\mathcal{K}} \omega'$ *does not hold.*
(F2) If $\omega \models \mathcal{K}$ *and* $\omega' \not\models \mathcal{K}$*, then* $\omega \prec_{\mathcal{K}} \omega'$*.*
(F3) If $\mathcal{K} \equiv \mathcal{K}'$*, then* $\preceq_{\mathcal{K}} = \preceq_{\mathcal{K}'}$*.*

An assignment $\preceq_{(.)}\colon \mathfrak{B} \to \mathcal{P}(\Omega \times \Omega)$ *is called min-friendly if* $\preceq_{\mathcal{K}}$ *is min-friendly for all* $\mathcal{K} \in \mathfrak{B}$*.*

Intuitively, faithful assignments provide information about which of the two interpretations is "closer to $\mathcal{K}$-modelhood". Consequently, the actual $\mathcal{K}$-models are

[5] If $\preceq$ is total, this definition is equivalent to the *absence* of any $\omega'' \in \Omega'$ with $\omega'' \prec \omega$.

$\preceq_{\mathcal{K}}$-minimal. The next definition captures the idea of an assignment adequately representing the behaviour of a revision operator.

Definition 8 (compatible). *A base change operator* $\circ$ *for a base logic* $\mathbb{B}$ *is called* compatible *with some assignment* $\preceq_{(.)}$ *for* $\mathbb{B}$ *if it satisfies*

$$[\![\mathcal{K} \circ \Gamma]\!] = \min\left([\![\Gamma]\!], \preceq_{\mathcal{K}}\right)$$

for all bases $\mathcal{K}$ *and* Γ *from* $\mathfrak{B}$*.*

Compatibility will be treated as a symmetric notion, so saying $\circ$ is compatible with $\preceq_{(.)}$ means the same as saying $\preceq_{(.)}$ is compatible with $\circ$ or simply saying that $\preceq_{(.)}$ and $\circ$ are compatible.

Theorem 9 ([7]). *A base change operator* $\circ$ *for a base logic* $\mathbb{B}$ *satisfies (G1)–(G6) if and only if* $\circ$ *is compatible with a min-friendly faithful assignment for* $\mathbb{B}$*.*

Example 10 ([7]). Let $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup)$ be an arbitrary base logic. We define the *trivial revision operator* $\circ^{\mathrm{fm}}$ for $\mathbb{B}$ by

$$\mathcal{K} \circ^{\mathrm{fm}} \Gamma = \begin{cases} \mathcal{K} \Cup \Gamma & \text{if } [\![\mathcal{K} \Cup \Gamma]\!] \text{ is consistent;} \\ \Gamma & \text{otherwise.} \end{cases}$$

To show satisfaction of (G1)–(G6), we construct a min-friendly faithful assignment $\preceq_{(.)}^{\mathrm{fm}}$ compatible with $\circ^{\mathrm{fm}}$. For each $\mathcal{K} \in \mathfrak{B}$ let $\omega_1 \preceq_{\mathcal{K}}^{\mathrm{fm}} \omega_2$ if $\omega_1 \models \mathcal{K}$ or $\omega_2 \not\models \mathcal{K}$. Obviously, the relation $\preceq_{\mathcal{K}}^{\mathrm{fm}}$ is a total preorder where ω_1, ω_2 are $\preceq_{\mathcal{K}}^{\mathrm{fm}}$-equivalent, if either $\omega_1, \omega_2 \models \mathcal{K}$ or $\omega_1, \omega_2 \not\models \mathcal{K}$ holds. One can show that the relation $\preceq_{\mathcal{K}}^{\mathrm{fm}}$ is min-complete and min-retractive. By construction of $\preceq_{(.)}^{\mathrm{fm}}$, we obtain that $\min([\![\Gamma]\!], \preceq_{\mathcal{K}}^{\mathrm{fm}}) = [\![\Gamma]\!]$ if $\mathcal{K} \Cup \Gamma$ is inconsistent. If $\mathcal{K} \Cup \Gamma$ is consistent, we obtain $\min([\![\Gamma]\!], \preceq_{\mathcal{K}}^{\mathrm{fm}}) = [\![\mathcal{K}]\!] \cap [\![\Gamma]\!] = [\![\mathcal{K} \Cup \Gamma]\!]$. In summary, the assignment $\preceq_{(.)}^{\mathrm{fm}}$ is min-friendly, and the base change operator $\circ^{\mathrm{fm}}$ is compatible with it.

6 Sentence Revision to Multiple Revision

In this section we consider the reducibility of sentence revision to multiple revision. First, we define the reduction of sentential base change operators to base change operators.

Definition 11 (Reducible). *Let* $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup)$ *be a sentential base logic and let* $*$ *be a sentential base change operator for* $\mathbb{B}$*. We say* $*$ *is* reducible to a base change operator $\circ$ for $\mathbb{B}$ *if for all* $\mathcal{K} \in \mathfrak{B}$ *and* $\psi \in \mathcal{L}$ *holds:*

$$K * \psi = K \circ \{\psi\}$$

Our main contribution is the insight that sentence revision is not reducible to multiple revision in the sense of Definition 11.

Theorem 12. *There is a sentential base logic $\mathbb{B}$ and a sentential base change operator for $\mathbb{B}$ that satisfies (S1)–(S6) which is **not** reducible to any base change operator for $\mathbb{B}$ that satisfies (G1)–(G6).*

In what remains, we will consider two proofs for Theorem 12. For that we construct a base logic and sentential base change operator that witness the result. We consider two such witnesses; one employs a cyclic situation and one employs an infinite descending chain (and no cyclic situation).

6.1 Proof of Theorem 12 via a Cyclic Situation

In the first proof of Theorem 12 we construct a sentential belief revision operator $*_c$ for which we will show that any base change operator $\circ_c$ to which $*_c$ is reducible is only compatible with assignments that contain a cycle. Non-reducibility follows then by a contradiction, because, as we will show, the setting is constituted in a way that any base change operator $\circ_c$ to which $*_c$ is reducible cannot contain the specified cycle and satisfy (G1)–(G6) at the same time. The feature that permits this contradiction is that base change operators have a higher expressiveness that allows performing revision by all bases (instead of just the singleton bases). Because of that, the mathematical object $\circ_c$ is more constrained (for fulfilling (G1)–(G6)) than $*_c$ (to fulfil (S1)–(S6)). More concretely, in the construction below, there are bases $\psi_{\mathrm{AB}},\ \psi_{\mathrm{BC}},\ \psi_{\mathrm{CA}}$ and $*_c$ chosen such that $\mathcal{K} *_c \psi_{\mathrm{XY}} = \{X\}$ holds. Now recall that a corresponding assignment for $\circ_{\mathrm{c}}$ encodes the behaviour of $\circ_{\mathrm{c}}$ via an order and selection of minimal elements, i.e., "$[\![\mathcal{K} \circ_{\mathrm{c}} \{\psi_{\mathrm{XY}}\}]\!] = \min\left([\![\Gamma]\!], <\right)$" (see Definition 8). The above sketched behaviour of $*_c$ forces that a corresponding assignment for $\circ_{\mathrm{c}}$ contains a cycle $A < B < C < A$. As we will see below, because of that cycle, there is a non-singleton base Γ with $[\![\mathcal{K} \circ_{\mathrm{c}} \Gamma]\!] = \emptyset$, while at the same time (G1)–(G6) postulate $[\![\mathcal{K} \circ_{\mathrm{c}} \Gamma]\!] \neq \emptyset$. Note that the situation above do not provide any conflict between the behaviour of $*_c$ and (S1)–(S6), especially, because Γ is a non-singleton base and hence, is Γ not in the domain of $*_c$.

The logic we are considering here has as interpretations Ω the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ and four distinct elements $\Box, A, B, C$. Formulas of the logic are the falsum $\bot$ (without any models), formulas of type ψ_x which have x as the one and only model, formulas of type ψ_{xy} which have $x, y \in \{A, B, C\}$ exactly as the one two models, and formulas of $\psi_{\mathbb{N}\setminus N}$ for finite sets $N \subsetneq \mathbb{N}$ that have $\{A, B, C, 1, 2, \ldots, \} \setminus N$ as models. As we will see, the formulas $\psi_{\mathrm{AB}},\ \psi_{\mathrm{BC}},\ \psi_{\mathrm{CA}}$ will later permit a circle situation between A, B and C. More formally, we're defining a logic $(\mathcal{L}, \Omega, \models)$ with:

$$\begin{aligned}\mathcal{L} = \{&\bot,\ \psi_{\Box},\ \psi_{\mathrm{A}},\ \psi_{\mathrm{B}},\ \psi_{\mathrm{C}},\ \psi_{\mathrm{AB}},\ \psi_{\mathrm{BC}},\ \psi_{\mathrm{CA}}\ \}\\ &\cup \{\psi_i \mid i \in \mathbb{N}\} \cup \{\ \psi_{\mathbb{N}\setminus N} \mid N \subseteq \mathbb{N} \text{ is finite}\}\\ \Omega = \{&\Box, A, B, C, 1, 2, 3, \ldots\}\end{aligned}$$

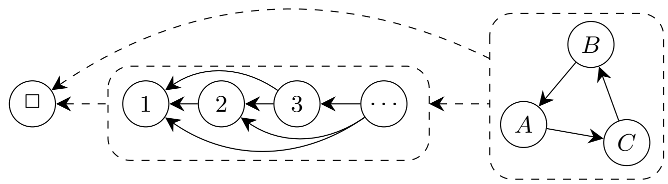

Fig. 1: Relation $\preceq_{\Box}$ imposed by $*_{\mathrm{c}}$ in the proof of Lemma 15.

The model-relation $\models$ is given by the following, where N ranges over all finite subsets of $\mathbb{N}$ and i ranges over the natural numbers:

$$
\begin{aligned}
&[\![\bot]\!] = \emptyset && [\![\psi_{\mathrm{A}}]\!] = \{A\} && [\![\psi_{\mathrm{AB}}]\!] = \{A, B\} \\
&[\![\psi_{\Box}]\!] = \{\Box\} && [\![\psi_{\mathrm{B}}]\!] = \{B\} && [\![\psi_{\mathrm{BC}}]\!] = \{B, C\} \\
&[\![\psi_{i}]\!] = \{i\} && [\![\psi_{\mathrm{C}}]\!] = \{C\} && [\![\psi_{\mathrm{CA}}]\!] = \{C, A\} \\
&[\![\psi_{\mathbb{N}\setminus N}]\!] = \{A, B, C, 1, 2, \dots, \} \setminus N
\end{aligned}
$$

Note that for every two formulas $\psi, \varphi \in \mathcal{L}$ there is a unique formula $\chi \in \mathcal{L}$ with $[\![\chi]\!] = [\![\psi]\!] \cap [\![\varphi]\!]$. For every two formulas $\psi, \varphi \in \mathcal{L}$ we denote with $\psi \owedge \varphi$ the respective "conjunction", e.g., we let $\psi_{\mathrm{AB}} \owedge \psi_{\mathrm{BC}} = \psi_{\mathrm{B}}$ or $\psi_{\mathbb{N}\setminus\{1,2\}} \owedge \psi_{\mathrm{BC}} = \psi_{\mathrm{BC}}$. The existence of $\owedge$ will allow us to extend the logic to a sentential base logic below. Furthermore, note that every formula $\psi_{\mathbb{N}\setminus N}$ has infinitely many models from $\mathbb{N}$, yet $[\![\psi_{\mathbb{N}\setminus N}]\!]$ has always a minimal element[6] from $\mathbb{N}$ as model. Moreover, every conjunction $\psi_{\mathbb{N}\setminus N} \owedge \psi_{\mathbb{N}\setminus M}$ is the formula $\psi_{\mathbb{N}\setminus(N \cup M)}$.

We now extend the logic $(\mathcal{L}, \Omega, \models)$ to a base logic $\mathbb{B}_{\mathrm{c}} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup)$. The set $\mathfrak{B}$ is defined by

$$\mathfrak{B} = \{\ \{\psi\}\ \mid \psi \in \mathcal{L}\} \cup \{\ \Gamma_{\mathrm{ABC}}\ \}\ ,$$

whereby $\Gamma_{\mathrm{ABC}} = \{\ \psi_{\mathbb{N}\setminus N} \mid N \subseteq \mathbb{N} \text{ is finite}\}$. Note that the models of Γ_{ABC} are $[\![\Gamma_{\mathrm{ABC}}]\!] = \{A, B, C\}$. The abstract union $\Cup$ is defined by setting $\{\psi\} \Cup \{\varphi\} = \{\psi \owedge \varphi\}$ and for all other bases:

$$\Gamma_{\mathrm{ABC}} \Cup \Gamma = \Gamma \Cup \Gamma_{\mathrm{ABC}} = \begin{cases} \Gamma & \text{if } \Gamma \in \{\{\psi_{\mathrm{A}}\},\{\psi_{\mathrm{B}}\},\{\psi_{\mathrm{C}}\},\{\psi_{\mathrm{AB}}\},\{\psi_{\mathrm{BC}}\},\{\psi_{\mathrm{CA}}\}\}; \\ \{\bot\} & \text{if } \Gamma \in \{\{\bot\}, \{\psi_{\Box}\}, \{\psi_1\}, \{\psi_2\}, \dots\}; \\ \Gamma_{\mathrm{ABC}} & \text{otherwise.} \end{cases}$$

One can verify that $[\![\Gamma_1 \Cup \Gamma_2]\!] = [\![\Gamma_1]\!] \cap [\![\Gamma_2]\!]$ holds, and thus, together with the existence of $\owedge$, this renders $\mathbb{B}_{\mathrm{c}}$ as a sentential base logic.

Lemma 13. $\mathbb{B}_{\mathrm{c}}$ *is a sentential base logic.*

We are now ready to define the sentential base change operator $*_{\mathrm{c}}$ for $\mathbb{B}_{\mathrm{c}}$. For $K_{\Box} = \{\psi_{\Box}\}$ and for all $K \neq \{\psi_{\Box}\}$, we let $*_{\mathrm{c}}$ be as follows (whereby one selects

[6] With respect to the usual order on $\mathbb{N}$

the uppermost matching case in the case distinctions):

$$K_{\square} *_{\mathrm{c}} \psi = \begin{cases} \{\psi_{\min(\mathbb{N}\setminus N)}\} & \text{if } \psi = \psi_{\mathbb{N}\setminus N}; \\ \{\psi_{\mathrm{A}}\} & \text{if } \psi = \psi_{\mathrm{AB}}; \\ \{\psi_{\mathrm{B}}\} & \text{if } \psi = \psi_{\mathrm{BC}}; \\ \{\psi_{\mathrm{C}}\} & \text{if } \psi = \psi_{\mathrm{CA}}; \\ \{\psi\} & \text{otherwise.} \end{cases} \quad \mathcal{K} *_{\mathrm{c}} \psi = \begin{cases} \mathcal{K} \Cup \{\psi\} & \text{if } [\![\mathcal{K} \Cup \{\psi\}]\!] \neq \emptyset; \\ \{\psi\} & \text{otherwise} \end{cases}$$

One can check case by case that (S1)–(S6) are satisfied.

Lemma 14. $*_{\mathrm{c}}$ *is a sentential base change operator for* $\mathbb{B}_{\mathrm{c}}$ *satisfying (S1)–(S6).*

We show that $*_{\mathrm{c}}$ cannot be reduced to any base change operator that satisfies (G1)–(G6). For that, we employ Theorem 9, and show that any candidate $\circ$ is not compatible with a min-friendly faithful assignment for $\mathbb{B}_{\mathrm{c}}$.

Lemma 15. $*_{\mathrm{c}}$ *is not reducible to any base change operator for* $\mathbb{B}_{\mathrm{c}}$ *that satisfies (G1)–(G6).*

Proof. Towards a contradiction, let $\circ_{\mathrm{c}}$ be a base change operator for $\mathbb{B}_{\mathrm{c}}$ that satisfies (G1)–(G6) and $*_{\mathrm{c}}$ is reducible to $\circ_{\mathrm{c}}$. Due to Theorem 9, there is a min-friendly faithful assignment $\preceq_{(.)}$ for $\mathbb{B}_{\mathrm{c}}$ that is compatible with $\circ_{\mathrm{c}}$. In the following, we write $\preceq_{\square}$ for $\preceq_{\{\psi_{\square}\}}$. Because $*_{\mathrm{c}}$ is reducible to $\circ_{\mathrm{c}}$, we have:

$$[\![\{\psi_{\square}\} \circ_{\mathrm{c}} \{\psi_{\mathrm{AB}}\}]\!] = \{A\} = \min([\![\{\psi_{\mathrm{AB}}\}]\!], \preceq_{\square})$$
$$[\![\{\psi_{\square}\} \circ_{\mathrm{c}} \{\psi_{\mathrm{BC}}\}]\!] = \{B\} = \min([\![\{\psi_{\mathrm{BC}}\}]\!], \preceq_{\square})$$
$$[\![\{\psi_{\square}\} \circ_{\mathrm{c}} \{\psi_{\mathrm{CA}}\}]\!] = \{C\} = \min([\![\{\psi_{\mathrm{CA}}\}]\!], \preceq_{\square})$$

Recall that $[\![\{\psi_{\mathrm{AB}}\}]\!] = \{A, B\}$, $[\![\{\psi_{\mathrm{BC}}\}]\!] = \{B, C\}$, and $[\![\{\psi_{\mathrm{CA}}\}]\!] = \{C, A\}$ holds. Consequently, we have:

$$A \preceq_{\square} B \qquad B \not\preceq_{\square} A \qquad B \preceq_{\square} C \qquad C \not\preceq_{\square} B \qquad C \preceq_{\square} A \qquad C \not\preceq_{\square} A$$

This yields that $\preceq_{\square}$ has the strict cycle $A \prec_{\square} B \prec_{\square} C \prec_{\square} A$. Next, observe that $[\![\Gamma_{\mathrm{ABC}}]\!] = \{A, B, C\}$. Yet, because of the strict cycle above, we obtain that $\min([\![\Gamma_{\mathrm{ABC}}]\!], \preceq_{\square}) = \emptyset$. The latter is impossible in $\preceq_{\square}$, as $\preceq_{\square}$ is min-friendly (and thus, min-complete). □

6.2 Proof of Theorem 12 via an Infinite Descending Chain

In the second proof of Theorem 12 we construct a sentential belief revision operator $*_d$ for which we will show that any base change operator $\circ_d$ to which $*_d$ is reducible is only compatible with assignments that contain an infinite descending chain. The proof strategy is similar as in Section 6.1, and works by showing that there is a non-singleton base Γ such that $[\![\mathcal{K} \circ_d \Gamma]\!] = \emptyset$ holds, when $*_d$ is reducible to $\circ_d$, yet (G1)–(G6) postulate $[\![\mathcal{K} \circ_d \Gamma]\!] \neq \emptyset$. Despite the structural similarity, we are required to construct a completely different logical setting to permit the required behaviour of $*_d$ and $\circ_d$.

We start, again, by defining a logic $(\mathcal{L}, \Omega, \models)$ by letting:

$$\mathcal{L} = \{\bot\} \cup \{\, [={}^1\!/_i],\ [\geq{}^1\!/_i] \mid i \in \mathbb{Z} \setminus \{0\}\,\} \qquad \Omega = \{\, {}^1\!/_i \mid i \in \mathbb{Z} \setminus \{0\}\}$$

The model-relation $\models$ is given by the following (where i ranges over $\mathbb{Z} \setminus \{0\}$):

$$[\![\bot]\!] = \emptyset \qquad [\![\,[={}^1\!/_i]\,]\!] = \{{}^1\!/_i\} \qquad [\![\,[\geq{}^1\!/_i]\,]\!] = \{{}^1\!/_n \in \Omega \mid {}^1\!/_n \geq {}^1\!/_i\}$$

That is the formulas of the logic $(\mathcal{L}, \Omega, \models)$ express either a fraction of the form $\pm{}^1\!/_i$ or an interval of all fractions of the form $\pm{}^1\!/_n$ between $\pm{}^1\!/_i$ and 1 (or nothing via the formula $\bot$). For every two formulas $\psi, \varphi \in \mathcal{L}$ we denote with $\psi \owedge \varphi$ the respective "conjunction" defined by:

$$\varphi \owedge \psi = \begin{cases} \varphi & \text{if } [\![\varphi]\!] \subseteq [\![\psi]\!]; \\ \psi & \text{if } [\![\psi]\!] \subsetneq [\![\varphi]\!]; \\ \bot & \text{otherwise.} \end{cases}$$

We extend $(\mathcal{L}, \Omega, \models)$ to a sentential base logic $\mathbb{B}_\mathrm{d} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup, \owedge)$. The set $\mathfrak{B}$ is defined by

$$\mathfrak{B} = \{\ \{\psi\}\ \mid \psi \in \mathcal{L}\} \cup \{\ \Gamma^\infty\ \}\ ,$$

whereby $\Gamma^\infty = \{\, [\geq -{}^1\!/_i] \mid i \in \mathbb{N}\}$. The sets of models of Γ^∞ consists of all positive fractions $[\![\Gamma^\infty]\!] = \{{}^1\!/_i \mid i \in \mathbb{N}\}$ from Ω. The abstract union $\Cup$ is defined by setting $\{\psi\} \Cup \{\varphi\} = \{\psi \owedge \varphi\}$ and for all other cases we have:

$$\Gamma^\infty \Cup \Gamma = \Gamma \Cup \Gamma^\infty = \begin{cases} \Gamma & \text{if } \Gamma = \{[={}^1\!/_i]\} \text{ or } \Gamma = \{[\geq{}^1\!/_i]\} \text{ for } i \in \mathbb{N}; \\ \Gamma^\infty & \text{if } \Gamma = \Gamma^\infty \text{ or } \Gamma = \{[\geq{}^1\!/_{-i}]\} \text{ for } i \in \mathbb{N}; \\ \{\bot\} & \text{otherwise.} \end{cases}$$

Again, one easily verifies that $[\![\Gamma_1 \Cup \Gamma_2]\!] = [\![\Gamma_1]\!] \cap [\![\Gamma_2]\!]$ holds, and thus, together with the existence of $\owedge$, we obtain the following.

Lemma 16. *$\mathbb{B}_\mathrm{d}$ is a sentential base logic.*

We define a sentential base change operator $*_\mathrm{d}$ for $\mathbb{B}_\mathrm{d}$. For $\mathcal{K}_\bot = \{\bot\}$ and all $K \neq \{\bot\}$, we let $*_\mathrm{d}$ be as follows:

$$\mathcal{K}_\bot *_\mathrm{d} \psi = \begin{cases} \{[={}^1\!/_i]\} & \text{if } \psi = [\geq{}^1\!/_i]; \\ \{\psi\} & \text{otherwise.} \end{cases} \quad \mathcal{K} *_\mathrm{d} \psi = \begin{cases} \mathcal{K} \Cup \{\psi\} & \text{if } [\![\mathcal{K} \Cup \{\psi\}]\!] \neq \emptyset; \\ \{\psi\} & \text{otherwise} \end{cases}$$

One can check case by case that (S1)–(S6) are satisfied.

Lemma 17. *$*_\mathrm{d}$ is a sentential base change operator for $\mathbb{B}_\mathrm{d}$ satisfies (S1)–(S6).*

To proof non-reducibility of $*_\mathrm{d}$, we show that any base change operator that embeds $*_\mathrm{d}$ does not satisfy (G1)–(G6).

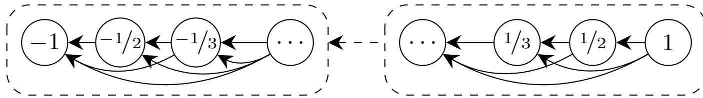

Fig. 2: Relation $\preceq_\bot$ imposed by $*_\mathrm{d}$ in the proof of Lemma 18.

Lemma 18. *$*_\mathrm{d}$ is not reducible to any base change operator for $\mathbb{B}_\mathrm{d}$ that satisfies (G1)–(G6).*

Proof. Towards a contradiction, let $\circ_\mathrm{d}$ be a base change operator for $\mathbb{B}_\mathrm{d}$ that satisfies (G1)–(G6) and $*_\mathrm{d}$ is reducible to $\circ_\mathrm{d}$. Due to Theorem 9, there is a min-friendly faithful assignment $\preceq_{(.)}$ for $\mathbb{B}_\mathrm{d}$ that is compatible with $\circ_\mathrm{d}$. The latter means especially, that $[\![\mathcal{K} \circ_\mathrm{d} \Gamma]\!] = \min\left([\![\Gamma]\!], \preceq_\mathcal{K}\right)$ holds for all $\mathcal{K}, \Gamma \in \mathfrak{B}$. In the following, we write $\preceq_\bot$ for $\preceq_{\mathcal{K}_\bot}$. Because $*_\mathrm{d}$ is reducible to $\circ_\mathrm{d}$, we obtain the following for all $i \in \mathbb{N}$:

$$[\![\, \mathcal{K}_\bot \circ_\mathrm{d} \{\, [\geq {}^1\!/i] \,\} \,]\!] = \{{}^1\!/i\} \qquad (\#)$$

Now suppose that there is some $i, j \in \mathbb{N}$ such that ${}^1\!/i < {}^1\!/j$ and ${}^1\!/j \preceq_\bot {}^1\!/i$. Because of the latter and ${}^1\!/j \in [\![[\geq {}^1\!/i]]\!]$, we have ${}^1\!/j \in \min([\![[\geq {}^1\!/i]]\!], \preceq_\bot)$. From compatibility of $\preceq_{(.)}$ with $\circ_\mathrm{d}$ we obtain ${}^1\!/j \in [\![\mathcal{K}_\bot \circ_\mathrm{d} \{[\geq {}^1\!/i]\}]\!]$. Thus, we obtain a contradiction of the latter with $(\#)$. This shows that ${}^1\!/j \preceq_\bot {}^1\!/i$ does not hold and consequently, $\preceq_\bot$ contains an infinite descending chain $\ldots \prec_\bot {}^1\!/3 \prec_\bot {}^1\!/2 \prec_\bot 1$. From this chain and $[\![\Gamma^\infty]\!] = \{1, {}^1\!/2, {}^1\!/3, \ldots\}$, we obtain that $\min([\![\Gamma^\infty]\!], \preceq_\bot) = \emptyset$ holds. The latter is impossible in $\preceq_\bot$, as $\preceq_\bot$ is min-friendly (and thus, $\min([\![\Gamma^\infty]\!], \preceq_\bot) \neq \emptyset$ holds due to min-completeness). □

7 Summary and Conclusion

In this paper, we demonstrated that sentence revision cannot be reduced to multiple revision. Together with Observation 5, this means that sentence revision and multiple revision are mutually different kinds of operations. We considered two examples that witness our observation: one example with a cyclic situation and one example without a cyclic situation. Having both examples is very important, as there is a debate whether belief revision with cyclic situations is appropriate or not [4]. Because we consider both cases, our result is independent of this debate. In summary, the main contribution is the following.

Observation 19. *Sentence revision and multiple revision are mutually irreducible, even in the presence or absence of cyclic situations.*

One might remark that the examples we discovered require "unnatural" logical settings and that sentence revision might be reducible to multiple revision. However, even for many natural settings, it is not immediately clear wether this is the case. Furthermore, the general lesson is that when one leaves safe grounds, one must be very careful in assuming intuitive results. This is in line with many other observations made in the area of belief change in recent years, e.g., representability [2, 3, 17–19] and semantics [4, 7].

References

1. Alchourrón, C.E., Gärdenfors, P., Makinson, D.: On the logic of theory change: Partial meet contraction and revision functions. Journal of Symbolic Logic **50**(22), 510–530 (1985). https://doi.org/10.2307/2274239
2. Baader, F., Wassermann, R.: Contractions based on optimal repairs. In: Marquis, P., Ortiz, M., Pagnucco, M. (eds.) Proceedings of the 21st International Conference on Principles of Knowledge Representation and Reasoning, KR 2024, Hanoi, Vietnam. November 2-8, 2024 (2024). https://doi.org/10.24963/KR.2024/9
3. Baader, F., Wassermann, R.: Gärdenfors's supplementary postulates for partial product contractions. In: Sauerwald, K., Thimm, M. (eds.) Proceedings of the 18th European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU 2025). Lecture Notes in Computer Science, vol. 16099, pp. 269–284. Springer (2025). https://doi.org/10.1007/978-3-032-05134-9_19
4. Delgrande, J.P., Peppas, P., Woltran, S.: General belief revision. Journal of the ACM **65**(5), 29:1–29:34 (2018). https://doi.org/10.1145/3203409
5. Falakh, F.M., Rudolph, S.: AGM revision in description logics under fixed-domain semantics. In: Arieli, O., Homola, M., Jung, J.C., Mugnier, M. (eds.) Proceedings of the 35th International Workshop on Description Logics (DL 2022) co-located with Federated Logic Conference (FLoC 2022). CEUR Workshop Proceedings, vol. 3263. CEUR-WS.org (2022), https://ceur-ws.org/Vol-3263/paper-8.pdf
6. Falakh, F.M., Rudolph, S., Sauerwald, K.: Semantic characterizations of AGM revision for Tarskian logics. In: Governatori, G., Turhan, A. (eds.) Proceedings of the 6th International Joint Conference on Rules and Reasoning (RuleML+RR 2022). Lecture Notes in Computer Science, vol. 13752, pp. 95–110. Springer (2022). https://doi.org/10.1007/978-3-031-21541-4_7
7. Falakh, F.M., Rudolph, S., Sauerwald, K.: AGM belief revision, semantically. ACM Trans. Comput. Log. **26**(4), 23:1–23:54 (2025). https://doi.org/10.1145/3763234
8. Fermé, E.L., Hansson, S.O.: Belief Change – Introduction and Overview. Springer Briefs in Intelligent Systems, Springer (2018). https://doi.org/10.1007/978-3-319-60535-7
9. Gärdenfors, P.: Knowledge in flux : modeling the dynamics of epistemic states. MIT Press Cambridge (1988)
10. Grove, A.: Two modellings for theory change. Journal of Philosophical Logic **17**(2), 157–170 (1988). https://doi.org/10.1007/BF00247909
11. Hansson, S.O.: A Textbook of Belief Dynamics: Theory Change and Database Updating. Springer (1999)
12. Katsuno, H., Mendelzon, A.O.: Propositional knowledge base revision and minimal change. Artificial Intelligence **52**(3), 263 – 294 (1991). https://doi.org/10.1016/0004-3702(91)90069-V
13. Kraus, S., Lehmann, D., Magidor, M.: Nonmonotonic reasoning, preferential models and cumulative logics. Artificial Intelligence **44**(1-2), 167–207 (1990). https://doi.org/10.1016/0004-3702(90)90101-5
14. Lewis, D.K.: Counterfactuals. Harvard University Press, Cambridge, Massachusetts (1973). https://doi.org/10.2307/2273738
15. Lindström, S.: A semantic approach to nonmonotonic reasoning: Inference operations and choice. Theoria **88**(3), 494–528 (2022). https://doi.org/https://doi.org/10.1111/theo.12405
16. Peppas, P.: The limit assumption and multiple revision. Journal of Logic and Computation **14**(3), 355–371 (2004). https://doi.org/10.1093/logcom/14.3.355

17. Ribeiro, M.M.: Belief Revision in Non-Classical Logics. Springer Briefs in Computer Science, Springer (2013). https://doi.org/10.1007/978-1-4471-4186-0
18. Ribeiro, M.M., Wassermann, R., Flouris, G., Antoniou, G.: Minimal change: Relevance and recovery revisited. Artificial Intelligence **201**, 59–80 (2013). https://doi.org/10.1016/j.artint.2013.06.001
19. Sauerwald, K.: Credibility-limited revision for epistemic spaces. In: Destercke, S., Martinez, M.V., Sanfilippo, G. (eds.) Proceedings of the 16th International Conference on Scalable Uncertainty Management (SUM 2024). pp. 429–444. Springer, Cambridge (2024). https://doi.org/10.1007/978-3-031-76235-2_32
20. Sernadas, A., Sernadas, C., Caleiro, C.: Synchronization of logics. Studia Logica **59**(1), 217–247 (1997). https://doi.org/10.1023/A:1004904401346
21. Tarski, A.: Logic Semantics, Metamathematics Papers From 1923 to 1938. Translated by J.H. Woodger. Clarendon Press (1956)