

DATABASE THEORY

Lecture 5: Complexity of FO Query Answering (II)

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Knowledge-Based Systems

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More recent versions of this slide deck might be available.
For the most current version of this course, see
https://iccl.inf.tu-dresden.de/web/Database_Theory/en

Review: Query Complexity

Query answering as decision problem

↪ consider Boolean queries

Various notions of complexity:

- Combined complexity (complexity w.r.t. size of query and database instance)
- Data complexity (worst case complexity for any fixed query)
- Query complexity (worst case complexity for any fixed database instance)

Various common complexity classes:

$$L \subseteq NL \subseteq P \subseteq NP \subseteq PSpace \subseteq ExpTime$$

Review: FO Combined Complexity

Theorem 4.1 The evaluation of FO queries is PSpace-complete with respect to combined complexity.

We have actually shown something stronger:

Theorem 4.2 The evaluation of FO queries is PSpace-complete with respect to query complexity.

This also holds true when restricting to domain-independent queries.

Data Complexity of FO Query Answering

The algorithm showed that FO query evaluation is in L

↪ can we do any better?

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↪ we need to define circuit complexities first

Boolean Circuits

Definition 5.1: A **Boolean circuit** is a finite, directed, acyclic graph where

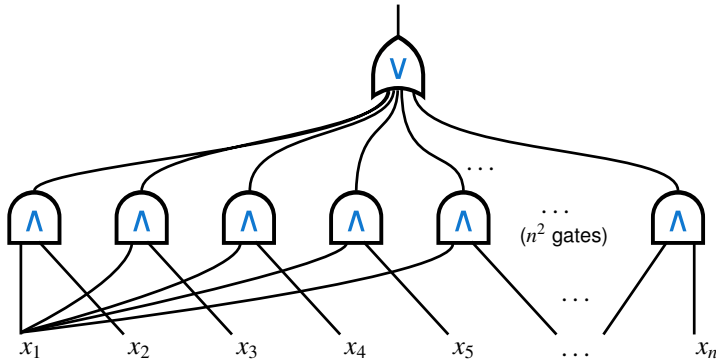
- each node that has no predecessors is an **input node**
- each node that is not an input node is one of the following types of **logical gate**: AND, OR, NOT
- one or more nodes are designated **output nodes**

↪ we will only consider Boolean circuits with exactly one output

↪ propositional logic formulae are Boolean circuits with one output and gates of fanout ≤ 1

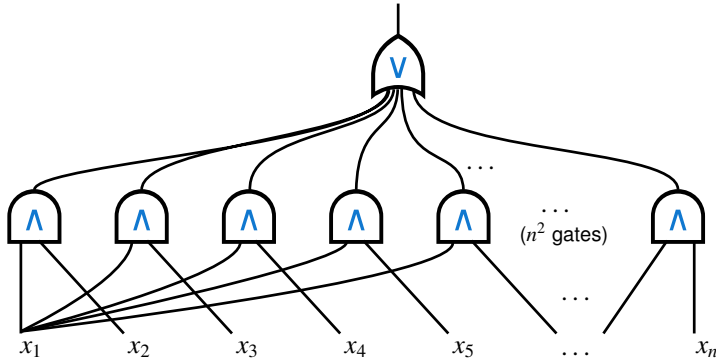
Example

A Boolean circuit over an input string $x_1x_2 \dots x_n$ of length n



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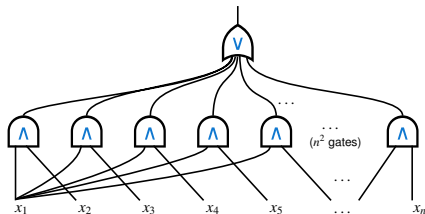


Corresponds to formula $(x_1 \wedge x_2) \vee (x_1 \wedge x_3) \vee \dots \vee (x_{n-1} \wedge x_n)$

\rightsquigarrow accepts all strings with at least two 1s

Circuits as a Model for Parallel Computation

Previous example:



$\leadsto n^2$ processors working in parallel
 \leadsto computation finishes in 2 steps

- **size:** number of gates = total number of computing steps
- **depth:** longest path of gates = time for parallel computation

\leadsto circuits as a refinement of polynomial time that takes parallelizability into account

Solving Problems With Circuits

Observation: the input size is “hard-wired” in circuits

~> each circuit only has a finite number of different inputs

~> not a computationally interesting problem

How can we solve interesting problems with Boolean circuits?

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How can we solve interesting problems with Boolean circuits?

Definition 5.2: A **uniform family** of Boolean circuits is a set of circuits C_n ($n \geq 0$) that can easily^a be computed from n .

A language $\mathcal{L} \subseteq \{0, 1\}^*$ is **decided by** a uniform family $(C_n)_{n \geq 0}$ of Boolean circuits if for each word w of length $|w|$:

$$w \in \mathcal{L} \quad \text{if and only if} \quad C_{|w|}(w) = 1$$

^aWe don't discuss the details here; see course Complexity Theory.

Measuring Complexity with Boolean Circuits

How to measure the computing power of Boolean circuits?

Relevant metrics:

- **size** of the circuit: overall number of gates
(as function of input size)
- **depth** of the circuit: longest path of gates
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- **fan in**: two inputs per gate or any number of inputs per gate?

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Important classes of circuits: **small-depth circuits**

Definition 5.3: $(C_n)_{n \geq 0}$ is a family of **small-depth circuits** if

- the size of C_n is polynomial in n ,
- the depth of C_n is poly-logarithmic in n , that is, $O(\log^k n)$.

The Complexity Classes NC and AC

Two important types of small-depth circuits:

Definition 5.4: NC^k is the class of problems that can be solved by uniform families of circuits $(C_n)_{n \geq 0}$ of fan-in ≤ 2 , size polynomial in n , and depth in $O(\log^k n)$.

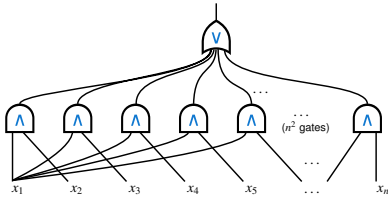
The class NC is defined as $NC = \bigcup_{k \geq 0} NC^k$.

(“Nick’s Class” named after Nicholas Pippenger by Stephen Cook)

Definition 5.5: AC^k and AC are defined like NC^k and NC, respectively, but for circuits with arbitrary fan-in.

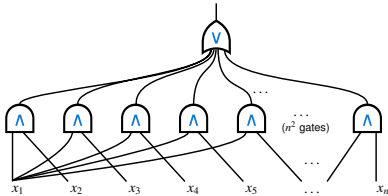
(A is for “Alternating”: AND-OR gates alternate in such circuits)

Example



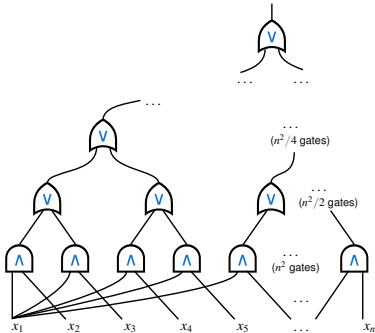
family of polynomial size,
constant depth,
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Example



family of polynomial size,
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We can eliminate arbitrary fan-ins by using more layers of gates:



family of polynomial size,
logarithmic depth,
bounded fan-in circuits
 \leadsto in NC^1

Relationships of Circuit Complexity Classes

The previous sketch can be generalised:

$$\text{NC}^0 \subseteq \text{AC}^0 \subseteq \text{NC}^1 \subseteq \text{AC}^1 \subseteq \dots \subseteq \text{AC}^k \subseteq \text{NC}^{k+1} \subseteq \dots$$

Only few inclusions are known to be proper: $\text{NC}^0 \subset \text{AC}^0 \subset \text{NC}^1$

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Direct consequence of above hierarchy: $\text{NC} = \text{AC}$

Interesting relations to other classes:

$$\text{NC}^0 \subset \text{AC}^0 \subset \text{NC}^1 \subseteq \text{L} \subseteq \text{NL} \subseteq \text{AC}^1 \subseteq \dots \subseteq \text{NC} \subseteq \text{P}$$

Intuition:

- Problems in NC are parallelisable (known from definition)
- Problems in $\text{P} \setminus \text{NC}$ are inherently sequential (educated guess)

However: it is not known if $\text{NC} \neq \text{P}$

Back to Databases ...

Theorem 5.6: The evaluation of FO queries is complete for (logtime uniform) AC^0 with respect to data complexity.

Proof:

- **Membership:** For a fixed Boolean FO query, provide a uniform construction for a small-depth circuit based on the size of a database
- **Hardness:** Show that circuits can be transformed into Boolean FO queries in logarithmic time (not on a standard TM ... not in this lecture)

From Query to Circuit

Assumptions:

- query and database schema is fixed
- database instance (and thus active domain) are variable

Construct circuit uniformly based on size of active domain

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Sketch of construction:

- one input node for each possible database tuple (over given schema and active domain)
 \leadsto true or false depending on whether tuple is present or not
- Recursively, for each subformula, introduce a gate for each possible tuple (instantiation) of this formula
 \leadsto true or false depending on whether the subformula holds for this tuple or not
- Logical operators correspond to gate types: basic operators obvious, \forall as generalised conjunction, \exists as generalised disjunction
- subformula with n free variables $\leadsto |\mathbf{adom}|^n$ gates
 \leadsto especially: $|\mathbf{adom}|^0 = 1$ output gate for Boolean query

Example

We consider the formula

$$\exists z.(\exists x.\exists y.R(x, y) \wedge S(y, z)) \wedge \neg R(a, z)$$

Over the database instance:

R:

<i>a</i>	<i>a</i>
<i>a</i>	<i>b</i>

S:

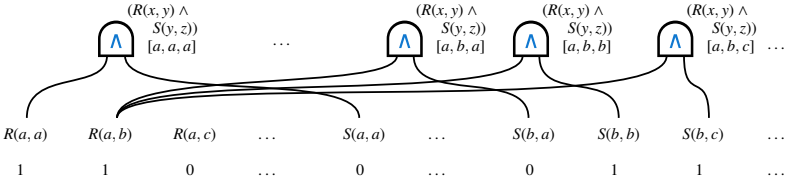
<i>b</i>	<i>b</i>
<i>b</i>	<i>c</i>

Active domain: $\{a, b, c\}$

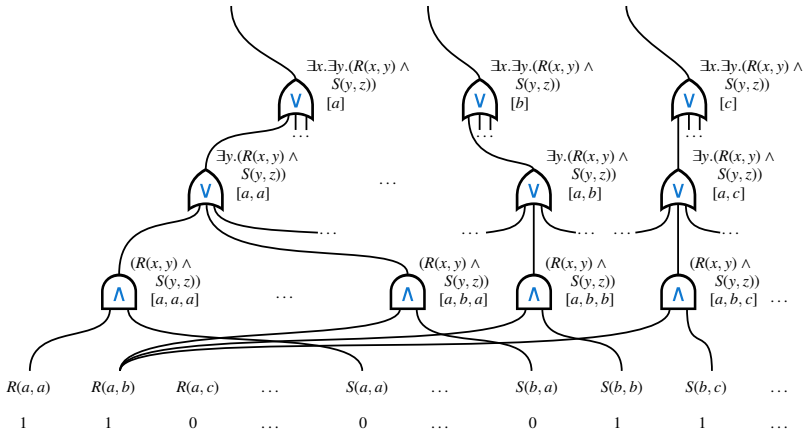
Example: $\exists z. (\exists x. \exists y. R(x, y) \wedge S(y, z)) \wedge \neg R(a, z)$

$R(a, a)$	$R(a, b)$	$R(a, c)$...	$S(a, a)$...	$S(b, a)$	$S(b, b)$	$S(b, c)$...
1	1	0	...	0	...	0	1	1	...

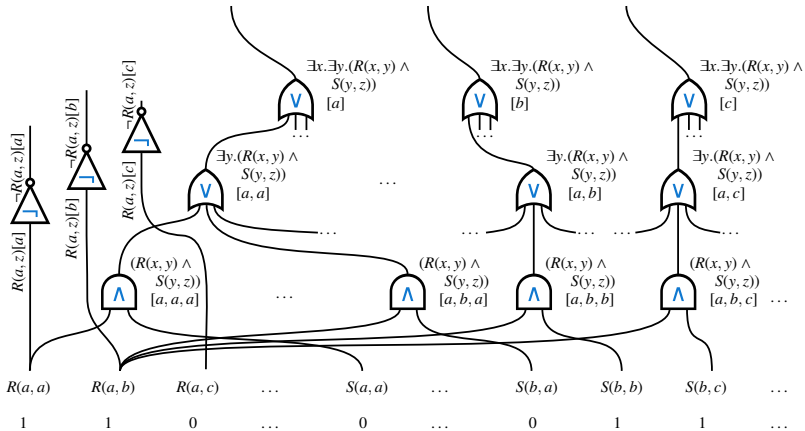
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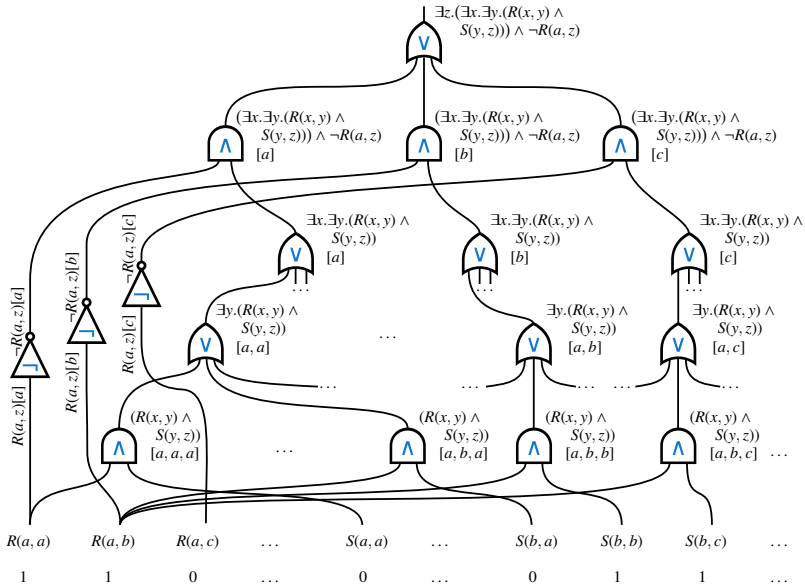
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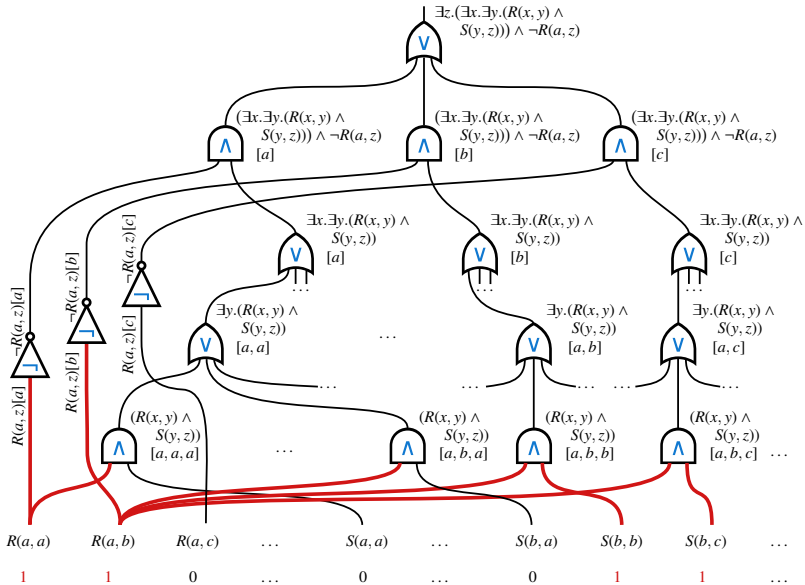
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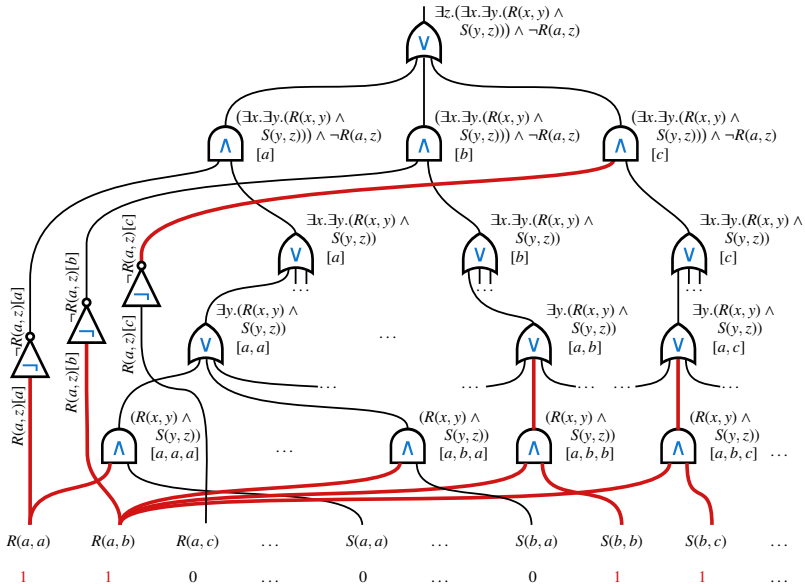
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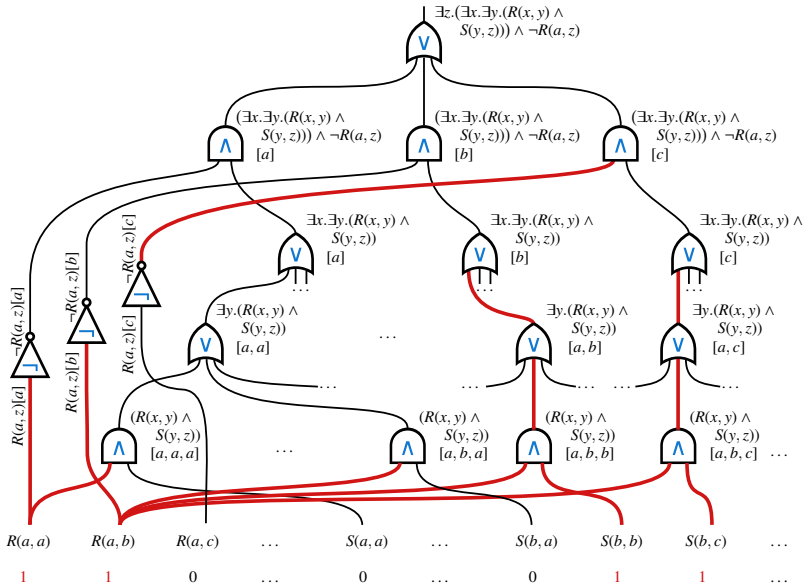
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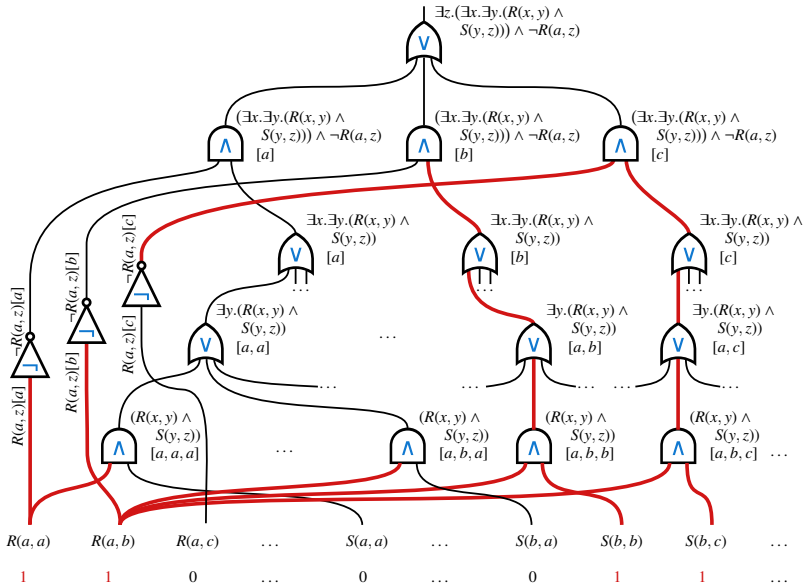
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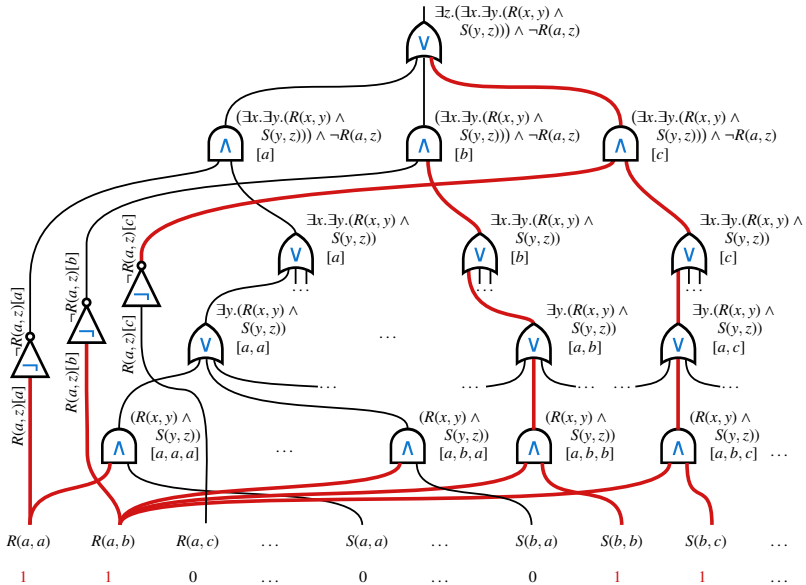
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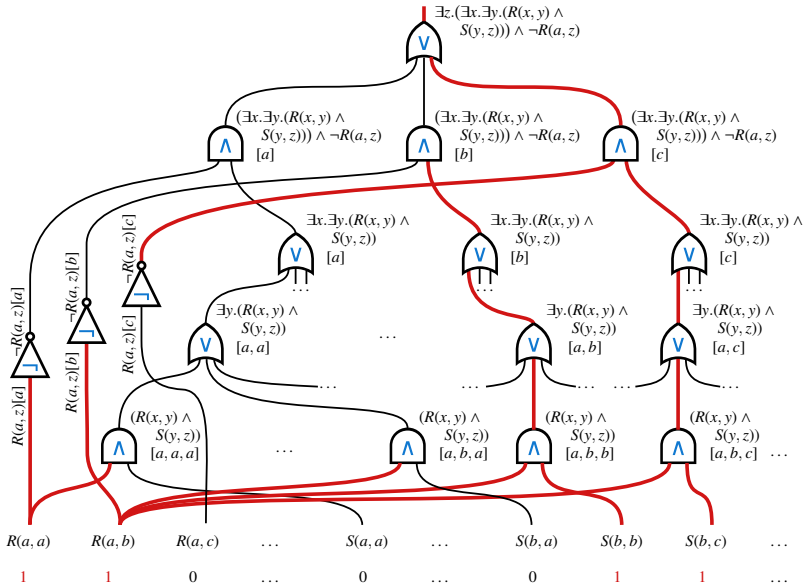
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Summary and Outlook

The evaluation of FO queries is

- PSpace-complete for combined complexity
- PSpace-complete for query complexity
- AC^0 -complete for data complexity

Circuit complexities help to identify highly parallelisable problems in P

Open questions:

- Are there query languages with lower complexities? (next lecture)
- Which other computing problems are interesting?
- How can we study the expressiveness of query languages?