TU Dresden, Fakultät Informatik
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Complexity Theory
Exercise 5: Space Complexity
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Exercise 5.1. Let $\mathbf{A}_{\text{LBA}}$ be the word problem of deterministic linear bounded automata.
Definition. A *deterministic linear bounded automaton* (LBA) is a tuple

$$\mathcal{M} = \langle Q, \Sigma, \Gamma, \delta, q_0, Q_F, E_L, E_R \rangle$$

where

- Q is a finite set of states, Σ and Γ the input and tape alphabet such that $\Sigma \subseteq \Gamma$, $\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, N, R\}$ the (deterministic) transition function, $q_0 \in Q$ the initial state, $Q_F \subseteq Q$ the set of final states, and
- $E_L, E_R \in \Sigma$ are special markers, marking the *left end* (E_L) and the *right end* (E_R) of the (usable) tape.

For all transitions $\langle q, a \rangle \mapsto \langle q', b, M \rangle \in \delta$,

- if $a = E_L$, then $b = E_L$ and $M = R$, and
- if $a = E_R$, then $b = E_R$ and $M = L$.

Inputs for an LBA $\mathcal{M}$ have the shape $E_L a_1 a_2 \cdots a_n E_R$ such that $a_i \notin \{E_L, E_R\}$ for $i = 1, 2, \ldots, n$.

Show that $\mathbf{A}_{\text{LBA}}$ is PSPACE-complete.

$$\mathbf{A}_{\text{LBA}} = \{\langle \mathcal{M}, w \rangle \mid \mathcal{M} \text{ is a (deterministic) LBA and } w \in \mathbf{L}(\mathcal{M})\}$$

Exercise 5.2. Consider the Japanese game *go-moku* that is played by two players X and O on a 19x19 board. Players alternately place markers on the board, and the first one to have five of its markers in a row, column, or diagonal wins.

Consider the generalised version of go-moku on an $n \times n$ board. Say that a *position* of go-moku is a placement of markers on such a board as it could occur during the game, together with a marker which player moves next. We define

$$\mathbf{GM} = \{\langle B \rangle \mid B \text{ is a position of go-moku where X has a winning strategy}\}.$$

Show that **GM** is in PSPACE.

Exercise 5.3. Show that the universality problem of nondeterministic finite automata

$$\mathbf{ALL}_{\text{NFA}} = \{\langle \mathcal{A} \rangle \mid \mathcal{A} \text{ an NFA accepting every valid input}\}$$

is in PSPACE.

Hint:

Prove that, if $L(\mathcal{A}) \neq \Sigma^*$ and $\mathcal{A}$ has n states, then there exists a word $w \in \Sigma^*$ of length at most 2^n such that $w \notin L(\mathcal{A})$. Then, use this fact to give a non-deterministic algorithm whose space consumption is polynomially bounded. Finally, apply Savitch's Theorem.

Exercise 5.4. Show that the composition of logspace reductions again yields a logspace reduction.

Exercise 5.5. Show that the word problem $\mathsf{A}_{\mathsf{NFA}}$ of non-deterministic finite automata is NL-complete.

Exercise 5.6. Show that

$$\mathsf{BIPARTITE} = \{ \langle G \rangle \mid G \text{ a finite bipartite graph} \}$$

is in NL. For this show that $\overline{\mathsf{BIPARTITE}} \in \mathsf{NL}$ and use $\mathrm{NL} = \mathrm{coNL}$. **Hint:** Show that a graph G is bipartite if and only if it does not contain a cycle of odd length.