Finite and Algorithmic Model Theory

Lecture 3 (Dresden 26.10.22, Long version)

Lecturer: Bartosz "Bart" Bednarczyk

Technische Universität Dresden & Uniwersytet Wrocławski

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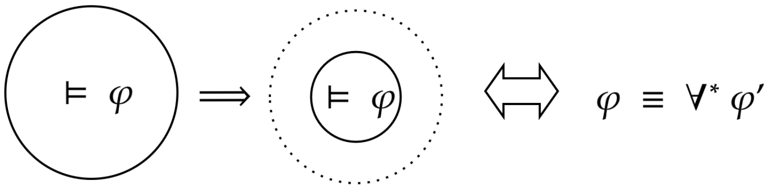

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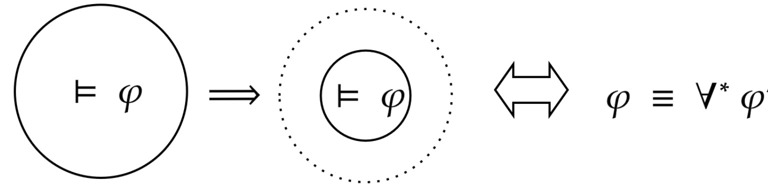

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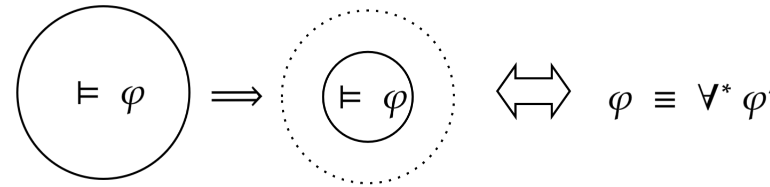

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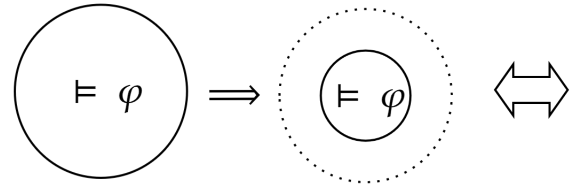

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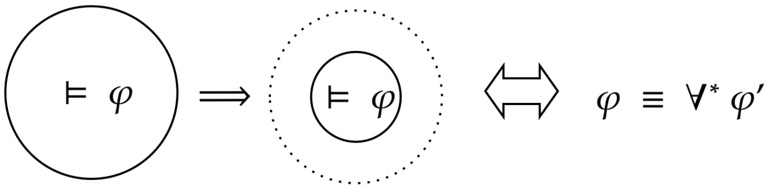

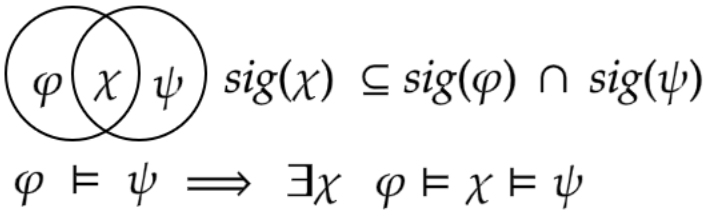

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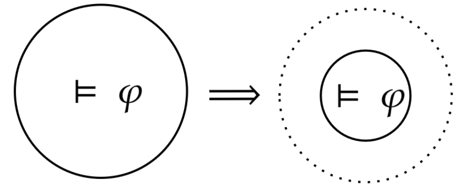

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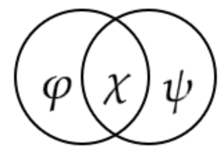

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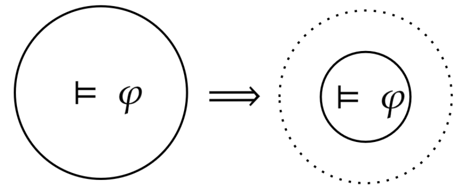

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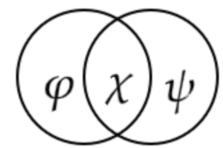
+ recent research papers.



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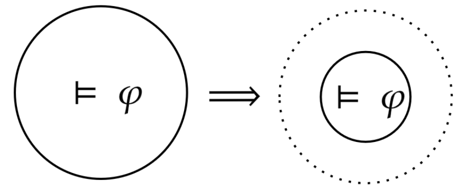

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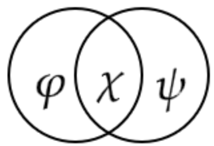

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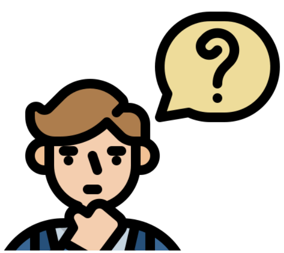

Feel free to ask questions and interrupt me!

Don't be shy! If needed send me an email (bartosz.bednarczyk@cs.uni.wroc.pl) or approach me after the lecture!

Reminder: this is an advanced lecture. Target: people that had fun learning logic during BSc studies!

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fresh constants | make them different | iterate through τ | positive facts | negative facts

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Alternative definition: $\mathrm{D}(\mathfrak{A}) := \{\varphi \in \mathrm{FO}[\tau_A] \mid \mathfrak{A}_A \models \varphi,\ \varphi \text{ is quantifier free}\}$

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[a] i.e. $\mathfrak{A} \models \varphi$ and $\mathfrak{B} \subseteq \mathfrak{A}$ then $\mathfrak{B} \models \varphi$

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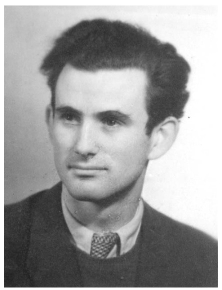

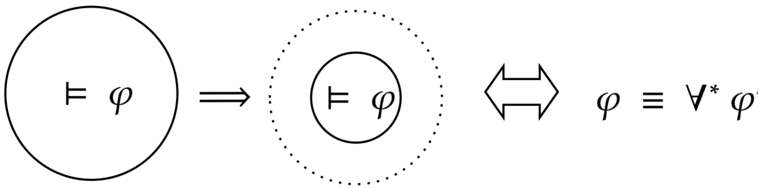

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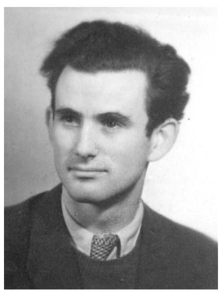

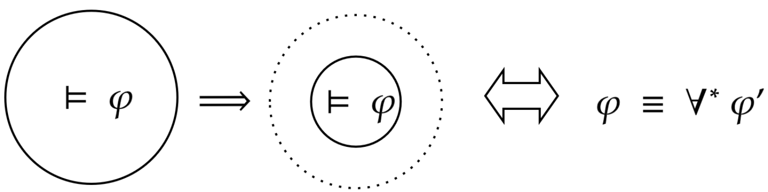

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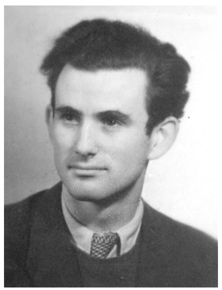

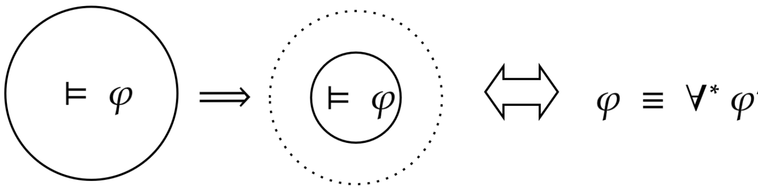

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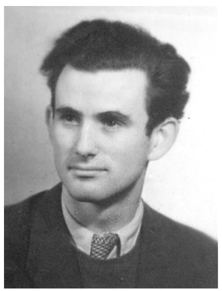

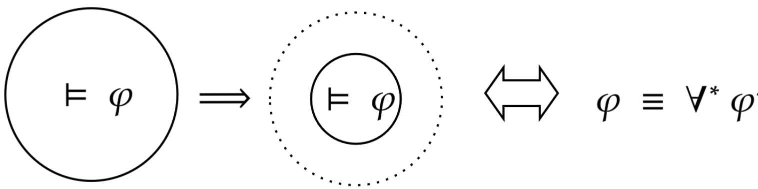

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Preservation Theorems

Common theme: Formulae having certain properties are precisely these of a certain fragment of FO

Theorem (Łoś-Tarski 1954)

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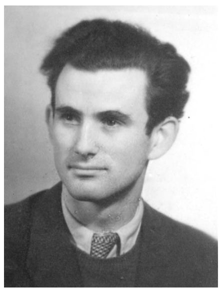
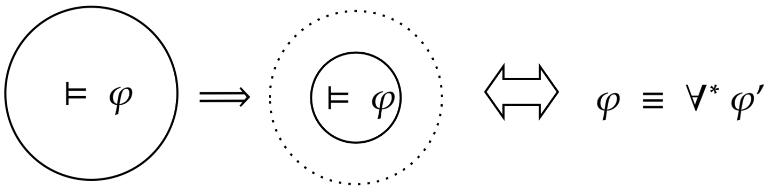
[a] i.e. $\mathfrak{A} \models \varphi$ and $\mathfrak{B} \subseteq \mathfrak{A}$ then $\mathfrak{B} \models \varphi$

[b] (possibly negated) atomic symbols $+ \wedge$, $\vee$ and $\forall$





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- Open problem: Is there a non-trivial $\mathcal{L} \subseteq$ FO (without equality) without Łoś-Tarski? [B. 2022]

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def of $\models$ magic assumption φ diagrams contradiction def of $\models$

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def of $\models$ | magic | assumption φ | diagrams | contradiction | def of $\models$

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But as diagrams are closed under conjunction, we get a

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Let $\mathfrak{A} \models \Psi$. We want to show $\mathfrak{A} \models \varphi$. It suffices to find a model $\mathfrak{B}$ of φ containing $\mathfrak{A}$ as a substructure.

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How to find such $\mathfrak{B}$? Show that $\mathrm{D}(\mathfrak{A}) \cup \{\varphi\}$ is satisfiable!

Ad absurdum, assume that $\mathrm{D}(\mathfrak{A}) \cup \{\varphi\}$ has no model. So $\varphi \models \neg\mathrm{D}(\mathfrak{A})$ holds, i.e. $\varphi \models \neg \bigwedge_{\psi(\bar{a}) \in \mathrm{D}(\mathfrak{A})} \psi(\bar{a})$.

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From $\xi(\bar{a}) \in \mathrm{D}(\mathfrak{A})$ we infer $\mathfrak{A} \models \exists \bar{x} \xi(\bar{x})$. A contradiction with $\mathfrak{A} \models \Psi$. $\square$

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Theorem (Tait 1933)

There is an FO formula that is preserved under substructures of finite structures but it is not equivalent (in the finite) to any universal formula.

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Consider $\tau = \{\mathtt{min}^{(0)}, \mathtt{max}^{(0)}, <^{(2)}, \mathrm{Next}^{(2)}, \mathrm{P}^{(1)}\}$. Let φ_0 be a universal stating that

$\mathfrak{A} \models \varphi_0$ iff $<^{\mathfrak{A}}$ is a strict linear order with the minimal/maximal elements $\mathtt{min}^{\mathfrak{A}}$, $\mathtt{max}^{\mathfrak{A}}$, and $\mathrm{Next}^{\mathfrak{A}} \subseteq <^{\mathfrak{A}}$.

Moreover, take $\varphi_1 := \forall x \forall y\ \mathrm{Next}(x, y) \leftrightarrow (x < y \wedge \neg(\exists z\ x < z \wedge z < y))$.

Note: if $\mathfrak{A} \models \varphi_0 \wedge \varphi_1$, then $\mathrm{Next}^{\mathfrak{A}}$ is the induced successor of $<^{\mathfrak{A}}$. Finally, let $\varphi := \varphi_0 \wedge (\varphi_1 \rightarrow \exists x\ \mathrm{P}(x))$.

Observation (The set of finite models of φ is closed under substructures.)

Take a finite $\mathfrak{A} \models \varphi$ and $\mathfrak{B} \subseteq \mathfrak{A}$. Observe that $\mathfrak{B} \models \varphi_0$ (because φ_0 is universal). If $\mathfrak{B} \not\models \varphi_1$ we are done. If $\mathfrak{B} \models \varphi_1$ then $\mathfrak{A} = \mathfrak{B}$, concluding $\mathfrak{B} \models \varphi$. $\square$

universals are preserved under $\subseteq$

finiteness

Failure of Łoś-Tarski in the finite. (Part II)

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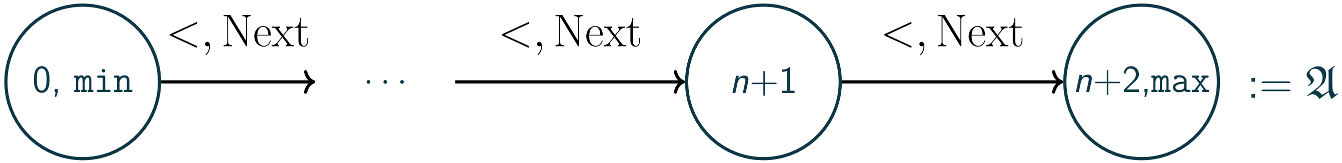

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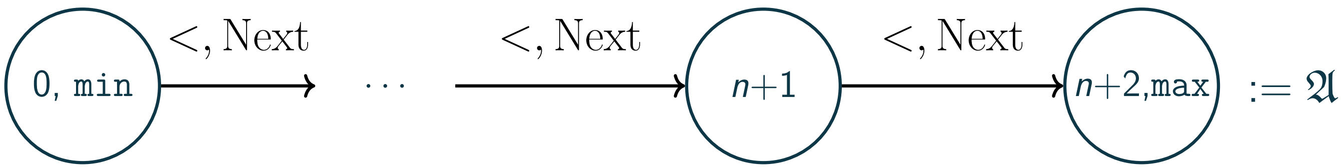

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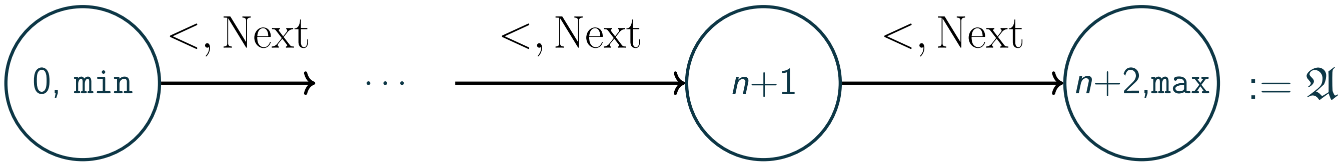

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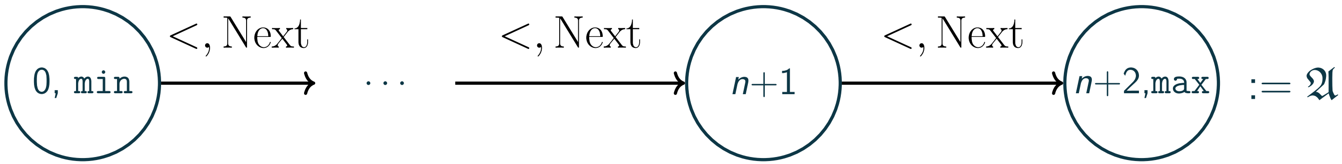

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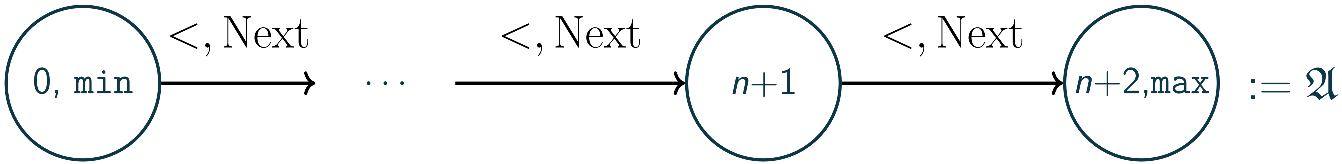

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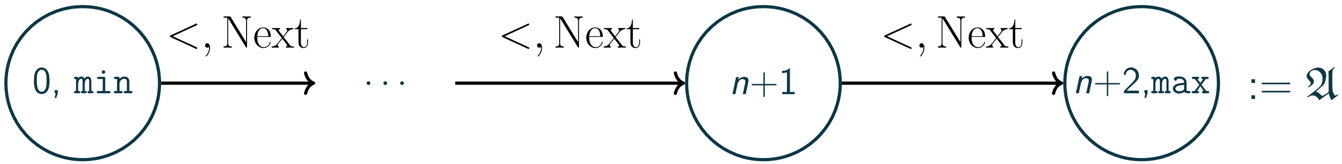

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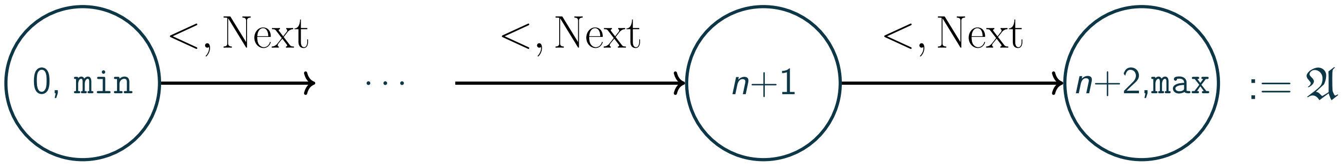

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contradiction def of P when $\mathrm{P}^{\mathfrak{A}} = \emptyset$ witness

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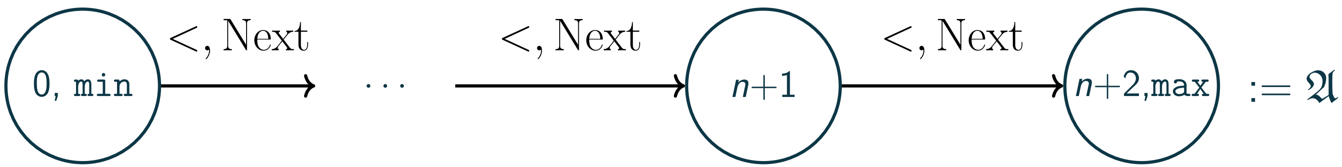

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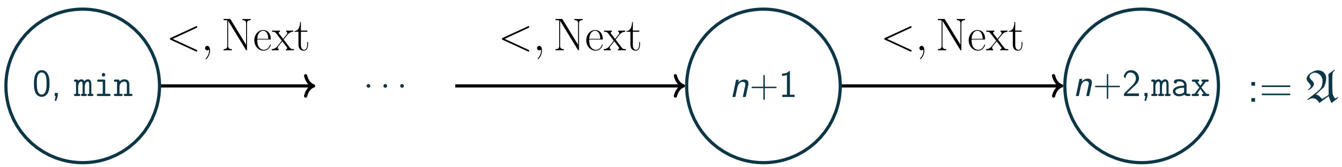

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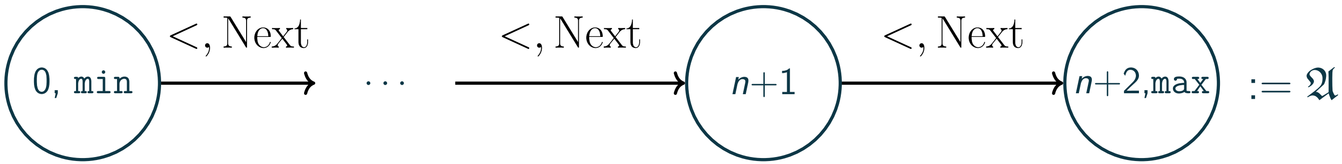

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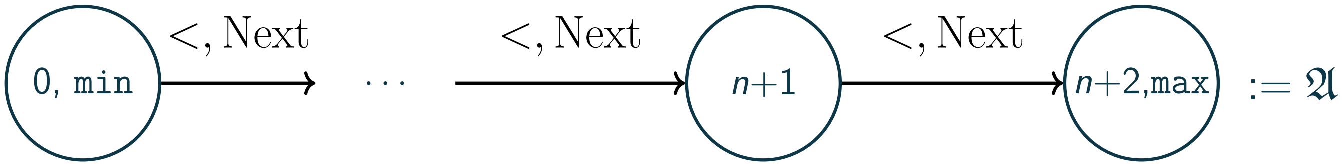

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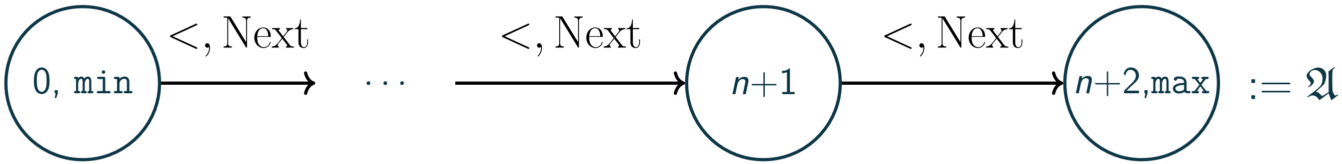

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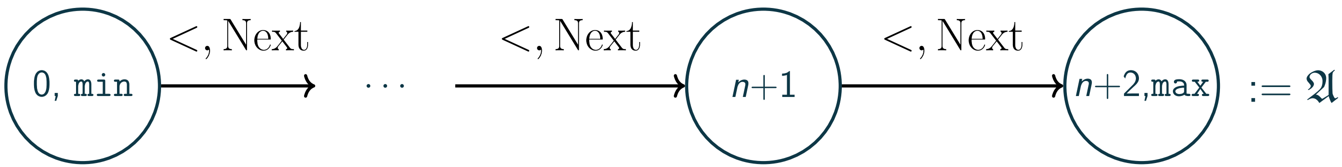

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contradiction | def of P | when $\mathrm{P}^{\mathfrak{A}} = \emptyset$ | witness | select suitable b and make it satisfy P | def of φ

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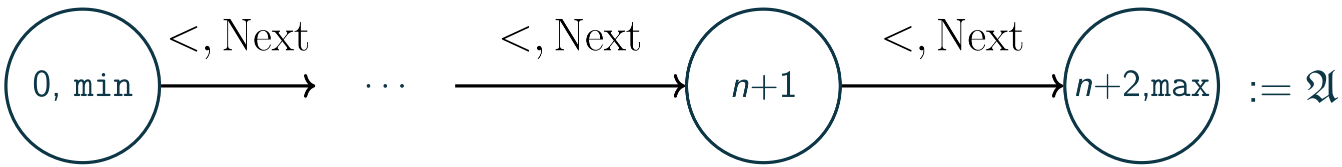

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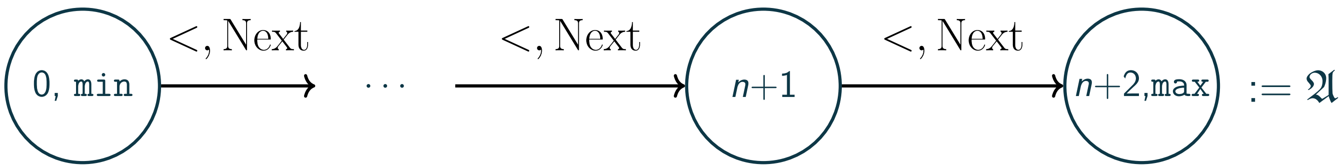
Lemma (φ is not equivalent (in the finite) to any universal formula.)

Ad absurdum, there exists quantifier-free $\chi(\bar{x})$ with n variables so that $\varphi \equiv_{\mathsf{fin}} \forall \bar{x} \; \chi(\bar{x})$. Take $\mathfrak{A}$ as below.



By construction $\mathfrak{A} \models \varphi_0 \land \varphi_1$. Moreover, observe that $(\mathfrak{A}, \mathrm{P}^{\mathfrak{A}}) \models \varphi$ iff $\mathrm{P}^{\mathfrak{A}} \neq \emptyset$.

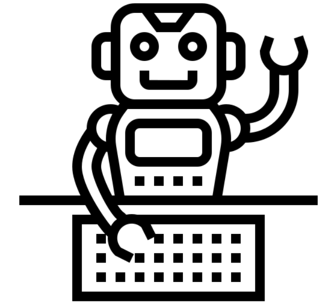
Then $(\mathfrak{A}, \emptyset) \not\models \varphi$ implies $(\mathfrak{A}, \emptyset) \not\models \forall \bar{x} \; \chi(\bar{x})$. Thus $(\mathfrak{A}, \emptyset) \models \neg\chi(\bar{a})$ for suitable $\bar{a}$.

Take b to be different from $\bar{a}$, $\mathtt{max}^{\mathfrak{A}}$ and $\mathtt{min}^{\mathfrak{A}}$ (we have enough elements!). Then $(\mathfrak{A}, \{b\}) \models \varphi$.

But $(\mathfrak{A}, \{b\}) \models \neg\chi(\bar{a})$ ($\mathfrak{A} \restriction \bar{a}$ was not touched!). But it means $(\mathfrak{A}, \{b\}) \not\models \forall \bar{x} \; \chi(\bar{x}) \equiv \varphi$. A contradiction!

contradiction | def of P | when $\mathrm{P}^{\mathfrak{A}} = \emptyset$ | witness | select suitable b and make it satisfy P | def of φ

Can we make Łoś-Tarski theorem computable?



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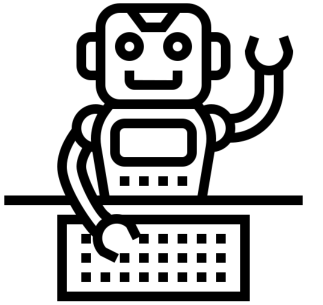
Input: First-Order φ closed under substructures (in the general setting).



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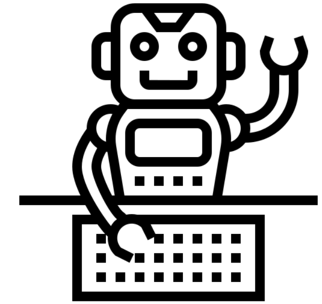
Output: the equivalent universal formula.



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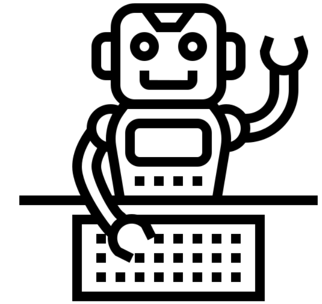

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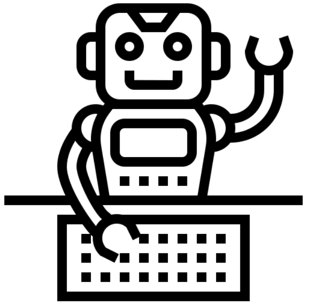

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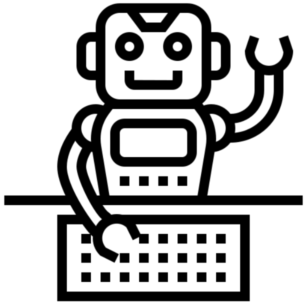

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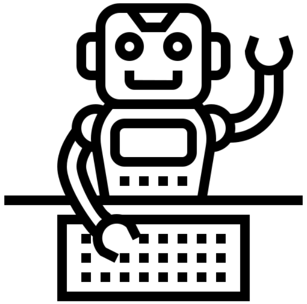

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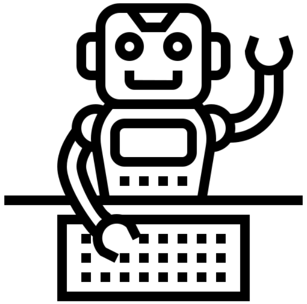

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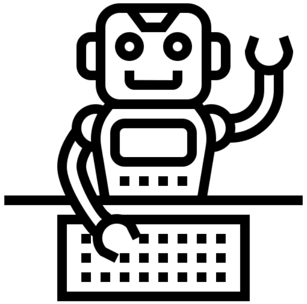

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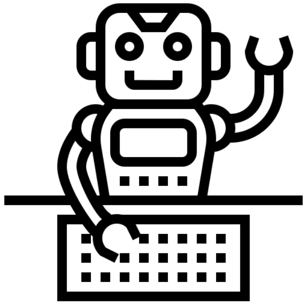

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- A notable example of classical MT theorem that works in the finite, c.f. [`Rossmann's paper`]

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