TECHNISCHE UNIVERSITÄT DRESDEN

COMPLEXITY THEORY

Lecture 13: Space Hierarchy and Gaps

Markus Krötzsch
Knowledge-Based Systems

TU Dresden, 26th Nov 2019

Review

Review: Time Hierarchy Theorems

Time Hierarchy Theorem 12.12 If $f, g : \mathbb{N} \to \mathbb{N}$ are such that f is time-constructible, and $g \cdot \log g \in o(f)$, then

$$\mathsf{DTime}_*(g) \subsetneq \mathsf{DTime}_*(f)$$

Nondeterministic Time Hierarchy Theorem 12.14 If $f, g : \mathbb{N} \to \mathbb{N}$ are such that f is time-constructible, and $g(n+1) \in o(f(n))$, then

$$\mathsf{NTime}_*(g) \subsetneq \mathsf{NTime}_*(f)$$

In particular, we find that P ≠ ExpTime and NP ≠ NExpTime:

$$\mathsf{L} \subseteq \mathsf{NL} \subseteq \mathsf{P} \subseteq \mathsf{NP} \subseteq \mathsf{PSpace} \subseteq \mathsf{ExpTime} \subseteq \mathsf{NExpTime} \subseteq \mathsf{ExpSpace}$$

≠

≠

A Hierarchy for Space

Space Hierarchy

For space, we can always assume a single working tape:

- Tape reduction leads to a constant-factor increase in space
- Constant factors can be eliminated by space compression

Therefore, $\mathsf{DSpace}_k(f) = \mathsf{DSpace}_1(f)$.

Space turns out to be easier to separate – we get:

Space Hierarchy Theorem 13.1: If $f, g : \mathbb{N} \to \mathbb{N}$ are such that f is space-constructible, and $g \in o(f)$, then

$$\mathsf{DSpace}(g) \subsetneq \mathsf{DSpace}(f)$$

Challenge: TMs can run forever even within bounded space.

Proving the Space Hierarchy Theorem (1)

Space Hierarchy Theorem 13.1: If $f, g : \mathbb{N} \to \mathbb{N}$ are such that f is space-constructible, and $g \in o(f)$, then

$$\mathsf{DSpace}(g) \subsetneq \mathsf{DSpace}(f)$$

Proof: Again, we construct a diagonalisation machine $\mathcal{D}$. We define a multi-tape TM $\mathcal{D}$ for inputs of the form $\langle \mathcal{M}, w\rangle$ (other cases do not matter), assuming that $|\langle \mathcal{M}, w\rangle| = n$

- Compute $f(n)$ in unary to mark the available space on the working tape
- Initialise a separate countdown tape with the largest binary number that can be written in $f(n)$ space
- Simulate $\mathcal{M}$ on $\langle \mathcal{M}, w\rangle$, making sure that only previously marked tape cells are used
- Time-bound the simulation using the content of the countdown tape by decrementing the counter in each simulated step
- If $\mathcal{M}$ rejects (in this space bound) or if the time bound is reached without $\mathcal{M}$ halting, then accept; otherwise, if $\mathcal{M}$ accepts or uses unmarked space, reject

Proving the Space Hierarchy Theorem (1)

Proof (continued): It remains to show that $\mathcal{D}$ implements diagonalisation:

$\mathbf{L}(\mathcal{D}) \in \mathsf{DSpace}(f)$:

- f is space-constructible, so both the marking of tape symbols and the initialisation of the counter are possible in $\mathsf{DSpace}(f)$
- The simulation is performed so that the marked $O(f)$-space is not left

There is w such that $\langle \mathcal{M}, w\rangle \in \mathbf{L}(\mathcal{D})$ iff $\langle \mathcal{M}, w\rangle \notin \mathbf{L}(\mathcal{M})$:

- As for time, we argue that some w is long enough to ensure that f is sufficiently larger than g, so $\mathcal{D}$'s simulation can finish.
- The countdown measures $2^{f(n)}$ steps. The number of possible distinct configurations of $\mathcal{M}$ on w is $|Q| \cdot n \cdot g(n) \cdot |\Gamma|^{g(n)} \in 2^{O(g(n)+\log n)}$, and due to $f(n) \geq \log n$ and $g \in o(f)$, this number is smaller than $2^{f(n)}$ for large enough n.
- If $\mathcal{M}$ has d tape symbols, then $\mathcal{D}$ can encode each in $\log d$ space, and due to $\mathcal{M}$'s space bound $\mathcal{D}$'s simulation needs at most $\log d \cdot g(n) \in o(f(n))$ cells.

Therefore, there is w for which $\mathcal{D}$ simulates $\mathcal{M}$ long enough to obtain (and flip) its output, or to detect that it is not terminating (and to accept, flipping again). $\square$

Space Hierarchies

Like for time, we get some useful corollaries:

Corollary 13.2: PSpace $\subsetneq$ ExpSpace

Proof: As for time, but easier. $\square$

Corollary 13.3: NL $\subsetneq$ PSpace

Proof: Savitch tells us that NL $\subseteq \mathsf{DSpace}(\log^2 n)$. We can apply the Space Hierarchy Theorem since $\log^2 n \in o(n)$. $\square$

Corollary 13.4: For all real numbers $0 < a < b$, we have $\mathsf{DSpace}(n^a) \subsetneq \mathsf{DSpace}(n^b)$.

In other words: The hierarchy of distinct space classes is very fine-grained.

The Gap Theorem

Why Constructibility?

The hierarchy theorems require that resource limits are given by constructible functions

Do we really need this?

Yes. The following theorem shows why (for time):

Special Gap Theorem 13.5: There is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that $\mathrm{DTime}(f(n)) = \mathrm{DTime}(2^{f(n)})$.

This has been shown independently by Boris Trakhtenbrot (1964) and Allan Borodin (1972).

Reminder: For this we continue to use the strict definition of $\mathrm{DTime}(f)$ where no constant factors are included (no hidden $O(f)$). This simplifes proofs; the factors are easy to add back.

Proving the Gap Theorem

Special Gap Theorem 13.5: There is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that $\mathrm{DTime}(f(n)) = \mathrm{DTime}(2^{f(n)})$.

Proof idea: We divide time into exponentially long intervals of the form:

$$[0, n], \quad [n+1, 2^n], \quad [2^n+1, 2^{2^n}], \quad [2^{2^n}+1, 2^{2^{2^n}}], \quad \cdots$$

(for some appropriate starting value n)

We are looking for gaps of time where no TM halts, since:

- for every finite set of TMs,
- and every finite set of inputs to these TMs,
- there is some interval of the above form $[m+1, 2^m]$

such none of the TMs halts in between $m+1$ and 2^m steps on any of the inputs.

The task of f is to find the start m of such a gap for a suitable set of TMs and words

Gaps in Time

We consider an (effectively computable) enumeration of all Turing machines:

$$\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2, \ldots$$

Definition 13.6: For arbitrary numbers $i, a, b \in \mathbb{N}$ with $a \leq b$, we say that $\mathrm{Gap}_i(a, b)$ is true if:

- Given any TM $\mathcal{M}_j$ with $0 \leq j \leq i$,
- and any input string w for $\mathcal{M}_j$ of length $|w| = i$,

$\mathcal{M}_j$ on input w will halt in less than a steps, in more than b steps, or not at all.

Lemma 13.7: Given $i, a, b \geq 0$ with $a \leq b$, it is decidable if $\mathrm{Gap}_i(a, b)$ holds.

Proof: We just need to ensure that none of the finitely many TMs $\mathcal{M}_0, \ldots, \mathcal{M}_i$ will halt after a to b steps on any of the finitely many inputs of length i. This can be checked by simulating TM runs for at most b steps. □

Find the Gap

We can now define the value $f(n)$ of f for some $n \geq 0$:

Let $\mathsf{in}(n)$ denote the number of runs of TMs $\mathcal{M}_0, \ldots, \mathcal{M}_n$ on words of length n, i.e.,

$$\mathsf{in}(n) = |\Sigma_0|^n + \cdots + |\Sigma_n|^n \quad \text{where } \Sigma_i \text{ is the input alphabet of } \mathcal{M}_i$$

We recursively define a series of numbers $k_0, k_1, k_2, \ldots$ by setting $k_0 = 2n$ and $k_{i+1} = 2^{k_i}$ for $i \geq 0$, and we consider the following list of intervals:

$$\begin{array}{ccccc} [k_0+1, k_1], & [k_1+1, k_2], & \cdots, & [k_{\mathsf{in}(n)}+1, k_{\mathsf{in}(n)+1}] \\ \| & \| & & \| \\ [2n+1, 2^{2n}], & [2^{2n}+1, 2^{2^{2n}}], & \cdots, & [2^{\cdot^{\cdot^{2n}}}+1, 2^{2^{\cdot^{\cdot^{2n}}}}] \end{array}$$

Let $f(n)$ be the least number k_i with $0 \leq i \leq \mathsf{in}(n)$ such that $\mathsf{Gap}_n(k_i + 1, k_{i+1})$ is true.

Properties of f

We first establish some basic properties of our definition of f:

Claim: The function f is well-defined.

Proof: For finding $f(n)$, we consider $\mathsf{in}(n) + 1$ intervals. Since there are only $\mathsf{in}(n)$ runs of TMs $\mathcal{M}_0, \ldots \mathcal{M}_n$, at least one interval remains a "gap" where no TM run halts. □

Claim: The function f is computable.

Proof: We can compute $\mathsf{in}(n)$ and k_i for any i, and we can decide $\mathsf{Gap}_n(k_i + 1, k_{i+1})$. □

Papadimitriou: "notice the fantastically fast growth, as well as the decidedly unnatural definition of this function."

Finishing the Proof

We can now complete the proof of the theorem:

Claim: $\mathsf{DTime}(f(n)) = \mathsf{DTime}(2^{f(n)})$.

Consider any $\mathbf{L} \in \mathsf{DTime}(2^{f(n)})$.
Then there is an $2^{f(n)}$-time bounded TM $\mathcal{M}_j$ with $\mathbf{L} = \mathbf{L}(\mathcal{M}_j)$.

For any input w with $|w| \geq j$:

- The definition of $f(|w|)$ took the run of $\mathcal{M}_j$ on w into account
- $\mathcal{M}_j$ on w halts after less than $f(|w|)$ steps, or not until after $2^{f(|w|)}$ steps (maybe never)
- Since $\mathcal{M}_j$ runs in time $\mathsf{DTime}(2^{f(n)})$, it must halt in $\mathsf{DTime}(f(n))$ on w

For the finitely many inputs w with $|w| < j$:

- We can augment the state space of $\mathcal{M}_j$ to run a finite automaton to decide these cases
- This will work in $\mathsf{DTime}(f(n))$

Therefore we have $\mathbf{L} \in \mathsf{DTime}(f(n))$. □

Discussion: The case $|w| < j$

Borodin says: It is meaningful to state complexity results if they hold for "almost every" input (i.e., for all but a finite number)

Papadimitriou says: These words can be handled since we can check the length and then recognise the word in less than $2j$ steps

Really?

- If we do these $< 2j$ steps before running $\mathcal{M}_j$, the modified TM runs in $\mathsf{DTime}(f(n) + 2j)$
- This does not show $\mathbf{L} \in \mathsf{DTime}(f(n))$

A more detailed argument:

- Make the intervals larger: $[k_i + 1, 2^{k_i+2n} + 2n]$, that is $k_{i+1} = 2^{k_i+2n} + 2n$.
- Select $f(n)$ to be $k_i + 2n + 1$ if the least gap starts at $k_i + 1$.

The same pigeon hole argument as before ensures that an empty interval is found.

But now the $f(n)$ time bounded machine $\mathcal{M}_j$ from the proof will be sure to stop after $f(n) - 2n - 1$ steps, so a shift of $2j \leq 2n$ to account for the finitely many cases will not make it use more than $f(n)$ steps either

Discussion: Generalising the Gap Theorem

- Our proof uses the function $n \mapsto 2^n$ to define intervals
- Any other computable function could be used without affecting the argument

This leads to a generalised Gap Theorem:

Gap Theorem 13.8: For every computable function $g : \mathbb{N} \to \mathbb{N}$ with $g(n) \geq n$, there is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that DTime$(f(n))$ = DTime$(g(f(n)))$.

Example 13.9: There is a function f such that

$$\mathsf{DTime}(f(n)) = \mathsf{DTime}\left(\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{f(n)\text{ times}}\right)$$

Moreover, the Gap Theorem can also be shown for space (and for other resources) in a similar fashion (space is a bit easier since the case of short words $|w| < j$ is easy to handle in very little space)

Discussion: Significance of the Gap Theorem

What have we learned?

- More time (or space) does not always increase computational power
- However, this only works for extremely fast-growing, very unnatural functions

"Fortunately, the gap phenomenon cannot happen for time bounds t that anyone would ever be interested in"[1]

Main insight: better stick to constructible functions

[1] Allender, Loui, Reagan: Complexity Theory. In Computing Handbook, 3rd ed., CRC Press, 2014

Summary and Outlook

Hierarchy theorems tell us that more time/space leads to more power:

$$\mathsf{L} \subseteq \mathsf{NL} \subseteq \mathsf{P} \subseteq \mathsf{NP} \subseteq \mathsf{PSpace} \subseteq \mathsf{ExpTime} \subseteq \mathsf{NExpTime} \subseteq \mathsf{ExpSpace}$$

However, they don't help us in comparing different resources and machine types (P vs. NP, or PSpace vs. ExpTime)

With non-constructible functions as time/space bounds, arbitrary (constructible or not) boosts in resources do not lead to more power

What's next?

- The inner structure of NP revisited
- Computing with oracles (reprise)
- The limits of diagonalisation, proved by diagonalisation