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Complexity Theory

Exercise 2: Time Complexity, PTime, and NP

Exercise 2.1. A language $\mathbf{L} \in \mathrm{P}$ is complete for P under polynomial-time reductions if $\mathbf{L}' \leq_p \mathbf{L}$ for every $\mathbf{L}' \in \mathrm{P}$. Show that every language in P except $\emptyset$ and Σ^* is complete for P under polynomial-time reductions.

Exercise 2.2. Show that P is closed under concatenation and star.

Exercise 2.3. An undirected graph $G = (V, E)$ is *connected* iff for every two nodes $x, y \in V$ there exist edges $e_0, e_1, \dots, e_n \in E$ with (1) $x \in e_0$, (2) $y \in e_n$, and (3) $e_{i-1} \cap e_i \neq \emptyset$ ($0 < i \leq n$).

$$\textbf{Connected} := \{\langle G\rangle \mid G \text{ is a connected undirected graph}\}$$

Show that **Connected** is in P.

Exercise 2.4. Let $G_i = (V_i, E_i)$ ($i = 1, 2$) be directed graphs, i. e., $E \subseteq V \times V$.

A bijective function $\iota : V_1 \to V_2$ is called a *graph isomorphism* between G_1 and G_2 if $(v_1, v_2) \in E_1$ iff $(\iota(v_1), \iota(v_2)) \in E_2$. G_1 and G_2 are said to be *isomorphic* iff there is a graph isomorphism between G_1 and G_2.

$$\textbf{Iso} := \{\langle G, H\rangle \mid G \text{ and } H \text{ are isomorphic}\}$$

A non-empty relation $R \subseteq V_1 \times V_2$ is called a *graph bisimulation* between G_1 and G_2 iff for every $(v_1, v_2) \in R$,

1. if $(v_1, w_1) \in E_1$, then there is a node $w_2 \in V_2$ with $(v_2, w_2) \in E_2$ and $(w_1, w_2) \in R$, and
2. if $(v_2, w_2) \in E_2$, then there is a node $w_1 \in V_1$ with $(v_1, w_1) \in E_1$ and $(w_1, w_2) \in R$.

G_1 and G_2 are said to be *bisimilar* iff there is a graph bisimulation between G_1 and G_2.

$$\textbf{Bisimilarity} := \{\langle G, H\rangle \mid G \text{ and } H \text{ are bisimilar}\}$$

Show that **Iso** and **Bisimilarity** are in NP.

Exercise 2.5. We recall some definitions.

- Given some language $\mathbf{L}$. $\mathbf{L} \in \mathrm{coNP}$ if and only if $\overline{\mathbf{L}} \in \mathrm{NP}$.
- $\mathbf{L}$ is coNP-hard if and only if $\mathbf{L}' \leq_p \mathbf{L}$ for every $\mathbf{L}' \in \mathrm{coNP}$.
- $\mathbf{L}$ is coNP-complete if and only if $\mathbf{L} \in \mathrm{coNP}$ and $\mathbf{L}$ is coNP-hard.

Show that if any coNP-complete problem is in NP, then $\mathrm{NP} = \mathrm{coNP}$.

Exercise 2.6. If G is an undirected graph, a *vertex cover* of G is a subset of the nodes where every edge of G touches one of those nodes. The vertex cover problem asks whether a graph contains a vertex cover of a specified size.

$$\textbf{Vertex-Cover} = \{\langle G, k\rangle \mid G \text{ is an undirected graph that has a } k\text{-node vertex cover}\}$$

Show that **Vertex-Cover** is NP-complete.

Hint:
Try to find a reduction from 3-Sat