Problem 2.1

Consider the set of clauses

\[ \mathcal{F} = \{ [p(f(Y)), q(Y), r(b)], \neg p(b), \neg q(a), \neg r(a) \} \]

and the equational system

\[ \mathcal{E} = \{ (\forall X) f(X) \approx X, a \approx b \} \]

Show by paramodulation, resolution and factoring that \( \mathcal{F} \cup \mathcal{E} \cup \mathcal{E}_\approx \) is unsatisfiable. Also give the mgu \( \theta \) used in every step.

Problem 2.2

Let \( \mathcal{R} \) be a term rewriting system and let \( s \) and \( t \) be terms. Prove that:

1. \( s \rightarrow_{\mathcal{R}} t \) implies \( s \approx_{\mathcal{E}_\mathcal{R}} t \).
2. \( s \leftrightarrow_{*_{\mathcal{R}}} t \) implies \( s \approx_{\mathcal{E}_\mathcal{R}} t \).

Problem 2.3

A non terminating term rewriting system can be confluent. True or false? Prove it.

Problem 2.4

Prove that a term rewriting system \( \mathcal{R} \) is Church-Rosser if and only if it is confluent.
Problem 2.5

Consider the following term rewriting system:

\[ f(f(X, Y), Z) \rightarrow f(X, f(Y, Z)); \]
\[ f(X, 1) \rightarrow X. \]

1. Is it terminating? Justify your answer.
2. Compute all the critical pairs, and show how you got them.
3. Can you orientate the critical pairs, i.e., add a rule \( s \rightarrow t \) or \( t \rightarrow s \) for each critical pair \( \langle s, t \rangle \), such that termination is preserved? (If it is possible, do it . . .)

Note: When executing the completion algorithm you have to go on trying to build critical pairs with the iteratively added rules.

Problem 2.6

Let \( \mathcal{R} \) be a term rewriting system and \( >/2 \) a termination ordering.
If for all rules \( l \rightarrow r \in \mathcal{R} \) the relation \( l > r \) holds, then \( \mathcal{R} \) is terminating.

Problem 2.7

Consider the term rewriting system

\[ \mathcal{R} = \{ f(g(X)) \rightarrow g(X), \quad (1) \]
\[ g(h(X)) \rightarrow g(X) \}\]
\[ (2) \]

Show that \( \mathcal{R} \) is canonical.