Human Reasoning – Weak Completion Semantics

Steffen Hölldobler
International Center for Computational Logic
Technische Universität Dresden
Germany

▶ Programs
▶ Weak Completion
▶ Three-Valued Logics
▶ Three-Valued Interpretations
▶ Model Intersection
▶ Computing Least Models
▶ Reasoning wrt Least Models
▶ Abduction
▶ Skeptical Reasoning

"Logic is everywhere …"
Literature

► Stenning, van Lambalgen: Human Reasoning and Cognitive Science
MIT Press: 2008

► H., Kencana Ramli: Logic Programs under Three-Valued Łukasiewicz’s Semantics.
LNCS 5649, 464-478: 2009

► Łukasiewicz: O logice trójwartościowej. Ruch Filozoficzny 5, 169-171: 1920
(On Three-Valued Logic. In: Jan Łukasiewicz Selected Works (Borkowski, ed.)
North Holland, 87-88: 1990)

► Kencana Ramli: Logic Programs and Three-Valued Consequence Operators
Motivation

- We would like to model sentences like
  - If she has an essay to write then she will study late in the library
  - She has an essay to write
  - She does not have an essay to write
  - Nothing abnormal is known

- We would like to adequately model the suppression task
  - Our system shall qualitatively produce the same answers as humans
Programs

► We consider propositional logic

► (Program) clauses are expressions of the forms

- \( A \leftarrow L_1 \land \ldots \land L_n \) (called rules)
- \( A \leftarrow \top \) (called facts)
- \( A \leftarrow \bot \) (called assumptions)

where \( n \geq 1 \), \( A \) is an atom, and each \( L_i \), \( 1 \leq i \leq n \), is a literal

- \( A \) is called head
- \( L_1 \land \ldots \land L_n \) as well as \( \top \) and \( \bot \) are called bodies

of the corresponding clauses

► A (logic) program is a finite set of clauses

► Notation In the sequel, \( \mathcal{P} \) denotes a program
Let $P$ consist of the following clauses:

\[
\begin{align*}
\ell & \leftarrow e \land \neg ab_1 \\
\ell & \leftarrow t \land \neg ab_2 \\
e & \leftarrow \top \\
ab_1 & \leftarrow \bot \\
ab_2 & \leftarrow \bot
\end{align*}
\]
Defined Atoms and Assumed Negations

- Let $\mathcal{P}$ be a program and $A$ an atom
- $A$ is defined in $\mathcal{P}$ iff $\mathcal{P}$ contains a rule or a fact with head $A$
- $A$ is undefined in $\mathcal{P}$ iff $A$ is not defined in $\mathcal{P}$
- The definition of $A$ in $\mathcal{P}$ ($\text{def}(A, \mathcal{P})$) is defined as follows
  $$\text{def}(A, \mathcal{P}) = \{ A \leftarrow \text{Body} \mid A \leftarrow \text{Body} \text{ is a rule or a fact occurring in } \mathcal{P} \}$$
- $\neg A$ is assumed in $\mathcal{P}$ iff $\mathcal{P}$ contains an assumption with head $A$ and $A$ is undefined in $\mathcal{P}$
Defined Atoms and Assumed Negations – Examples

Let $\mathcal{P} = \{\ell \leftarrow e \land \neg ab_1, \ell \leftarrow t \land \neg ab_2, e \leftarrow \top, ab_1 \leftarrow \bot, ab_2 \leftarrow \bot\}$

- $\ell$ and $e$ are defined
- $\text{def}(\ell, \mathcal{P}) = \{\ell \leftarrow e \land \neg ab_1, \ell \leftarrow t \land \neg ab_2\}$
- $\text{def}(e, \mathcal{P}) = \{e \leftarrow \top\}$
- $t$, $ab_1$ and $ab_2$ are undefined
- $\neg ab_1$ and $\neg ab_2$ are assumed

Let $\mathcal{P} = \{a \leftarrow \top, a \leftarrow d, b \leftarrow \top, b \leftarrow \bot, c \leftarrow \bot\}$

- $a$ and $b$ are defined
- $\text{def}(a, \mathcal{P}) = \{a \leftarrow \top, a \leftarrow d\}$
- $\text{def}(b, \mathcal{P}) = \{b \leftarrow \top\}$
- $d$ and $c$ are undefined
- $\neg c$ is assumed
Program Completion


Let $\mathcal{P}$ be a program. Consider the following transformation

1. Replace all clauses with the same head $A \leftarrow Body_1, A \leftarrow Body_2, \ldots$ by $A \leftarrow Body_1 \lor Body_2 \lor \ldots$
2. If an atom $A$ is not the head of any clause in $\mathcal{P}$ then add $A \leftarrow \bot$
3. Replace all occurrences of $\leftarrow$ by $\leftrightarrow$

The resulting set is called completion of $\mathcal{P}$ or $c\mathcal{P}$

If step 2 is omitted then the resulting set is called weak completion of $\mathcal{P}$ or $wc\mathcal{P}$
Program Completion – Example

Let $P$ consist of the following clauses

$$
\ell \leftarrow e \land \neg ab_1 \\
\ell \leftarrow t \land \neg ab_2 \\
e \leftarrow \top \\
ab_1 \leftarrow \bot \\
ab_2 \leftarrow \bot
$$

The completion of $P$ consists of the following formulas

$$
\ell \leftrightarrow (e \land \neg ab_1) \lor (t \land \neg ab_2) \\
e \leftrightarrow \top \\
ab_1 \leftrightarrow \bot \\
ab_2 \leftrightarrow \bot \\
t \leftrightarrow \bot
$$

The weak completion of $P$ is obtained by deleting the last formula.
Three-Valued Logics

- **Truth tables**

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<tr>
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- **Łukasiewicz (Ł) logic** \{¬, ∧, ∨, ←Ł, ↔Ł\}  
- **We will omit the Ł if it can be determined from the context**
Semantic Equivalence and Some Laws

- Two formulas $F$ and $G$ are **semantically equivalent** ($F \equiv G$) iff for all interpretations $I$ we find $I(F) = I(G)$

<table>
<thead>
<tr>
<th>Laws</th>
<th>$\mathcal{L}$</th>
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<tbody>
<tr>
<td>Double Neg.</td>
<td>$\neg\neg F \equiv F$</td>
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<tr>
<td>Idempotency</td>
<td>$F \land F \equiv F$</td>
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<tr>
<td>Commutativity</td>
<td>$F \land G \equiv G \land F$</td>
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<tr>
<td>Absorption</td>
<td>$(F \land G) \lor F \equiv F$</td>
</tr>
<tr>
<td>de Morgan</td>
<td>$\neg(F \land G) \equiv \neg F \lor \neg G$</td>
</tr>
<tr>
<td>Associativity</td>
<td>$(F \lor G) \land H \equiv F \land (G \land H)$</td>
</tr>
<tr>
<td>Distributivity</td>
<td>$F \land (G \lor H) \equiv (F \land G) \lor (F \land H)$</td>
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<td></td>
<td>$F \lor (G \land H) \equiv (F \lor G) \land (F \lor H)$</td>
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<tr>
<td></td>
<td>$(F \lor G) \lor H \equiv F \lor (G \lor H)$</td>
</tr>
<tr>
<td>Equivalence</td>
<td>$F \leftrightarrow G \equiv (F \leftrightarrow G) \land (G \leftrightarrow F)$</td>
</tr>
<tr>
<td>Implication</td>
<td>$F \rightarrow G \equiv \neg F \lor G$</td>
</tr>
<tr>
<td>Contraposition</td>
<td>$G \rightarrow F \equiv \neg F \rightarrow \neg G$</td>
</tr>
<tr>
<td>Syllogism</td>
<td>$(F \rightarrow G) \land (G \rightarrow H) \equiv F \rightarrow H$</td>
</tr>
<tr>
<td>Excluded Middle</td>
<td>$F \lor \neg F \equiv \top$</td>
</tr>
<tr>
<td>Contradiction</td>
<td>$F \land \neg F \equiv \bot$</td>
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</tbody>
</table>
Facts versus Assumptions

Consider

\[ P = \{a \leftarrow \top, \ a \leftarrow d, \ b \leftarrow \top, \ b \leftarrow \bot, \ c \leftarrow \bot\} \]

Then

\[ wcP = \{a \leftrightarrow \top \lor d, \ b \leftrightarrow \top \lor \bot, \ c \leftrightarrow \bot\} \]

\[ \equiv \{a \leftrightarrow \top, \ b \leftrightarrow \top, \ c \leftrightarrow \bot\} \]

If \( A \leftarrow \top \in P \), then \( A \leftrightarrow \top \in wcP \)

If \( \neg A \) is assumed in \( P \), then \( A \leftrightarrow \bot \in wcP \)

Assumptions may be overridden
Three-Valued Interpretations

- A (three-valued) interpretation assigns to each formula a value from \{\top, \bot, U\}
- It is represented by \langle I^\top, I^\bot \rangle, where
  - \( I^\top \) contains all ground atoms which are mapped to \( \top \)
  - \( I^\bot \) contains all ground atoms which are mapped to \( \bot \)
  - \( I^\top \cap I^\bot = \emptyset \)
  - All ground atoms which occur neither in \( I^\top \) nor \( I^\bot \) are mapped to \( U \)
- **Notation** In the sequel, \( I, J \) denote interpretations \langle I^\top, I^\bot \rangle, \langle J^\top, J^\bot \rangle, resp.
- The intersection \( I \cap J \) is defined as \langle I^\top \cap J^\top, I^\bot \cap J^\bot \rangle
Three-Valued Interpretations – Examples

Consider

\[ \begin{aligned}
\mathcal{P} & \quad wc\mathcal{P} & \quad c\mathcal{P} \\
\ell & \leftarrow e \land \neg ab_1 & \ell & \leftrightarrow (e \land \neg ab_1) \lor (t \land \neg ab_2) & \ell & \leftrightarrow (e \land \neg ab_1) \lor (t \land \neg ab_2) \\
\ell & \leftarrow t \land \neg ab_2 & e & \leftrightarrow \top & e & \leftrightarrow \top \\
e & \leftarrow \top & ab_1 & \leftrightarrow \bot & ab_1 & \leftrightarrow \bot \\
ab_1 & \leftarrow \bot & ab_2 & \leftrightarrow \bot & ab_2 & \leftrightarrow \bot \\
ab_2 & \leftarrow \bot & t & \leftrightarrow \bot & t & \leftrightarrow \bot 
\end{aligned} \]

Then

\[ \begin{aligned}
I & \quad I(\mathcal{P}) \quad I(wc\mathcal{P}) \quad I(c\mathcal{P}) \\
\langle \{e, ab_1\}, \emptyset \rangle & \quad T \quad \bot \quad \bot \\
\langle \{e, \ell\}, \{ab_1, ab_2\}\rangle & \quad T \quad T \quad U \\
\langle \{e, \ell, t\}, \{ab_1, ab_2\}\rangle & \quad T \quad T \quad \bot \\
\langle \{e, \ell\}, \{ab_1, ab_2, t\}\rangle & \quad T \quad T \quad T
\end{aligned} \]
An interpretation $I$ is a (three-valued) model for a program $P$ ($I \models P$) iff $I(P) = \top$

Let $P = \{p \leftarrow q\}$

$\langle \emptyset, \emptyset \rangle \models P$

$\langle \{p, q\}, \emptyset \rangle \models P$

$\langle \emptyset, \{p, q\} \rangle \models P$
Programs under Łukasiewicz Logic

- **Theorem**  The model intersection property holds for \( \mathcal{P} \)
i.e., \( \cap \{ I \mid I \models \mathcal{P} \} \models \mathcal{P} \)

- **Example**  Consider \( \mathcal{P} = \{ p \leftarrow q \} \)
  - The least model of \( \mathcal{P} \) under Łukasiewicz logic is \( \langle \emptyset, \emptyset \rangle \)

- **Theorem**  The model intersection property holds for \( wc\mathcal{P} \) as well

- **Notation**  \( \mathcal{M}_\mathcal{P} \) denotes the least model of \( wc\mathcal{P} \)

- **\( \mathcal{P} \) entails \( F \) under the weak completion semantics** \( (\mathcal{P} \models_{wcs} F) \)
  - iff \( \mathcal{M}_\mathcal{P}(F) = \top \)
Monotonicity

- Let \( \mathcal{P} \) and \( \mathcal{P}' \) be sets of formulas and \( G \) a formula
  
  A logic is **monotonic** if the following holds:
  
  If \( \mathcal{P} \models G \) then \( \mathcal{P} \cup \mathcal{P}' \models G \)

- Classical logic is monotonic

- A logic based on the weak completion semantics is non-monotonic

  ▶ Consider

  \[
  \begin{align*}
  \mathcal{P} &= \{ c \leftarrow \bot \} \\
  \mathcal{P}' &= \mathcal{P} \cup \{ c \leftarrow \top \}
  \end{align*}
  \]

  ▶ Then

  \[
  \begin{align*}
  wc\mathcal{P} &= \{ c \leftrightarrow \bot \} \models \neg c \\
  wc\mathcal{P}' &= \{ c \leftrightarrow \bot \lor \top \} \not\models \neg c \models c
  \end{align*}
  \]
Computing the Least Models of Weakly Completed Programs

Stenning, van Lambalgen 2008

Consider the following immediate consequence operator

\[ \Phi_P(I) = \langle J^\top, J^\perp \rangle \]

where

\[ J^\top = \{ A \mid \text{there exists } A \leftarrow Body \in P \text{ with } I(Body) = \top \} \]

\[ J^\perp = \{ A \mid \text{there exists } A \leftarrow Body \in P, \text{ for all } A \leftarrow Body \in P \text{ we find } I(Body) = \perp \} \]
Properties of the Stenning and van Lambalgen Operator

- **Proposition** For each program \( P \) the mapping \( \Phi_P \) is monotonic.

- **Proposition** For each program \( P \), \( M_P \) is the least fixed point of \( \Phi_P \).

- **Proposition** For each propositional program \( P \), the mapping \( \Phi_P \) is continuous.

- **Proposition** For each propositional program \( P \), the least fixed point of \( \Phi_P \) is \( \text{lub}(\{\Phi^n_P(\langle\emptyset, \emptyset\rangle) \mid n \in \mathbb{N}\}) \).

- Whereas the first two propositions hold also for first-order programs, the final two propositions do not hold for first-order programs.
Integrity Constraints

- An integrity constraint is an expression of the form $U \leftarrow L_1 \land \ldots \land L_n$, where each $L_i$, $1 \leq i \leq n$, is a literal.
- In the sequel, $\mathcal{IC}$ denotes a finite set of integrity constraints.
- Let $I$ be an interpretation.
  - $I$ satisfies $\mathcal{IC}$ iff for each $U \leftarrow Body \in \mathcal{IC}$ we find $I(Body) \in \{U, \bot\}$. 

Steffen Hölldobler
Human Reasoning – Weak Completion Semantics 20
Abductive Frameworks

Let $\mathcal{P}$ be a program

The set of abducibles of $\mathcal{P}$ is

$$A_{\mathcal{P}} = \{ A \leftarrow \top \mid A \text{ is undefined in } \mathcal{P} \} \cup \{ A \leftarrow \bot \mid A \text{ is undefined in } \mathcal{P} \} \cup \{ A \leftarrow \top \mid \neg A \text{ is assumed in } \mathcal{P} \}$$

undefined ground atoms

defeaters of assumptions

An abductive framework $\langle \mathcal{P}, A, IC, \models_{wcs} \rangle$ consists of

- a program $\mathcal{P}$
- a set of abducibles $A \subseteq A_{\mathcal{P}}$
- a set $IC$ of integrity constraints
- the entailment relation $\models_{wcs}$
Abductive Frameworks – Example

Suppose \( \mathcal{P} \) contains the following clauses

\[
\begin{align*}
\ell & \leftarrow e \land \neg ab_1 \\
ab_1 & \leftarrow \bot \\
\ell & \leftarrow t \land \neg ab_2 \\
ab_2 & \leftarrow \bot
\end{align*}
\]

Let \( \mathcal{A} = \{e \leftarrow \top, e \leftarrow \bot, t \leftarrow \top, t \leftarrow \bot\} \) and \( \mathcal{IC} = \emptyset \)
Observations and Explanations

- An observation $\mathcal{O}$ is a set of ground literals.

- $\mathcal{O}$ is explainable in the abductive framework $\langle \mathcal{P}, \mathcal{A}, \mathcal{IC}, \models_{wcs} \rangle$ iff there exists $\mathcal{E} \subseteq \mathcal{A}$ called explanation such that
  - $\mathcal{M}_{\mathcal{P} \cup \mathcal{E}} \models_{wcs} L$ for all $L \in \mathcal{O}$
  - $\mathcal{M}_{\mathcal{P} \cup \mathcal{E}}$ satisfies $\mathcal{IC}$

- Sometimes explanations are required to be minimal
  - An explanation is minimal if it cannot be subsumed by another explanation.

- Is $\mathcal{P} \cup \mathcal{E}$ satisfiable?
Suppose $\mathcal{P}$ contains the following clauses

$$
\begin{align*}
\ell & \leftarrow e \land \neg ab_1 \\
ab_1 & \leftarrow \bot \\
\ell & \leftarrow t \land \neg ab_2 \\
ab_2 & \leftarrow \bot 
\end{align*}
$$

Let $\mathcal{A} = \{e \leftarrow \top, \ e \leftarrow \bot, \ t \leftarrow \top, \ t \leftarrow \bot\}$ and $\mathcal{IC} = \emptyset$

Let $\mathcal{O} = \{\ell\}$

Are there any (minimal) explanations?

$\triangleright \mathcal{E}_1 = \{e \leftarrow \top\}$

$\triangleright \mathcal{E}_2 = \{t \leftarrow \top\}$
Skeptical and Credulous Consequences

- Let $\langle \mathcal{P}, \mathcal{A}, \mathcal{IC}, \models_{wcs} \rangle$ be an abductive framework, $\mathcal{O}$ an observation, and $F$ a formula.

- $F$ follows credulously from $\mathcal{P}$ and $\mathcal{O}$ iff there exists an explanation $\mathcal{E}$ for $\mathcal{O}$ such that $\mathcal{P} \cup \mathcal{E} \models_{wcs} F$.

- $F$ follows skeptically from $\mathcal{P}$ and $\mathcal{O}$ iff for all explanations $\mathcal{E}$ for $\mathcal{O}$ we find $\mathcal{P} \cup \mathcal{E} \models_{wcs} F$. 

Skeptical and Credulous Consequences

- Suppose $P$ contains the following clauses:

$$
\begin{align*}
\ell & \leftarrow e \land \neg ab_1 \\
ab_1 & \leftarrow \bot \\
\ell & \leftarrow t \land \neg ab_2 \\
ab_2 & \leftarrow \bot
\end{align*}
$$

- Let $A = \{ e \leftarrow \top, \ e \leftarrow \bot, \ t \leftarrow \top, \ t \leftarrow \bot \}$ and $IC = \emptyset$

- Let $O = \{ \ell \}$

- The minimal explanations are $E_1 = \{ e \leftarrow \top \}$ and $E_2 = \{ t \leftarrow \top \}$

  - $e$ follows credulously
  - $e$ does not follow skeptically

- Byrne 1989 Only 13% conclude that she has an essay to write