Chapter 4

Declarative Interpretation
Outline

- Algebras (which provide a semantics of terms)
- Interpretations (which provide a semantics of programs)
- Soundness of SLD-resolution
- Completeness of SLD-resolution
- Least Herbrand models
- Computing least Herbrand models
What is an Interpretation?

direct(frankfurt, san_francisco).
direct(frankfurt, chicago).
direct(san_francisco, honolulu).
direct(honolulu, maui).

collection(X, Y) :- direct(X, Y).
collection(X, Y) :- direct(X, Z), collection(Z, Y).

\[ D = \{FRA, DRS, ORD, SFO, \ldots\} \]
\[ \text{frankfurt}_j = FRA, \text{chicago}_j = ORD, \text{san-francisco}_j = SFO, \ldots \]
\[ \text{direct}_i = \{(FRA, SFO), (FRA, ORD), \ldots\} \]
\[ \text{collection}_i = \{(FRA, SFO), (FRA, ORD), (FRA, HNL), \ldots\} \]
What is an Interpretation?

\[
\begin{align*}
\text{add} (X, 0, X) . \\
\text{add} (X, s(Y), s(Z)) & : \text{add} (X, Y, Z) . \\
D & = \mathbb{N} \\
0_J & = 0 \\
s_J : \mathbb{N} & \to \mathbb{N} \text{ such that } s_J(n) = n + 1 \\
\text{add}_J & = \{(0, 0, 0), (1, 0, 1), (0, 1, 1), (1, 1, 2), \ldots\}
\end{align*}
\]
Another Example

\[
\begin{align*}
\text{add}(X, 0, X). \\
\text{add}(X, \text{s}(Y), \text{s}(Z)) & : \text{add}(X, Y, Z).
\end{align*}
\]

\[D = \{0, \text{s}(0), \text{s}(\text{s}(0)), \ldots\}\]

\[0_J = 0\]

\[s_J : D \rightarrow D \text{ such that } s_J(t) = \text{s}(t)\]

\[\text{add}_J = \{(0, 0, 0), (\text{s}(0), 0, \text{s}(0)), (0, \text{s}(0), \text{s}(0)), (\text{s}(0), \text{s}(0), \text{s}(\text{s}(0))), \ldots\}\]

(This will be called a “Herbrand model”.)
Algebras

$V$ set of variables, $F$ ranked alphabet of function symbols: An algebra $J$ for $F$ (or pre-interpretation for $F$) consists of:

1. domain $\iff$ non-empty set $D$
2. assignment of a mapping $f_J : D^n \rightarrow D$

$\text{to every } f \in F^{(n)} \text{ with } n \geq 0$

State $\sigma$ over $D$ $\iff$ mapping $\sigma : V \rightarrow D$

Extension of $\sigma$ to $TU_{F,V}$ $\iff$ $\sigma : TU_{F,V} \rightarrow D$ such that for every $f \in F^{(n)}$

$\sigma(f(t_1, ..., t_n)) = f_J(\sigma(t_1), ..., \sigma(t_n))$
Interpretations

$F$ ranked alphabet of function symbols, $\Pi$ ranked alphabet of predicate symbols:

An interpretation $I$ for $F$ and $\Pi$ consists of:

1. algebra $J$ for $F$ (with domain $D$)
2. assignment of a relation $p_I \subseteq \underbrace{D \times \ldots \times D}_n$

... to every $p \in \Pi^{(n)}$ with $n \geq 0$
Herbrand Universes and Bases

Recall $TU_{F,V} :\iff$ term universe over function symbols $F$, variables $V$

$TB_{\Pi,F,V} :\iff$ term base (i.e., all atoms) over predicate symbols $\Pi$ and $F$, $V$

- Herbrand universe $HU_F :\iff TU_{F,\emptyset}$
- Herbrand base $HB_{\Pi,F} :\iff TB_{\Pi,F,\emptyset}$
Interpretations (Example)

Let $P_{\text{add}}$ “add-program”.

$I_1, I_2, I_3, I_4, I_5,$ and $I_6$ are interpretations for \{s, 0\} and \{\text{add}\}:

$I_1$: $D_{I_1} = \mathbb{N}, \ 0_{I_1} = 0, \ s_{I_1}(n) = n + 1$ for each $n \in \mathbb{N}, \ add_{I_1} = \{(m, n, m + n) \mid m, n \in \mathbb{N}\}$

$I_2$: $D_{I_2} = \mathbb{N}, \ 0_{I_2} = 0, \ s_{I_2}(n) = n + 1$ for each $n \in \mathbb{N}, \ add_{I_2} = \{(m, n, m \ast n) \mid m, n \in \mathbb{N}\}$

$I_3$: $D_{I_3} = \text{HU}_\{s, 0\}, \ 0_{I_3} = 0, \ s_{I_3}(t) = s(t)$ for each $t \in \text{HU}_\{s, 0\}$,

\[ add_{I_3} = \{(s^m(0), s^n(0), s^{m+n}(0)) \mid m, n \in \mathbb{N}\} \]

$I_4$: $D_{I_4} = \text{HU}_\{s, 0\}, \ 0_{I_4} = 0, \ s_{I_4}(t) = s(t)$ for each $t \in \text{HU}_\{s, 0\}, \ add_{I_4} = \emptyset$

$I_5$: $D_{I_5} = \text{HU}_\{s, 0\}, \ 0_{I_5} = 0, \ s_{I_5}(t) = s(t)$ for each $t \in \text{HU}_\{s, 0\}, \ add_{I_5} = (\text{HU}_\{s, 0\})^3$

$I_6$: $D_{I_6} = \{0, 1\}, \ 0_{I_6} = 0, \ s_{I_6}(n) = n$ for each $n \in \{0, 1\}, \ add_{I_6} = \{(m, n, m) \mid m, n \in \{0, 1\}\}$
Logical Truth (I)

$E$ expression $\iff E$ atom, query, clause, or resultant

$E$ expression, $I$ interpretation, $\sigma$ state:

$E$ true in $I$ under $\sigma$, written: $I \models_{\sigma} E$

$\iff$

by case analysis on $E$:

- $I \models_{\sigma} p(t_1, \ldots, t_n) \iff (\sigma(t_1), \ldots, \sigma(t_n)) \in p_I$
- $I \models_{\sigma} A_1, \ldots, A_n \iff I \models_{\sigma} A_i$ for every $i = 1, \ldots, n$
- $I \models_{\sigma} A \leftarrow B \iff \text{if } I \models_{\sigma} B \text{ then } I \models_{\sigma} A$
- $I \models_{\sigma} A \leftarrow B \iff \text{if } I \models_{\sigma} B \text{ then } I \models_{\sigma} A$
Logical Truth (II)

$E$ expression, $I$ interpretation:
Let $x_1, ..., x_k$ be the variables occurring in $E$.

- $\forall x_1, ..., \forall x_k E$ universal closure of $E$ (abbreviated $\forall E$)
- $\exists x_1, ..., \exists x_k E$ existential closure of $E$ (abbreviated $\exists E$)
- $I \models \forall E :\iff I \models_\sigma E$ for every state $\sigma$
- $I \models \exists E :\iff I \models_\sigma E$ for some state $\sigma$
- $E$ true in $I$ (or: $I$ model of $E$), written: $I \models E :\iff I \models \forall E$
Logical Truth (III)

$S, T$ sets of expressions, $I$ interpretation:
- $I$ model of $S$, written: $I \models S :\iff I \models E$ for every $E \in S$
- $T$ semantic (or: logical) consequence of $S$, written $S \models T$ :
  $:\iff$ every model of $S$ is a model of $T$

$P$ program, $Q_0$ query, $\theta$ substitution:
- $\theta \mid_{\text{Var}(Q_0)}$ correct answer substitution of $Q_0$ :
  $\iff P \models Q_0\theta$
- $Q_0\theta$ correct instance of $Q_0$ :
  $\iff P \models Q_0\theta$
Models (Example)

Let $P_{\text{add}}$ “add-program” and let $I_1, I_2, I_3, I_4, I_5, \text{ and } I_6$ be the interpretations from slide 8.

- $I_1 \models P_{\text{add}}$ (since $I_1 \models_\sigma c$ for every clause $c \in P_{\text{add}}$ and state $\sigma: V \rightarrow \mathbb{N}$:
  (i) $(\sigma(x), \sigma(0), \sigma(x)) \in \text{add}_{I_1}$ and
  (ii) if $(\sigma(x), \sigma(y), \sigma(z)) \in \text{add}_{I_1}$ then $(\sigma(x), \sigma(y)+1, \sigma(z)+1) \in \text{add}_{I_1}$

- $I_2 \not\models P_{\text{add}}$ (e.g. let $\sigma(x) = 1$, then $I_2 \not\models_\sigma \text{add}(x, 0, x)$
  since $(\sigma(x), \sigma(0), \sigma(x)) = (1, 0, 1) \not\in \text{add}_{I_2}$

- $I_3 \models P_{\text{add}}$ (like for $I_1$; we call $I_3$ a (least) Herbrand model)

- $I_4 \not\models P_{\text{add}}$ (e.g. let $\sigma(x) = s(0)$, then $I_4 \not\models_\sigma \text{add}(x, 0, x)$
  since $(\sigma(x), \sigma(0), \sigma(x)) = (s(0), 0, s(0)) \not\in \text{add}_{I_4}$

- $I_5 \models P_{\text{add}}$ (like for $I_1$; we call $I_5$ a Herbrand model)

- $I_6 \models P_{\text{add}}$ (like for $I_1$)
Semantic Consequences (Example)

Let $P_{\text{add}}$ “add-program”.

- $P_{\text{add}} \models add(x, 0, x)$
  (for every interpretation $I$: if $I \models P_{\text{add}}$ then $I \models add(x, 0, x)$, since $add(x, 0, x) \in P_{\text{add}}$)

- $P_{\text{add}} \models add(x, s(0), s(x))$
  (for every interpretation $I$: if $I \models P_{\text{add}}$ then $I \models add(x, 0, x)$
  and $I \models add(x, s(0), s(x)) \leftarrow add(x, 0, x)$ (instance of clause), thus $I \models add(x, s(0), s(x))$)

- $P_{\text{add}} \not\models add(0, x, x)$
  (consider interpretation $I_6$ from slide 8 with $I_6 \models P_{\text{add}}$:
  $I_6 \not\models add(0, x, x)$, since e.g. $I_6 \not\models_{\sigma} add(0, x, x)$ for $\sigma(x) = 1$,
  since $(\sigma(0), \sigma(x), \sigma(x)) = (0, 1, 1) \not\in add_{I_6}$)
Towards Soundness of SLD-Resolution (I)

Lemma 4.3 (i)

Let \( Q \xrightarrow{c} Q' \) be an SLD-derivation step and \( Q\theta \leftarrow Q' \) the resultant associated with it. Then

\[ c \models Q\theta \leftarrow Q' \]

Proof.

Let \( Q = A, B, C \) with selected atom \( B \). Let \( H \leftarrow B \) be the input clause and \( Q' = (A, B, C)\theta \). Then

\[ c \models H \leftarrow B \]

(variant of \( c \))

implies \( c \models H\theta \leftarrow B\theta \) (instance)

implies \( c \models B\theta \leftarrow B\theta \) (\( \theta \) unifier)

implies \( c \models (A, B, C)\theta \leftarrow (A, B, C)\theta \) (“context” unchanged)
Towards Soundness of SLD-Resolution (II)

Lemma 4.3 (ii)

Let $\xi$ be an SLD-derivation of $P \cup \{Q_0\}$. For $i \geq 0$ let $R_i$ be the resultant of level $i$ of $\xi$. Then $P \models R_i$

Proof.

Let $\xi = Q_0 \Rightarrow Q_1 \ldots Q_n \Rightarrow Q_{n+1} \ldots$ Induction on $i \geq 0$:

- $i = 0$: $R_0 = Q_0 \leftarrow Q_0 = "true", \text{ thus } P \models R_0$
- $i = 1$: $R_1 = Q_0 \theta_1 \leftarrow Q_1; \text{ by Lemma 4.3 (i): } P \models R_1$
- $i \geq i + 1$: $R_{i+1} = Q_0 \theta_1 \ldots \theta_i \leftarrow Q_{i+1}$ is a semantic consequence of resultant $Q_i \theta_{i+1} \leftarrow Q_{i+1}$ associated with $(i + 1)$-st derivation step and $R_i \theta_{i+1} = Q_0 \theta_1 \ldots \theta_i \leftarrow Q_i \theta_{i+1}$, thus by Lemma 4.3 (i) and induction hypothesis: $P \models R_{i+1}$
Theorem 4.4

If there exists a successful SLD-derivation of $P \cup \{Q_0\}$ with $\text{CAS } \theta$, then $P \models Q_0 \theta$.

Proof.
Let $\xi = Q_0 \implies \ldots \implies \Box$ be successful SLD-derivation.
Lemma 4.3 (ii) applied to the resultant of level $n$ of $\xi$ implies $P \models Q_0 \theta_1 \ldots \theta_n$ and $Q_0 \theta_1 \ldots \theta_n = Q_0(\theta_1 \ldots \theta_n|_{\text{Var}(Q_0)}) = Q_0 \theta$. 
Corollary 4.5

If there exists a successful SLD-derivation of $P \cup \{Q_0\}$, then $P \models \exists Q_0$.

Proof.
Theorem 4.4 implies $P \models Q_0 \theta$ for some $\text{CAS } \theta$.
Then, $P \models Q_0 \theta$
implies for every interpretation $I$: if $I \models P$, then $I \models Q_0 \theta$
implies for every interpretation $I$: if $I \models P$, then $I \models \forall (Q_0 \theta)$
implies for every interpretation $I$: if $I \models P$, then $I \models \exists Q_0$
implies $P \models \exists Q_0$
Towards Completeness of SLD-Resolution

To show completeness of SLD-resolution we need to syntactically characterize the set of semantically derivable queries.

The concepts of term models and implication trees serve this purpose.
Term Models

$V$ set of variables, $F$ function symbols, $\Pi$ predicate symbols:

The term algebra $J$ for $F$ is defined as follows:
1. domain $D = TU_{F,V}$
2. mapping $f_J : (TU_{F,V})^n \to TU_{F,V}$ assigned to every $f \in F^n$ with
   $f_J(t_1, \ldots, t_n) \iff f(t_1, \ldots, t_n)$

A term interpretation $I$ for $F$ and $\Pi$ consists of:
1. term algebra for $F$
2. $I \subseteq TB_{\Pi,F,V}$ (set of atoms that are true; equivalent: assignment of a relation $p_I \subseteq (TU_{F,V})^n$
   to every $p \in \Pi^n$)

$I$ term model of a set $S$ of expressions $:\iff I$ term interpretation and model of $S$
Herbrand Models

The Herbrand algebra $J$ for $F$ is defined as follows:

1. domain $D = HU_F$
2. mapping $f_J : (HU_F)^n \rightarrow HU_F$ assigned to every $f \in F^n$ with $f_J(t_1, ..., t_n) \leftrightarrow f(t_1, ..., t_n)$

A Herbrand interpretation $I$ for $F$ and $\Pi$ consists of:

1. Herbrand algebra for $F$
2. $I \subseteq HB_{\Pi,F}$ (set of ground atoms that are true)

$I$ Herbrand model of a set $S$ of expressions $\iff I$ Herbrand interpretation and model of $S$

$I$ least Herbrand model of a set $S$ of expressions $\iff I$ Herbrand model of $S$ and $I \subseteq I'$ for all Herbrand models $I'$ of $S$
Implication Trees

implication tree w.r.t. program $P$

: ⇔

• finite tree whose nodes are atoms

• if $A$ is a node with the direct descendants $B_1, ..., B_n$ then $A \leftarrow B_1, ..., B_n \in \text{inst}(P)$

• if $A$ is a leaf, then $A \leftarrow \in \text{inst}(P)$

$E$ expression, $S$ set of expressions:

• $\text{inst}(E) : \leftrightarrow \text{set of all instances of } E$

• $\text{inst}(S) : \leftrightarrow \text{set of all instances of Elements } E \in S$

• $\text{ground}(E) : \leftrightarrow \text{set of all ground instances of } E$

• $\text{ground}(S) : \leftrightarrow \text{set of all ground instances of Elements } E \in S$
Implication Trees (Example)

Let $P_{\text{add}}$ "add-program", $n \in \mathbb{N}$, $V$ set of variables, $t \in TU_{\{s,0\}, V}$, and

$$T = \begin{array}{c}
add(t, s^n(0), s^n(t)) \\
| \\
add(t, s^{n-1}(0), s^{n-1}(t)) \\
| \\
\vdots \\
| \\
add(t, s(0), s(t)) \\
| \\
add(t, 0, t)
\end{array}$$

If $t \in HU_{\{s,0\}}$, then $T$ is ground implication tree w.r.t. $P_{\text{add}}$. 

Lemma 4.7
Consider term interpretation $I$, atom $A$, program $P$

- $I \models A$ iff $\text{inst}(A) \subseteq I$
- $I \models P$ iff for every $A \leftarrow B_1, \ldots, B_n \in \text{inst}(P)$: if $\{B_1, \ldots, B_n\} \subseteq I$ then $A \in I$

Lemma 4.12
The term interpretation $C(P) \iff \{A \mid A$ is the root of some implication tree w.r.t. $P\}$ is a model of $P$. 
Lemma 4.26
Consider Herbrand interpretation $I$, atom $A$, program $P$

- $I \models A$ iff $\text{ground}(A) \subseteq I$
- $I \models P$ iff for every $A \leftarrow B_1, \ldots, B_n \in \text{ground}(P)$, $\{B_1, \ldots, B_n\} \subseteq I$ implies $A \in I$

Lemma 4.28
The Herbrand interpretation $M(P) : \leftrightarrow \{A \mid A \text{ is the root of some ground implication tree w.r.t. } P\}$ is a model of $P$. 
Example

Let $P_{\text{add}}$ “add-program”, and $V$ set of variables.

The term interpretation

\[
C(P_{\text{add}}) = \{\text{add}(t, s^n(0), s^n(t)) \mid n \in \mathbb{N}, t \in TU_{\{s,0\},V}\}
\]

\[
= \{\text{add}(s^m(v), s^n(0), s^{n+m}(v)) \mid m, n \in \mathbb{N}, v \in V \cup \{0\}\}
\]

and the Herbrand interpretation

\[
M(P_{\text{add}}) = \{\text{add}(t, s^n(0), s^n(t)) \mid n \in \mathbb{N}, t \in HU_{\{s,0\}}\}
\]

\[
= \{\text{add}(s^m(0), s^n(0), s^{n+m}(0)) \mid m, n \in \mathbb{N}\}
\]

are models of $P_{\text{add}}$. 
Correct Answer Substitutions versus Computed Answer Substitutions (Example)

Let $P_{\text{add}}$ “add-program”, and $Q = add(u, s(0), s(u))$ query.

- $\theta = \{u/s^2(v)\}$ correct answer substitution of $Q$, since $P_{\text{add}} \models Q\theta = add(s^2(v), s(0), s^3(v))$ (in analogy to slide 13 with $x = s^2(v)$).

- SLD-derivation of $P_{\text{add}} \cup \{Q\}$:

  \[
  \begin{align*}
  \theta_1 & \vdash add(u, 0, u) \to□ \\
  \theta_2 & \text{ with } \theta_1 = \{x/u, y/0, z/u\} \text{ and } \theta_2 = \{x/u\}, \\
  \end{align*}
  \]

  thus $\eta = (\theta_1 \theta_2)_{\{u\}} = \epsilon$ is a computed answer substitution of $Q$.

- Thus, $Q\eta$ more general than $Q\theta$.

- In fact, no SLD-derivation of $P_{\text{add}} \cup \{Q\}$ can deliver correct answer substitution $\theta$. 
Completeness of SLD-Resolution for Implication Trees

Query $Q$ is $n$-deep.

$\iff$

every atom in $Q$ is the root of an implication tree,

and $n$ is the total number of nodes in these trees

Lemma 4.15

Suppose $Q^0$ is $n$-deep for some $n \geq 0$. Then for every selection rule $\mathcal{R}$ there exists a successful SLD-derivation of $P \cup \{Q\}$ with $\text{CAS } \eta$ such that $Q_\eta$ is more general than $Q^0$. 
Completeness of SLD-Resolution (I)

Theorem 4.13
Suppose that $\theta$ is a correct answer substitution of $Q$. Then for every selection rule $\mathcal{R}$ there exists a successful SLD-derivation of $P \cup \{Q\}$ with $\text{CAS } \eta$ such that $Q\eta$ is more general than $Q\theta$.

Proof. Let $Q = A_1, ..., A_m$. Then: $\theta$ correct answer substitution of $A_1, ..., A_m$
implies $P \models A_1\theta, ..., A_m\theta$
implies for every interpretation $I$: if $I \models P$, then $I \models A_1\theta, ..., A_m\theta$
implies $C(P) \models A_1\theta, ..., A_m\theta$ (since $C(P) \models P$ by Lemma 4.12)
implies $\text{inst}(A_i\theta) \subseteq C(P)$ for every $i = 1, ..., m$ (by Lemma 4.7)
implies $A_i\theta \in C(P)$ for every $i = 1, ..., m$
implies $A_1\theta, ..., A_m\theta$ is $n$-deep for some $n \geq 0$ (by def. of $C(P)$)
implies claim (by Lemma 4.15)
Completeness of SLD-Resolution (II)

**Corollary 4.16**
Suppose $P \models \exists Q$.
Then there exists a successful SLD-derivation of $P \cup \{Q\}$.

Proof. $P \models \exists Q$
implies $P \models Q\theta$ for some substitution $\theta$
implies $\theta$ correct answer substitution of $Q$
implies claim (by Theorem 4.13)
Least Herbrand Model

Theorem 4.29 \( \mathcal{M}(P) \) is the least Herbrand model of \( P \).

Proof. Let \( I \) be a Herbrand model of \( P \) and let \( A \in \mathcal{M}(P) \).

We prove \( A \in I \) by induction on the number \( i \) of nodes in the ground implication tree w.r.t. \( P \) with root \( A \). Then \( \mathcal{M}(P) \subseteq I \).

\( i = 1 \): A leaf implies \( A \leftarrow \in \text{ground}(P) \)

implies \( I \models A \) (since \( I \models P \))

implies \( A \in I \)

\( i \to i+1 \): A has direct descendants \( B_1, \ldots, B_n \) (roots of subtrees)

implies \( A \leftarrow B_1, \ldots, B_n \in \text{ground}(P) \) and \( B_1, \ldots, B_n \in I \) (induction hypothesis)

implies \( I \models B_1, \ldots, B_n \)

implies \( I \models A \) (since \( I \models P \))

implies \( A \in I \)
Ground Equivalence

Theorem 4.30  For every ground atom $A$: $P \models A$ iff $\mathcal{M}(P) \models A$.

Proof. “only if”: $P \models A$ and $\mathcal{M}(P) \models P$ implies $\mathcal{M}(P) \models A$ (semantic consequence).

“if”: Show for every interpretation $I$: $I \models P$ implies $I \not\models A$.

Let $I_H = \{A \mid A$ ground atom and $I \models A\}$ Herbrand interpretation.

\[ I \models P \]

implies $I \models B \leftarrow B_1, \ldots, B_n$ for all $B \leftarrow B_1, \ldots, B_n \in \text{ground}(P)$

implies if $I \models B_1, \ldots, I \models B_n$ then $I \models B$ for all ...

implies if $B_1 \in I_H, \ldots, B_n \in I_H$ then $B \in I_H$ for all ... (Def. $I_H$)

implies $I_H \models P$ (by Lemma 4.26; thus $I_H$ Herbrand model)

implies $A \in I_H$ (since $A \in \mathcal{M}(P)$ and $\mathcal{M}(P)$ least Herbrand model)

implies $I \models A$ (by Def. $I_H$)
Complete Partial Orderings

Let \((\mathcal{A}, \sqsubseteq)\) be a partial ordering (cf. Slide 18 for Chapter 2).

- \textbf{a least element} of \(X \subseteq \mathcal{A}\)
  \[\iff a \in X, a \sqsubseteq x \text{ for all } x \in X\]

- \textbf{a least upper bound} of \(X \subseteq \mathcal{A}\) (Notation: \(a = \sqcup X\))
  \[\iff a \in \mathcal{A}, x \sqsubseteq a \text{ for all } x \in X \text{ and } a \text{ is the least element of } \mathcal{A} \text{ with this property}\]

\((\mathcal{A}, \sqsubseteq)\) \textbf{complete partial ordering} (\text{CPO}) \(\iff\)

- \(\mathcal{A}\) contains a least element (denoted by \(\emptyset\))

- for every increasing sequence \(a_0 \sqsubseteq a_1 \sqsubseteq a_2 \ldots\) of elements of \(\mathcal{A}\),
  the set \(X = \{a_0, a_1, a_2, \ldots\}\) has a least upper bound
Some Properties of Operators

Let \((A, \sqsubseteq)\) be a \text{CPO}.

operator \(T: A \rightarrow A\) monotonic
\[\iff l \sqsubseteq J \implies T(l) \sqsubseteq T(J)\]

operator \(T: A \rightarrow A\) finitary
\[\iff \text{for every infinite sequence } I_0 \sqsubseteq I_1 \sqsubseteq \ldots, \quad \bigcup_{n=0}^{\infty} T(I_n) \text{ exists and } T \left( \bigcup_{n=0}^{\infty} I_n \right) \sqsubseteq \bigcup_{n=0}^{\infty} T(I_n)\]

operator \(T: A \rightarrow A\) continuous :\(\iff T\) monotonic and finitary

\(I\) pre-fixpoint of \(T :\(\iff T(I) \sqsubseteq I\)

\(I\) fixpoint of \(T :\(\iff T(I) = I\)
Iterating Operators

Let \((\mathcal{A}, \sqsubseteq)\) be a \texttt{CPO}, \(T: \mathcal{A} \to \mathcal{A}\), and \(I \in \mathcal{A}\).

- \(T^0 (I) \iff I\)
- \(T^{(n+1)} (I) \iff T(T^n (I))\)
- \(T^w (I) \iff \bigcup_{n=0}^{\infty} T^n (I)\)

\(T^a :\iff T^a (\emptyset)\) (for \(a = 0, 1, 2, \ldots, w\))

By the definition of a \texttt{CPO}:
If the sequence \(T^0 (I), T^1 (I), T^2 (I), \ldots\) is increasing, then \(T^w (I)\) exists.

\textbf{Theorem 4.22}

If \(T\) is a continuous operator on a \texttt{CPO}, then \(T^w\) exists and is the least prefixpoint of \(T\) and the least fixpoint of \(T\).
Consider the CPO \( \{ I \mid I \text{Herbrand interpretation}\}, \subseteq \).
Let \( P \) be a program and \( I \) a Herbrand interpretation. Then
\[
T_P(I) :\iff \{ A \mid A \leftarrow B_1, ..., B_n \in \text{ground}(P), \{B_1, ..., B_n\} \subseteq I \}
\]

**Lemma 4.33**

(i) \( T_P \) is finitary.
(ii) \( T_P \) is monotonic.
Lemma 4.32

A Herbrand interpretation $I$ is a model of $P$ iff

$$T_P(I) \subseteq I$$

Proof.

$I \models P$

iff for every $A \leftarrow B_1, \ldots, B_n \in \text{ground}(P)$:

$$\{B_1, \ldots, B_n\} \subseteq I \text{ implies } A \in I \quad \text{(by Lemma 4.26)}$$

iff for every ground atom $A$: $A \in T_P(I)$ implies $A \in I$

iff $T_P(I) \subseteq I$
Characterization Theorem

Theorem 4.34

\[ \mathcal{M}(P) \]

- (i) = least Herbrand model of \( P \)
- (ii) = least pre-fixpoint of \( T_P \)
- (iii) = least fixpoint of \( T_P \)
- (iv) = \( T_P^w \)
- (v) = \( \{ A \mid A \text{ ground atom, } P \models A \} \)

Foundations of Logic Programming

Declarative Interpretation
Success Sets

success set of a program $P :\iff$

$\{ A \mid A$ ground atom, $\exists$ successful SLD-derivation of $P \cup \{ A \} \}$

Theorem 4.37

For a ground atom $A$, the following are equivalent:

(i) $\mathcal{M}(P) \models A$
(ii) $P \models A$
(iii) Every SLD-tree for $P \cup \{ A \}$ is successful
(iv) $A$ is in the success set of $P$
Objectives

- Algebras (which provide a semantics of terms)
- Interpretations (which provide a semantics of programs)
- Soundness of SLD-resolution
- Completeness of SLD-resolution
- Least Herbrand models
- Computing least Herbrand models