

COMPLEXITY THEORY

Lecture 10: Polynomial Space

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For the most current version of this course, see
https://iccl.inf.tu-dresden.de/web/Complexity_Theory/en

The Class PSpace

We defined PSpace as:

$$\text{PSpace} = \bigcup_{d \geq 1} \text{DSpace}(n^d)$$

and we observed that

$$P \subseteq NP \subseteq \text{PSpace} = \text{NPSpace} \subseteq \text{ExpTime}.$$

We can also define a corresponding notion of PSpace-hardness:

Definition 10.1:

- A language **H** is **PSpace-hard**, if $L \leq_p H$ for every language $L \in \text{PSpace}$.
- A language **C** is **PSpace-complete**, if **C** is PSpace-hard and $C \in \text{PSpace}$.

Quantified Boolean Formulae (QBF)

A **QBF** is a formula of the following form:

$$Q_1 X_1 . Q_2 X_2 . \dots . Q_\ell X_\ell . \varphi[X_1, \dots, X_\ell]$$

where $Q_i \in \{\exists, \forall\}$ are quantifiers, X_i are propositional logic variables, and φ is a propositional logic formula with variables X_1, \dots, X_ℓ and constants \top (true) and \perp (false)

Semantics:

- Propositional formulae without variables (only constants \top and \perp) are evaluated as usual
 - $\exists X . \varphi[X]$ is true if either $\varphi[X/\top]$ or $\varphi[X/\perp]$ are true
 - $\forall X . \varphi[X]$ is true if both $\varphi[X/\top]$ and $\varphi[X/\perp]$ are true
- (where $\varphi[X/\top]$ is “ φ with X replaced by \top , and similar for \perp)

Deciding QBF Validity

TRUE QBF

Input: A quantified Boolean formula φ .

Problem: Is φ true (valid)?

Observation: We can assume that the quantified formula is in CNF or 3-CNF (same transformations possible as for propositional logic formulae)

Consider a propositional logic formula φ with variables X_1, \dots, X_ℓ :

Example 10.2: The QBF $\exists X_1 \dots \exists X_\ell. \varphi$ is true if and only if φ is satisfiable.

Example 10.3: The QBF $\forall X_1 \dots \forall X_\ell. \varphi$ is true if and only if φ is a tautology.

The Power of QBF

Theorem 10.4: **TRUE QBF** is PSpace-complete.

Proof:

(1) **TRUE QBF** \in PSpace:

Give an algorithm that runs in polynomial space.

(2) **TRUE QBF** is PSpace-hard:

Proof by reduction from the word problem of any polynomially space-bounded TM.

□

Solving **TRUE QBF** in PSpace

```
01 TRUEQBF( $\varphi$ ) {  
02   if  $\varphi$  has no quantifiers :  
03     return “evaluation of  $\varphi$ ”  
04   else if  $\varphi = \exists X.\psi$  :  
05     return (TRUEQBF( $\psi[X/\top]$ ) OR TRUEQBF( $\psi[X/\perp]$ ))  
06   else if  $\varphi = \forall X.\psi$  :  
07     return (TRUEQBF( $\psi[X/\top]$ ) AND TRUEQBF( $\psi[X/\perp]$ ))  
08 }
```

- Evaluation in line 03 can be done in polynomial space
- Recursions in lines 05 and 07 can be executed one after the other, reusing space
- Maximum depth of recursion = number of variables (linear)
- Store one variable assignment per recursive call

→ polynomial space algorithm

PSpace-Hardness of **TRUE QBF**

Express TM computation in logic, similar to Cook-Levin

Given:

An arbitrary polynomially space-bounded NTM, that is:

- a polynomial p
- a p -space bounded 1-tape NTM $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}})$

Intended reduction

Given a word w , define a QBF $\varphi_{p, \mathcal{M}, w}$ such that

$\varphi_{p, \mathcal{M}, w}$ is true if and only if \mathcal{M} accepts w in space $p(|w|)$.

Notes

- We show the reduction for NTMs, which is more than needed, but makes little difference in logic and allows us to reuse our previous formulae from Cook-Levin
- The proof actually shows many reductions, one for every polyspace NTM, showing PSpace-hardness from first principles

Review: Encoding Configurations

Use propositional variables for describing configurations:

Q_q for each $q \in Q$ means “ \mathcal{M} is in state $q \in Q$ ”

P_i for each $0 \leq i < p(n)$ means “the head is at Position i ”

$S_{a,i}$ for each $a \in \Gamma$ and $0 \leq i < p(n)$ means “tape cell i contains Symbol a ”

Represent configuration $(q, p, a_0 \dots a_{p(n)})$

by assigning truth values to variables from the set

$$\bar{C} := \{Q_q, P_i, S_{a,i} \mid q \in Q, \quad a \in \Gamma, \quad 0 \leq i < p(n)\}$$

using the truth assignment β defined as

$$\beta(Q_s) := \begin{cases} 1 & s = q \\ 0 & s \neq q \end{cases} \quad \beta(P_i) := \begin{cases} 1 & i = p \\ 0 & i \neq p \end{cases} \quad \beta(S_{a,i}) := \begin{cases} 1 & a = a_i \\ 0 & a \neq a_i \end{cases}$$

Review: Validating Configurations

We define a formula $\text{Conf}(\bar{C})$ for a set of configuration variables

$$\bar{C} = \{Q_q, P_i, S_{a,i} \mid q \in Q, \quad a \in \Gamma, \quad 0 \leq i < p(n)\}$$

as follows:

$\text{Conf}(\bar{C}) :=$

“the assignment is a valid configuration”:

$$\bigvee_{q \in Q} (Q_q \wedge \bigwedge_{q' \neq q} \neg Q_{q'})$$

“TM in exactly one state $q \in Q$ ”

$$\wedge \bigvee_{p < p(n)} (P_p \wedge \bigwedge_{p' \neq p} \neg P_{p'})$$

“head in exactly one position $p < p(n)$ ”

$$\wedge \bigwedge_{0 \leq i < p(n)} \bigvee_{a \in \Gamma} (S_{a,i} \wedge \bigwedge_{b \neq a \in \Gamma} \neg S_{b,i})$$

“exactly one $a \in \Gamma$ in each cell”

Review: Validating Configurations

For an assignment β defined on variables in \bar{C} define

$$\text{conf}(\bar{C}, \beta) := \left\{ \begin{array}{l} \beta(Q_q) = 1, \\ (q, p, w_0 \dots w_{p(n)}) \mid \beta(P_p) = 1, \\ \beta(S_{w_i, i}) = 1 \text{ for all } 0 \leq i < p(n) \end{array} \right\}$$

Note: β may be defined on other variables besides those in \bar{C} .

Lemma 10.5: If β satisfies $\text{Conf}(\bar{C})$ then $|\text{conf}(\bar{C}, \beta)| = 1$.

We can therefore write $\text{conf}(\bar{C}, \beta) = (q, p, w)$ to simplify notation.

Observations:

- $\text{conf}(\bar{C}, \beta)$ is a potential configuration of \mathcal{M} , but it may not be reachable from the start configuration of \mathcal{M} on input w .
- Conversely, every configuration $(q, p, w_1 \dots w_{p(n)})$ induces a satisfying assignment β for which $\text{conf}(\bar{C}, \beta) = (q, p, w_1 \dots w_{p(n)})$.

Review: Transitions Between Configurations

Consider the following formula $\text{Next}(\bar{C}, \bar{C}')$ defined as

$$\text{Conf}(\bar{C}) \wedge \text{Conf}(\bar{C}') \wedge \text{NoChange}(\bar{C}, \bar{C}') \wedge \text{Change}(\bar{C}, \bar{C}').$$

$$\text{NoChange} := \bigvee_{0 \leq p < p(n)} \left(P_p \wedge \bigwedge_{i \neq p, a \in \Gamma} (S_{a,i} \rightarrow S'_{a,i}) \right)$$

$$\text{Change} := \bigvee_{0 \leq p < p(n)} \left(P_p \wedge \bigvee_{\substack{q \in Q \\ a \in \Gamma}} (Q_q \wedge S_{a,p} \wedge \bigvee_{(q', b, D) \in \delta(q, a)} (Q'_{q'} \wedge S'_{b,p} \wedge P'_{D(p)})) \right)$$

where $D(p)$ is the position reached by moving in direction D from p .

Lemma 10.6: For any assignment β defined on $\bar{C} \cup \bar{C}'$:

β satisfies $\text{Next}(\bar{C}, \bar{C}')$ if and only if $\text{conf}(\bar{C}, \beta) \vdash_{\mathcal{M}} \text{conf}(\bar{C}', \beta)$

Review: Start and End

Defined so far:

- $\text{Conf}(\bar{C})$: \bar{C} describes a potential configuration
- $\text{Next}(\bar{C}, \bar{C}')$: $\text{conf}(\bar{C}, \beta) \vdash_{\mathcal{M}} \text{conf}(\bar{C}', \beta)$

Start configuration: Let $w = w_0 \cdots w_{n-1} \in \Sigma^*$ be the input word

$$\text{Start}_{\mathcal{M}, w}(\bar{C}) := \text{Conf}(\bar{C}) \wedge Q_{q_0} \wedge P_0 \wedge \bigwedge_{i=0}^{n-1} S_{w_i, i} \wedge \bigwedge_{i=n}^{p(n)-1} S_{\perp, i}$$

Then an assignment β satisfies $\text{Start}_{\mathcal{M}, w}(\bar{C})$ if and only if \bar{C} represents the start configuration of \mathcal{M} on input w .

Accepting stop configuration:

$$\text{Acc-Conf}(\bar{C}) := \text{Conf}(\bar{C}) \wedge Q_{q_{\text{accept}}}$$

Then an assignment β satisfies $\text{Acc-Conf}(\bar{C})$ if and only if \bar{C} represents an accepting configuration of \mathcal{M} .

Simulating Polynomial Space Computations

For Cook-Levin, we used one set of configuration variables for every computing step:
polynomially time \leadsto polynomially many variables

Problem: For polynomial space, we have $2^{O(p(n))}$ possible steps ...

What would Savitch do?

Define a formula $\text{CanYield}_i(\bar{C}_1, \bar{C}_2)$ to state that \bar{C}_2 is reachable from \bar{C}_1 in at most 2^i steps:

$$\text{CanYield}_0(\bar{C}_1, \bar{C}_2) := (\bar{C}_1 = \bar{C}_2) \vee \text{Next}(\bar{C}_1, \bar{C}_2)$$

$$\text{CanYield}_{i+1}(\bar{C}_1, \bar{C}_2) := \exists \bar{C}. \text{Conf}(\bar{C}) \wedge \text{CanYield}_i(\bar{C}_1, \bar{C}) \wedge \text{CanYield}_i(\bar{C}, \bar{C}_2)$$

But what is $\bar{C}_1 = \bar{C}_2$ supposed to mean here? It is short for:

$$\bigwedge_{q \in Q} Q_q^1 \leftrightarrow Q_q^2 \wedge \bigwedge_{0 \leq i < p(n)} P_i^1 \leftrightarrow P_i^2 \wedge \bigwedge_{a \in \Gamma, 0 \leq i < p(n)} S_{a,i}^1 \leftrightarrow S_{a,i}^2$$

Putting Everything Together

We define the formula $\varphi_{p,\mathcal{M},w}$ as follows:

$$\varphi_{p,\mathcal{M},w} := \exists \bar{C}_1. \exists \bar{C}_2. \text{Start}_{\mathcal{M},w}(\bar{C}_1) \wedge \text{Acc-Conf}(\bar{C}_2) \wedge \text{CanYield}_{dp(n)}(\bar{C}_1, \bar{C}_2)$$

where we select d to be the least number such that \mathcal{M} has less than $2^{dp(n)}$ configurations in space $p(n)$.

Lemma 10.7: $\varphi_{p,\mathcal{M},w}$ is satisfiable if and only if \mathcal{M} accepts w in space $p(|w|)$.

Did we do it?

Note: we used only existential quantifiers when defining $\varphi_{p,\mathcal{M},w}$:

$$\text{CanYield}_0(\bar{C}_1, \bar{C}_2) := (\bar{C}_1 = \bar{C}_2) \vee \text{Next}(\bar{C}_1, \bar{C}_2)$$

$$\text{CanYield}_{i+1}(\bar{C}_1, \bar{C}_2) := \exists \bar{C}. \text{Conf}(\bar{C}) \wedge \text{CanYield}_i(\bar{C}_1, \bar{C}) \wedge \text{CanYield}_i(\bar{C}, \bar{C}_2)$$

$$\varphi_{p,\mathcal{M},w} := \exists \bar{C}_1. \exists \bar{C}_2. \text{Start}_{\mathcal{M},w}(\bar{C}_1) \wedge \text{Acc-Conf}(\bar{C}_2) \wedge \text{CanYield}_{dp(n)}(\bar{C}_1, \bar{C}_2)$$

Now that's quite interesting . . .

- With only (non-negated) \exists quantifiers, **TRUE QBF** coincides with **SAT**
- **SAT** is in NP
- So we showed that the word problem for PSpace NTMs to be in NP

So we found that **NP = PSpace!**

Strangely, most textbooks claim that this is not known to be true . . .

Are we up for the next Turing Award, or did we make a [mistake](#)?

Size

How big is $\varphi_{p, \mathcal{M}, w}$?

$$\text{CanYield}_0(\bar{C}_1, \bar{C}_2) := (\bar{C}_1 = \bar{C}_2) \vee \text{Next}(\bar{C}_1, \bar{C}_2)$$

$$\text{CanYield}_{i+1}(\bar{C}_1, \bar{C}_2) := \exists \bar{C}. \text{Conf}(\bar{C}) \wedge \text{CanYield}_i(\bar{C}_1, \bar{C}) \wedge \text{CanYield}_i(\bar{C}, \bar{C}_2)$$

$$\varphi_{p, \mathcal{M}, w} := \exists \bar{C}_1. \exists \bar{C}_2. \text{Start}_{\mathcal{M}, w}(\bar{C}_1) \wedge \text{Acc-Conf}(\bar{C}_2) \wedge \text{CanYield}_{dp(n)}(\bar{C}_1, \bar{C}_2)$$

Size of CanYield_{i+1} is more than twice the size of CanYield_i

\leadsto Size of $\varphi_{p, \mathcal{M}, w}$ is in $2^{O(p(n))}$. Oops.

A correct reduction: We redefine CanYield by setting

$$\text{CanYield}_{i+1}(\bar{C}_1, \bar{C}_2) :=$$

$$\exists \bar{C}. \text{Conf}(\bar{C}) \wedge$$

$$\forall \bar{Z}_1. \forall \bar{Z}_2. (((\bar{Z}_1 = \bar{C}_1 \wedge \bar{Z}_2 = \bar{C}) \vee (\bar{Z}_1 = \bar{C} \wedge \bar{Z}_2 = \bar{C}_2)) \rightarrow \text{CanYield}_i(\bar{Z}_1, \bar{Z}_2))$$

Size

Let's analyse the size more carefully this time:

$$\begin{aligned} \text{CanYield}_{i+1}(\bar{C}_1, \bar{C}_2) &:= \\ \exists \bar{C}. \text{Conf}(\bar{C}) \wedge \\ \forall \bar{Z}_1. \forall \bar{Z}_2. ((\bar{Z}_1 = \bar{C}_1 \wedge \bar{Z}_2 = \bar{C}) \vee (\bar{Z}_1 = \bar{C} \wedge \bar{Z}_2 = \bar{C}_2)) &\rightarrow \text{CanYield}_i(\bar{Z}_1, \bar{Z}_2) \end{aligned}$$

- $\text{CanYield}_{i+1}(\bar{C}_1, \bar{C}_2)$ extends $\text{CanYield}_i(\bar{C}_1, \bar{C}_2)$ by parts that are linear in the size of configurations \leadsto growth in $O(p(n))$
- Maximum index i used in $\varphi_{p, \mathcal{M}, w}$ is $dp(n)$, that is in $O(p(n))$
- Therefore: $\varphi_{p, \mathcal{M}, w}$ has size $O(p^2(n))$ – and thus can be computed in polynomial time

Exercise:

Why can we just use $dp(n)$ in the reduction? Don't we have to compute it somehow? Maybe even in polynomial time?

The Power of QBF

Theorem 10.4: **TRUE QBF** is PSpace-complete.

Proof:

(1) **TRUE QBF** \in PSpace:

Give an algorithm that runs in polynomial space.

(2) **TRUE QBF** is PSpace-hard:

Proof by reduction from the word problem of any polynomially space-bounded TM.

□

A More Common Logical Problem in PSpace

Recall standard first-order logic:

- Instead of propositional variables, we have **atoms** (predicates with constants and variables)
- Instead of propositional evaluations we have **first-order structures** (or **interpretations**)
- First-order **quantifiers** can be used on variables
- **Sentences** are formulae where all variables are quantified
- A sentence can be **satisfied** or not by a given first-order structure

FOL MODEL CHECKING

Input: A first-order sentence φ and a finite first-order structure \mathcal{I} .

Problem: Is φ satisfied by \mathcal{I} ?

First-Order Logic is PSpace-complete

Theorem 10.8: FOL MODEL CHECKING is PSpace-complete.

Proof:

- (1) **FOL MODEL CHECKING** \in PSpace:
Give algorithm that runs in polynomial space.
- (2) **FOL MODEL CHECKING** is PSpace-hard:
Proof by reduction **TRUE QBF** \leq_p **FOL MODEL CHECKING**.

□

Checking FOL Models in Polynomial Space (Sketch)

```
01 EVAL( $\varphi, \mathcal{I}$ ) {
02   switch ( $\varphi$ ) :
03     case  $p(c_1, \dots, c_n)$  : return  $\langle c_1, \dots, c_n \rangle \in p^{\mathcal{I}}$ 
04     case  $\neg\psi$  : return NOT EVAL( $\psi, \mathcal{I}$ )
05     case  $\psi_1 \wedge \psi_2$  : return EVAL( $\psi_1, \mathcal{I}$ ) AND EVAL( $\psi_2, \mathcal{I}$ )
06     case  $\exists x.\psi$  :
07       for  $c \in \Delta^{\mathcal{I}}$  :
08         if EVAL( $\psi[x \mapsto c], \mathcal{I}$ ) : return TRUE
09       // eventually, if no success:
10       return FALSE
11 }
```

- We can assume φ only uses \neg , \wedge and \exists (easy to get)
- We use $\Delta^{\mathcal{I}}$ to denote the (finite!) domain of \mathcal{I}
- We allow domain elements to be used like constants in the formula

Hardness of FOL MODEL CHECKING

Given: a QBF $\varphi = Q_1 X_1 \cdot \dots \cdot Q_\ell X_\ell . \psi$

FOL Model Checking Problem:

- Interpretation domain $\Delta^{\mathcal{I}} := \{0, 1\}$
- Single predicate symbol true with interpretation $\text{true}^{\mathcal{I}} = \{\langle 1 \rangle\}$
- FOL formula φ' is obtained by replacing variables in input QBF with corresponding first-order expressions:

$$Q_1 x_1 \cdot \dots \cdot Q_\ell x_\ell . \psi[X_1 \mapsto \text{true}(x_1), \dots, X_\ell \mapsto \text{true}(x_\ell)]$$

Lemma 10.9: $\langle \mathcal{I}, \varphi' \rangle \in \text{FOL MODEL CHECKING}$ if and only if $\varphi \in \text{TRUE QBF}$.

First-Order Logic is PSpace-complete

Theorem 10.8: FOL MODEL CHECKING is PSpace-complete.

Proof:

- (1) **FOL MODEL CHECKING** \in PSpace:
Give algorithm that runs in polynomial space.
- (2) **FOL MODEL CHECKING** is PSpace-hard:
Proof by reduction **TRUE QBF** \leq_p **FOL MODEL CHECKING**.

□

FOL MODEL CHECKING: Practical Significance

Why is **FOL MODEL CHECKING** a relevant problem?

Correspondence with database query answering:

- Finite first-order interpretation = database
- First-order logic formula = database query
- Satisfying assignments (for non-sentences) = query results

Known correspondence:

As a query language, FOL has the same expressive power as (basic) SQL (relational algebra).

Corollary 10.10: Answering SQL queries over a given database is PSpace-complete.

Games

Games as Computational Problems

Many single-player games relate to NP-complete problems:

- Sudoku
- Minesweeper
- Tetris
- ...

Decision problem: *Is there a solution?*

(For Tetris: is it possible to clear all blocks?)

What about *two-player games*?

- Two players take moves in turns
- The players have different goals
- The game ends if a player wins

Decision problem: *Does Player 1 have a winning strategy?*

In other words: can Player 1 enforce winning, whatever Player 2 does?

Example: The Formula Game

A contrived game, to illustrate the idea:

- Given: a propositional logic formula φ with consecutively numbered variables X_1, \dots, X_ℓ .
- Two players take turns in selecting values for the next variable:
 - Player 1 sets X_1 to true or false
 - Player 2 sets X_2 to true or false
 - Player 1 sets X_3 to true or false
 - ...

until all variables are set.

- Player 1 wins if the assignment makes φ true.
Otherwise, Player 2 wins.

Deciding the Formula Game

FORMULA GAME

Input: A formula φ .

Problem: Does Player 1 have a winning strategy on φ ?

Theorem 10.11: **FORMULA GAME** is PSpace-complete.

Proof sketch: **FORMULA GAME** is essentially the same as **TRUE QBF**.

Having a winning strategy means: there is a truth value for X_1 , such that, for all truth values of X_2 , there is a truth value of X_3, \dots such that φ becomes true.

If we have a QBF where quantifiers do not alternate, we can add dummy quantifiers and variables that do not change the semantics to get the same alternating form as for the Formula Game. \square

Example: The Geography Game

A children's game:

- Two players are taking turns naming cities.
- Each city must start with the last letter of the previous.
- Repetitions are not allowed.
- The first player who cannot name a new city loses.

A mathematicians' game:

- Two players are marking nodes on a directed graph.
- Each node must be a successor of the previous one.
- Repetitions are not allowed.
- The first player who cannot mark a new node loses.

Decision problem (**GENERALISED**) **GEOGRAPHY**:

given a graph and start node, does Player 1 have a winning strategy?

GEOGRAPHY is PSpace-complete

Theorem 10.12: GENERALISED GEOGRAPHY is PSpace-complete.

Proof:

(1) **GEOGRAPHY** \in PSpace:

Give algorithm that runs in polynomial space.

It is not difficult to provide a recursive algorithm similar to the one for **TRUE QBF** or **FOL MODEL CHECKING**.

(2) **GEOGRAPHY** is PSpace-hard:

Proof by reduction **FORMULA GAME** \leq_p **GEOGRAPHY**.

□

GEOGRAPHY is PSpace-hard

Let φ with variables X_1, \dots, X_ℓ be an instance of **FORMULA GAME**.

Without loss of generality, we assume:

- ℓ is odd (Player 1 gets the first and last turn)
- φ is in CNF

We now build a graph that encodes **FORMULA GAME** in terms of **GEOGRAPHY**

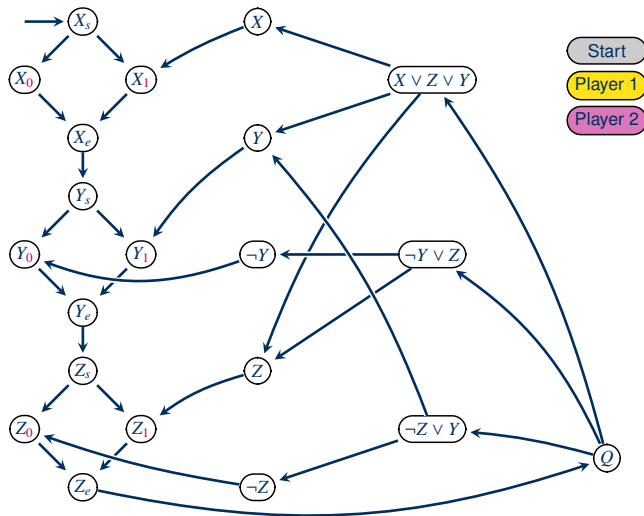
- The left-hand side of the graph is a chain of diamond structures that represent the choices that players have when assigning truth values
- The right-hand side of the graph encodes the structure of φ : Player 2 may choose a clause (trying to find one that is not true under the assignment); Player 1 may choose a literal (trying to find one that is true under the assignment).

(see board or [Sipser, Theorem 8.14])

□

GEOGRAPHY is PSpace-hard: Example

We consider the formula $\exists X.\forall Y.\exists Z.(X \vee Z \vee Y) \wedge (\neg Y \vee Z) \wedge (\neg Z \vee Y)$



Summary and Outlook

TRUE QBF is PSpace-complete

FOL MODEL CHECKING and the related problem of SQL query answering are PSpace-complete

Some games are PSpace-complete

What's next?

- Some more remarks on games
- Logarithmic space
- Complements of space classes