TU Dresden, Fakultät Informatik
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Complexity Theory
Exercise 3: NP-Completeness

Exercise 3.1. Let

$$\mathsf{A}_{\mathsf{PNTM}} = \{\langle \mathcal{M}, p, w\rangle \mid \mathcal{M} \text{ is a non-deterministic TM that accepts } w \text{ in time } p(|w|) \text{ with } p \text{ a polynomial function}\}$$

Show that $\mathsf{A}_{\mathsf{PNTM}}$ is NP-complete.

Exercise 3.2. Show that the following problem is NP-complete:

Input: A propositional formula φ in CNF
Question: Does φ have at least 2 different satisfying assignments?

Exercise 3.3. Consider the problem **CLIQUE**:

Input: An undirected graph G and some $k \in \mathbb{N}$
Question: Does there exists a clique in G of size at least k?

Suppose **CLIQUE** can be solved in time $T(n)$ for some $T\colon \mathbb{N} \to \mathbb{N}$ with $T(n) \geq n$ for all $n \in \mathbb{N}$. Furthermore, show that then the optimisation problem

Input: An undirected graph G
Compute: A clique in G of maximal size

can be computed in time $\mathcal{O}(n \cdot T(n))$. You can assume that T is monotone.

Exercise 3.4. Show that if a language L is NP-complete, then $\overline{L}$ is coNP-complete.

Exercise 3.5. Show that if $\mathrm{P} = \mathrm{NP}$, then a polynomial-time algorithm exists that produces a satisfying assignment of a given satisfiable propositional formula.

Exercise 3.6. Show that finding paths of a given length in undirected graphs, i.e.,

$$\textsc{Path} = \{\, \langle G, s, t, k\rangle \mid G \text{ contains a simple path from } s \text{ to } t \text{ of length } k \,\}$$

is NP-complete.

$*$ **Exercise 3.7.** Let $A \subseteq 1^*$. Show that if A is NP-complete, then $\mathsf{P} = \mathsf{NP}$.
Proceed as follows: Consider a polynomial-time reduction f from SAT to A. For a formula φ, let φ_{0100} be the reduced formula where variables x_1, x_2, x_3, x_4 in φ are set to the values 0, 1, 0, 0, respectively. (The particular choice of 4 variables as well as of 0100 is arbitrary here) What happens when one applies f to all of these exponentially many reduced formulas?

Exercise 3.8. Show that P is closed under homomorphisms iff $\mathrm{P} = \mathrm{NP}$.