TECHNISCHE UNIVERSITÄT DRESDEN

International Center for Computational Logic

COMPLEXITY THEORY

Lecture 18: Questions and Answers

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Knowledge-Based Systems

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Question 1: The Logarithmic Hierarchy

Q1: The Logarithmic Hierarchy

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In detail, we can define:

- $\Sigma_0^{\mathsf{L}} = \Pi_0^{\mathsf{L}} = \mathsf{L}$
- $\Sigma_{i+1}^{\mathsf{L}} = \mathsf{NL}^{\Sigma_i^{\mathsf{L}}}$ alternatively: languages of log-space bounded Σ_{i+1} ATMs
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Therefore $\Sigma_i^{\mathsf{L}} = \Pi_i^{\mathsf{L}} = \mathsf{NL}$ for all $i \geq 1$.

The Logarithmic Hierarchy collapses on the first level.

Historic note: In 1987, just before the Immerman-Szelepcsényi Theorem was published, Klaus-Jörn Lange, Birgit Jenner, and Bernd Kirsig showed that the Logarithmic Hierarchy collapses on the second level [ICALP 1987].

Question 2: The Hardest Problems in P

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What we know about P and NP:

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Let's first recall the definitions:

Definition: A problem **L** is NP-hard if, for all problems **M** $\in$ NP, there is a polynomial many-one reduction **M** $\leq_m$ **L**.

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Example 18.1: We know that $\mathsf{L} \subseteq \mathsf{P} \subseteq \mathsf{NP}$ but we do not know if any of these subsumptions are proper. Suppose that the truth actually looks like this: $\mathsf{L} \subsetneq \mathsf{P} = \mathsf{NP}$. Then all non-trivial problems in P are NP-hard (why?), but not every such problem would be P-hard (why?).

Note: This is really about the different notions of reduction used to define hardness. If we used log-space reductions for P-hardness and NP-hardness, the claim would follow.

Question 3: Problems Harder than P

Q3: Problems harder than P

Polynomial time is an approximation of "practically tractable" problems:

- Many practical problems are in P, including many very simple ones (e.g., $\emptyset$)
- P-hard problems are as hard as any other problem in P, and
  P-complete problems therefore are the hardest problems in P
- However, there are even harder problems that are no longer in P

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So, clearly, problems that are not even in P must be P-hard, right?

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Can we find any problem that is surely harder than P? Yes, easily:

- The Halting Problem is undecidable and therefore not in P
- Any ExpTime-complete problem is not in P (Time Hierarchy Theorem); e.g., the Word Problem for DTMs with a (fixed) exponential time bound

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These concrete examples both are hard for P:

- The Word Problem for polynomially time-bounded DTMs log-space reduces to the Word Problem for exponential TMs (reduction: the identity function)
- This polytime Word Problem also log-space reduces to the Halting problem: a reduction merely has to modify the TM so that every rejecting halting configuration leads into an infinite loop

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- We can enumerate DTMs for all P-hard languages in ExpTime (how?)

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So, it's clear what we have to do now …

Q3: Are problems harder than P also hard for P?

Schöning to the rescue (see Theorem 15.2):

Corollary 18.2: Consider the classes C_1 = ExpPHard (P-hard problems in ExpTime) and C_2 = P. Both are classes of decidable languages. We find that for either class C_k:

- We can effectively enumerate TMs $\mathcal{M}_0^k, \mathcal{M}_1^k, \ldots$ such that $C_k = \{\mathbf{L}(\mathcal{M}_i^k) \mid i \geq 0)\}$.
- If **L** $\in C_k$ and **L'** differs from **L** on only a finite number of words, then **L'** $\in C_k$

Let $\mathbf{L_1} = \emptyset$, and let $\mathbf{L_2}$ be some ExpTime-complete problem. Clearly, $\mathbf{L_1} \notin$ ExpPHard and $\mathbf{L_2} \notin$ P (Time Hierarchy), hence there is a decidable language $\mathbf{L_d} \notin$ ExpPHard $\cup$ P.
Moreover, as $\emptyset \in$ P and $\mathbf{L_2}$ is not trivial, $\mathbf{L_d} \leq_p \mathbf{L_2}$ and hence $\mathbf{L_d} \in$ ExpTime. Therefore $\mathbf{L_d} \notin$ ExpPHard implies that $\mathbf{L_d}$ is not P-hard.

This idea of using Schöning's Theorem has been put forward by Ryan Williams (link). Our version is a modification requiring $C_1 \subseteq$ ExpTime.

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No, there are problems in ExpTime that are neither in P nor hard for P.

(Other arguments can even show the existence of undecidable sets that are not P-hard[1])

[1] Related note: the undecidable **UHalt** is not NP-hard, since it is a so-called sparse language.

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Discussion:

- Considering Questions 2 and 3, the use of the word hard is misleading, since we interpret it as difficult
- However, the actual meaning of difficult would be "not in a given class" (e.g., problems not in P are clearly more difficult than those in P)
- Our formal notion of hard also implies that a problem is difficult in some sense, but it also requires it to be universal in the sense that many other problems can be solved through it

What we have seen is that there are difficult problems that are not universal.

[1] Related note: the undecidable **UHALT** is not NP-hard, since it is a so-called sparse language.

Your Questions

Summary and Outlook

Answer 1: The Logarithmic Hierarchy collapses.

Answer 2: We don't know that NP-hard implies P-hard.

Answer 3: Being outside of P does not make a problem P-hard.

What's next?

- Holidays
- Circuits as a model of computation
- Randomness

Here's wishing you

a Merry Christmas, a Happy Hanukkah,

a Joyous Yalda, a Cheerful Dōngzhì,

a Great Feast of Juul,

and a Wonderful Winter Solstice,

respectively!