Technische Universität Dresden

Prof. Dr. Sebastian Rudolph

Formal Concept Analysis

Exercise Sheet 3, Winter Semester 2015/16

Exercise 1 (computing and drawing formal concepts)

Given the formal context in Table 1,

1. compute all formal concepts of that context,
2. draw and label the line diagram of the concept lattice, and
3. check if the diagram is correct.

Can you discover implications in that context? If yes, which? Do these implications hold in general? If not, how could they be become invalid?

Tabelle 1: Grobian Gans: *Die Ducks. Psychogramm einer Sippe.* Rowohlt, Reinbek bei Hamburg 1972, ISBN 3-499-11481-X

	generation			sex		financial status		
	older	middle	younger	male	female	rich	carefree	indebted
Tick			×	×			×	
Trick			×	×			×	
Track			×	×			×	
Donald		×		×				×
Daisy		×			×		×	
Gustav		×		×			×	
Dagobert	×			×		×		
Annette	×				×		×	
Primus v. Quack	×			×			×	

<u>Solution:</u>

1) We use the intersection method to compute all formal concepts manually.

   1. Write the attribute extents to a list

No.		extent	found as
e1	:=	{Dagobert, Annette, Primus v. Quack}	$\{\text{older}\}'$
e2	:=	{Donald, Daisy, Gustav}	$\{\text{middle}\}'$
e3	:=	{Tick, Trick, Track}	$\{\text{younger}\}'$
e4	:=	{Tick, Trick, Track, Donald, Gustav, Dagobert, Primus v. Quack}	$\{\text{male}\}'$
e5	:=	{Daisy, Annette}	$\{\text{female}\}'$
e6	:=	{Dagobert}	$\{\text{rich}\}'$
e7	:=	{Tick, Trick, Track, Daisy, Gustav, Annette, Primus v. Quack}	$\{\text{carefree}\}'$
e8	:=	{Donald}	$\{\text{indebted}\}'$

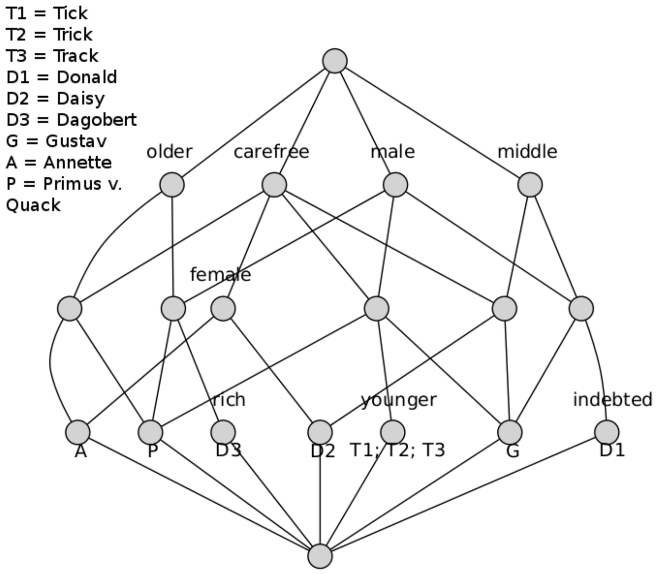
2. Compute all pairwise intersections of objects, and add G.

No.		extent	found as
e9	:=	$\emptyset$	e1 $\cap$ e2
e10	:=	{Dagobert, Primus v. Quack}	e1 $\cap$ e4
e11	:=	{Annette}	e1 $\cap$ e5
e12	:=	{Annette, Primus v. Quack}	e1 $\cap$ e7
e13	:=	{Donald, Gustav}	e2 $\cap$ e4
e14	:=	{Daisy}	e2 $\cap$ e5
e15	:=	{Daisy, Gustav}	e2 $\cap$ e7
e16	:=	{ Tick, Trick, Track, Gustav, Primus v. Quack}	e4 $\cap$ e7
e17	:=	{Primus v. Quack}	e7 $\cap$ e10
e18	:=	{Gustav}	e7 $\cap$ e13
e19	:=	{ Tick, Trick, Track, Donald, Daisy, Gustav, Dagobert, Annette, Primus v. Quack}	Add G

3. Compute the intents of the objects.

No.	(extents	,	intents)
e1	({Dagobert, Annette, Primus v. Quack}	,	{older})
e2	({Donald, Daisy, Gustav}	,	{middle})
e3	({Tick, Trick, Track}	,	{younger, male, carefree})
e4	({Tick, Trick, Track, Donald, Gustav, Dagobert, Primus v. Quack}	,	{male})
e5	({Daisy, Annette}	,	{female, carefree})
e6	({Dagobert}	,	{older, male, rich})
e7	({ Tick, Trick, Track, Daisy, Gustav, Annette, Primus v. Quack}	,	{carefree})
e8	({Donald}	,	{middle, male, indebted})
e9	($\emptyset$	,	M = set of all attributes)
e10	({Dagobert, Primus v. Quack}	,	{older, male})
e11	({Annette}	,	{older, female, carefree})
e12	({Annette, Primus v. Quack}	,	{older, carefree})
e13	({Donald, Gustav}	,	{middle, male})
e14	({Daisy}	,	{middle, female, carefree})
e15	({Daisy, Gustav}	,	{middle, carefree})
e16	({ Tick, Trick, Track, Gustav, Primus v. Quack}	,	{male, carefree})
e17	({Primus v. Quack}	,	{older, male, carefree})
e18	({Gustav}	,	{middle, male, carefree })
e19	(G = set of all objects	,	$\emptyset$)

2) Drawing and labelling line diagram of the concept lattice.



Exercise 2 (the basic theorem of formal concept analysis)

1. Show that $\underline{L} := (L, |)$ with $L := \{2^i 3^j 5^k \in \mathbb{N} \mid 0 \leq i, j \leq 2; 0 \leq k \leq 3\}$ is a complete lattice ($a|b$ is shorthand for the relation "a divides b").
2. Draw the line diagram for $\underline{L}$.
3. Which are the supremum-irreducible elements?
4. Which are the infimum-irreducible elements?
5. Give a formal context (G, M, I) such that its concept lattice is isomorphic to $\underline{L}$. Give the isomorphism explicitly.
6. How could the fact that $\underline{L}$ and $\mathfrak{B}(G, M, I)$ are isomorphic be shown using the Basic Theorem of Formal Concept Analysis?

<u>**Solution:**</u>

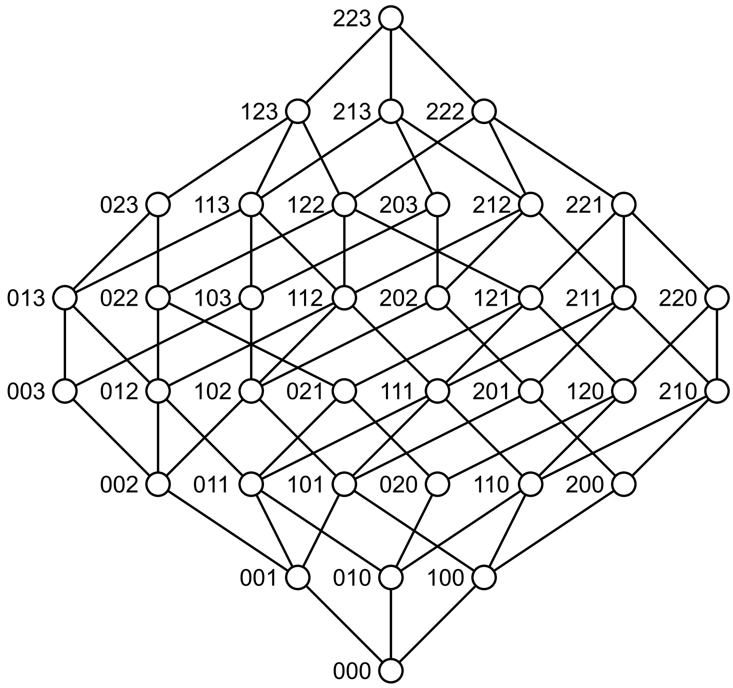
1) Since $\underline{L}$ is finite it is sufficient to show that $x \vee y$ and $x \wedge y$ exist for all $x, y \in \underline{L}$. Let us first state that

$$2^i 3^j 5^k | 2^\ell 3^m 5^n \text{ if and only if } i \leq \ell \text{ and } j \leq m \text{ and } k \leq n$$

holds. From that it follows that:

$$\begin{aligned} 2^i 3^j 5^k \wedge 2^\ell 3^m 5^n &= 2^{\min(i,\ell)} 3^{\min(j,m)} 5^{\min(k,n)} \\ 2^i 3^j 5^k \vee 2^\ell 3^m 5^n &= 2^{\max(i,\ell)} 3^{\max(j,m)} 5^{\max(k,n)} \end{aligned}$$

2) We are writing ijk as shorthand for $2^i 3^j 5^k$.



3, 4) An element of a finite lattice is $\vee$-irreducible ($\wedge$-irreducible), if and only if the corresponding node in the line diagram has exactly one lower (upper) neighbor. Thus, the following propositions hold:

- $\wedge$-irreducible elements are exactly the numbers $2^i3^j5^k$ with exactly two *maximal* exponents.
- $\vee$-irreducible elemente are exactly the numbers $2^i3^j5^k$ with exactly two *minimal* exponents. These are exactly the prime powers.

5) We set $\mathbb{K} := (L, L, |)$. We have to show that $\underline{\mathfrak{B}}(L, L, |)$ and $\underline{L}$ are isomorphic, i.e., there exists a bijective map $\varphi\colon \underline{\mathfrak{B}}(L, L, |) \to \underline{L}$ with

$$(A_1, B_1) \leq (A_2, B_2) \text{ if and only if } \varphi(A_1, B_1)|\varphi(A_2, B_2).$$

We now show that $\varphi\colon (A, B) \mapsto A \cap B$ is an isomorphism between $\underline{\mathfrak{B}}(L, L, |)$ and $\underline{L}$. Therefore, we use the following abbreviations:

$$\{ijk\}\downarrow := \{2^\ell 3^m 5^k \mid \ell \leq i, m \leq j, n \leq k\}$$
$$\{ijk\}\uparrow := \{2^\ell 3^m 5^k \mid \ell \geq i, m \geq j, n \geq k\}$$

Now $(A, B) \in \mathfrak{B}(L, L, I)$ holds if and only if exponents $0 \leq i \leq 2, 0 \leq j \leq 2, 0 \leq k \leq 3$ exist such that $A = \{ijk\}\downarrow$ and $B = \{ijk\}\uparrow$. This allows us to show that φ is surjective and injective and thus bijective.

Furthermore, for two arbitrary concepts $(A_1, B_1) =: (\{ijk\}\downarrow, \{ijk\}\uparrow)$ and $(A_2, B_2) =: (\{\ell mn\}\downarrow, \{\ell mn\}\uparrow)$ the following holds:

$$\begin{aligned}(A_1, B_1) \leq (A_2, B_2) :\iff &\ (\{ijk\}\downarrow, \{ijk\}\uparrow) \leq (\{\ell mn\}\downarrow, \{\ell mn\}\uparrow)\\ \iff &\ \{ijk\}\downarrow \subseteq \{\ell mn\}\downarrow\\ \iff &\ i \leq \ell \wedge j \leq m \wedge k \leq n\\ \iff &\ \underbrace{2^i3^j5^k}_{=\varphi(A_1,B_1)} \mid \underbrace{2^\ell 3^m 5^n}_{=\varphi(A_2,B_2)}\end{aligned}$$

6) According to the Basic Theorem of Formal Concept Analysis, we need to specify two maps $\tilde{\gamma}\colon L \to L$ and $\tilde{\mu}\colon L \to L$, such that $\tilde{\gamma}(L)$ is supremum-dense and $\tilde{\mu}(L)$ infimum-dense in $\underline{L}$ and $a|b$ holds if and only if $\tilde{\gamma}(a)|\tilde{\mu}(b)$ holds. This is the case with the identity function, respectively, i.e., $\tilde{\mu}\colon l \mapsto l$ and $\tilde{\gamma}\colon l \mapsto l$ for $l \in L$.