Technische Universität Dresden

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Formal Concept Analysis

Exercise Sheet 7, Winter Semester 2015/16

Definition (closure system and closure operator).

(a) A set $\mathfrak{A} \subseteq \mathfrak{P}(M)$ *is a* closure system *on the set* M*, iff* $M \in \mathfrak{A}$ *and* $\mathfrak{X} \subseteq \mathfrak{A} \implies \bigcap \mathfrak{X} \in \mathfrak{A}$.

(b) A closure operator φ *on* M *is a map* φ *which maps each subset* $X \subseteq M$ *onto the corresponding* closure $\varphi(X) \subseteq M$ *such that the following hold:*

1) $X \subseteq \varphi(X)$ *(extensive)*
2) $X \subseteq Y \implies \varphi(X) \subseteq \varphi(Y)$ *(monotone)*
3) $\varphi(\varphi(X)) = \varphi(X)$ *(idempotent)*

Exercise 1

Prove the following statements:

a) For any closure system $\mathfrak{A}$ on some set M, the mapping $\varphi_{\mathfrak{A}} : X \mapsto \bigcap_{X \subseteq Y \in \mathfrak{A}} Y$ is a closure operator on M.

b) For any closure operator φ on some set M, the family $\mathfrak{A}_\varphi := \{\varphi(X) \mid X \subseteq M\}$ is a closure system on M.

<u>**Solution:**</u>

a) We have to show that all three requirements for closure operators are satisfied:

- extensivity
  $X \subseteq \bigcap_{X \subseteq Y \in \mathfrak{A}} Y = \varphi_{\mathfrak{A}}(X)$
- monotonicity
  $X \subseteq Y$
  $\implies \{Z \mid X \subseteq Z \in \mathfrak{A}\} \supseteq \{Z \mid Y \subseteq Z \in \mathfrak{A}\}$
  $\implies \bigcap\{Z \mid X \subseteq Z \in \mathfrak{A}\} \subseteq \bigcap\{Z \mid Y \subseteq Z \in \mathfrak{A}\}$
  $\varphi_{\mathfrak{A}}(X) \subseteq \varphi_{\mathfrak{A}}(Y)$
- idempotency
  We first show that $\varphi_{\mathfrak{A}}(X) \in \mathfrak{A}$ for any X: in fact, this directly follows from the definition $\varphi_{\mathfrak{A}}(X) = \bigcap_{X \subseteq Y \in \mathfrak{A}} Y$ and that, thanks to $\mathfrak{A}$ being a closure system, intersecting any selection of sets from $\mathfrak{A}$ yields a set from $\mathfrak{A}$.

  Now for every $\varphi_{\mathfrak{A}}(X) \in \mathfrak{A}$ we obtain $\varphi_{\mathfrak{A}}(\varphi_{\mathfrak{A}}(X)) = \bigcap_{\varphi_{\mathfrak{A}}(X) \subseteq Y \in \mathfrak{A}} Y = \varphi_{\mathfrak{A}}(X) \cap \bigcap_{\varphi_{\mathfrak{A}}(X) \subset Y \in \mathfrak{A}} Y = \varphi_{\mathfrak{A}}(X)$, which concludes the proof.

b) We have to show that $M \in \mathfrak{A}_\varphi$ and that $\mathfrak{A}_\varphi$ is closed under intersection.
Note that due to extensivity $\varphi(M) = M$, therefore $M \in \mathfrak{A}_\varphi$.
Now, consider a family $\mathfrak{X} \subseteq \mathfrak{A}_\varphi$. We need to show that $\bigcap \mathfrak{X} \in \mathfrak{A}_\varphi$, i.e. that $\bigcap \mathfrak{X} = \varphi(Y)$ for some $Y \subseteq M$. We show the correspondence for $Y = \bigcap \mathfrak{X}$ by showing mutual inclusion of the two sets:
$\bigcap \mathfrak{X} \subseteq \varphi(\bigcap \mathfrak{X})$ is a direct consequence of extensivity.
To show the other direction, first note that $\varphi(X) = X$ for all $X \in \mathfrak{X}$. Next we observe that for any $X \in \mathfrak{X}$ holds $\bigcap \mathfrak{X} \subseteq X$ and hence, by monotonicity and the above observation $\varphi(\bigcap \mathfrak{X}) \subseteq \varphi(X) = X$. Since the latter holds for every $X \in \mathfrak{X}$, we can conclude $\varphi(\bigcap \mathfrak{X}) \subseteq \bigcap \mathfrak{X}$, which finishes our proof.

Exercise 2 (closure system)

Regard the "family context" $\mathbb{K}$:=({father, mother, daughter, son}, {old, young, male, female}, {(father, old), (father, male), (mother, old), (mother, female), (daughter, young), (daughter, female), (son, young), (son, male)}).

a) Explicitly list the elements of the map $\varphi\colon \mathfrak{P}(M) \to \mathfrak{P}(M)$ with $\varphi\colon B \mapsto B''$ and verify that φ is a closure operator.

b) Verify that the set of all concept intents of the family context is a closure system.

c) Draw a line diagram of the powerset of $\{\text{father}, \text{mother}, \text{daughter}, \text{son}\}$ and highlight the sets that have the same closure. Compare the diagram with the diagram of the concept lattice of the family context.

Solution:

(a)

$x \in \mathfrak{P}(M)$	,	$\varphi(x)$
($\emptyset$	,	$\emptyset$)
({o}	,	{o})
({y}	,	{y})
({m}	,	{m})
({f}	,	{f})
({o,y}	,	{o,y,m,f})
({o,m}	,	{o,m})
({o,f}	,	{o,f})
({y,m}	,	{y,m})
({y,f}	,	{y,f})
({m,f}	,	{o,y,m,f})
({o,y,m}	,	{o,y,m,f})
({o,y,f}	,	{o,y,m,f})
({o,m,f}	,	{o,y,m,f})
({y,m,f}	,	{o,y,m,f})
({o,y,m,f}	,	{o,y,m,f})

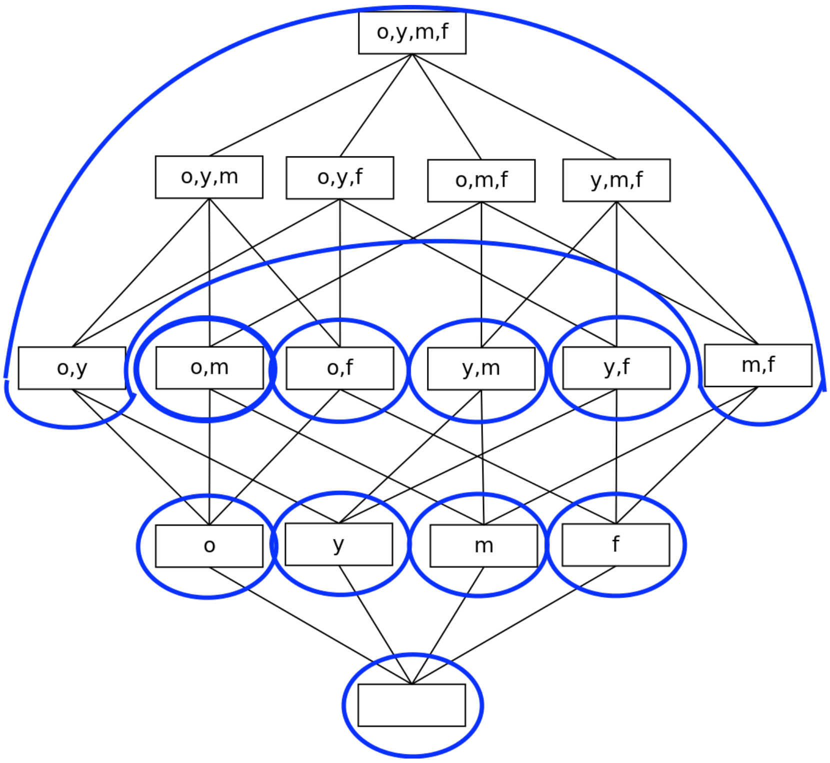
(b) The set of all concept intents of the family context is essentially the set whose elements are the sets at the right hand side of the above table. i.e if we denote it by $\mathfrak{S}$:= {$\emptyset$, {o}, {y}, {m}, {f}, {o,m}, {o,f}, {y,m}, {y,f}, {o,y,m,f} }. We want to show that:

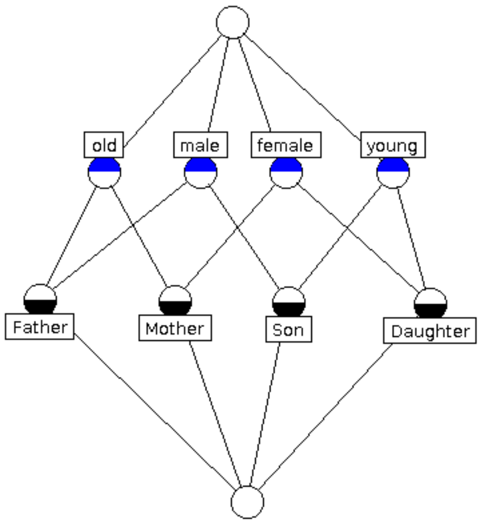
i $M \in \mathfrak{S}$. This is True since {o,y,m,f} $\subset \mathfrak{S}$.

ii For any set $\mathfrak{s} \subset \mathfrak{S}$, the intersection of $\mathfrak{s}$ is in $\mathfrak{S}$. We show this by exhaustive enumeration in the following table. Note that the table is symmetric across the main diagonal. The entries of the table are the intersections of all possible combinations of subsets of $\mathfrak{S}$ and are all in $\mathfrak{S}$.

$\cap$	$\emptyset$	{o}	{y}	{m}	{f}	{o,m}	{o,f}	{y,m}	{y,f}	{o,y,m,f}
$\emptyset$	$\emptyset$									
{o}	$\emptyset$	{o}								
{y}	$\emptyset$	$\emptyset$	{y}							
{m}	$\emptyset$	$\emptyset$	$\emptyset$	{m}						
{f}	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	{f}					
{o,m}	$\emptyset$	{o}	$\emptyset$	{m}	$\emptyset$	{o,m}				
{o,f}	$\emptyset$	{o}	$\emptyset$	$\emptyset$	{f}	{o}	{o,f}			
{y,m}	$\emptyset$	$\emptyset$	{y}	{m}	$\emptyset$	{m}	$\emptyset$	{y,m}		
{y,f}	$\emptyset$	$\emptyset$	{y}	$\emptyset$	{f}	$\emptyset$	{f}	$\emptyset$	{y,f}	
{o,y,m,f}	$\emptyset$	{o}	{y}	{m}	{f}	{o,m}	{o,f}	{y,m}	{y,f}	{o,y,m,f}

(c) The Line diagram for the power-set of $\{F, M, D, S\}$ is given below. The sets with the same closure are highlighted in Blue. Following that is the diagram of the concept lattice of the family context.


o,y,m,f
o,y,m
o,y,f
o,m,f
y,m,f
o,y
o,m
o,f
y,m
y,f
m,f
o
y
m
f


old
male
female
young
Father
Mother
Son
Daughter

Exercise 3 (Next-Closure)

	old (1)	young (2)	male (3)	female (4)
father	×		×	
mother	×			×
son		×	×	
daughter		×		×

Compute all concept intents of the above "family context" using the Next-Closure algorithm. Compare your result with the concept intents from Exercise 2.

A	i	$(A\cap\{1,2,\ldots,i-1\})\cup\{i\}$ $A+i$	$(A+i)''$ $A\oplus i$	$A <_i A\oplus i$?	new intent

Solution:

(a)

A	i	$(A\cap\{1,2,\ldots,i-1\})\cup\{i\}$ $A+i$	$(A+i)''$ $A\oplus i$	$A <_i A\oplus i$?	new intent
$\emptyset$	4	$\{4\}$	$\{4\}$	$\times$	$\{4\}$
$\{4\}$	3	$\{3\}$	$\{3\}$	$\times$	$\{3\}$
$\{3\}$	4	$\{3,4\}$	$\{1,2,3,4\}$		
$\{3\}$	2	$\{2\}$	$\{2\}$	$\times$	$\{2\}$
$\{2\}$	4	$\{2,4\}$	$\{2,4\}$	$\times$	$\{2,4\}$
$\{2,4\}$	3	$\{2,3\}$	$\{2,3\}$	$\times$	$\{2,3\}$
$\{2,3\}$	4	$\{2,3,4\}$	$\{1,2,3,4\}$		
$\{2,3\}$	1	$\{1\}$	$\{1\}$	$\times$	$\{1\}$
$\{1\}$	4	$\{1,4\}$	$\{1,4\}$	$\times$	$\{1,4\}$
$\{1,4\}$	3	$\{1,3\}$	$\{1,3\}$	$\times$	$\{1,3\}$
$\{1,3\}$	4	$\{1,3,4\}$	$\{1,2,3,4\}$		
$\{1,3\}$	2	$\{1,2\}$	$\{1,2,3,4\}$	$\times$	$\{1,2,3,4\}$

(b) Result found same as the one in 2(b).