EXERCISE 1

Science of Computational Logic

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Problem 1.1

In the lectures the following example from Description Logics was presented:

$$\mathcal{K}_T: \quad \begin{aligned} \text{woman} &\sqsubseteq \text{person}, \\ \text{man} &\sqsubseteq \text{person}, \\ \text{mother} &= \text{woman} \sqcap \exists \text{child} : \text{person}, \\ \text{father} &= \text{man} \sqcap \exists \text{child} : \text{person}, \\ \text{parent} &= \text{mother} \sqcup \text{father}, \\ \text{grandparent} &= \text{parent} \sqcap \exists \text{child} : \text{parent}, \\ \text{father\_without\_son} &= \text{father} \sqcap \forall \text{child} : \neg \text{man} \end{aligned}$$

$$\mathcal{K}_A: \quad \begin{aligned} &\text{parent(carl)},\ \text{parent(conny)}, \\ &\text{child(conny, joe)},\ \text{child(conny, carl)}, \\ &\text{man(joe)},\ \text{man(carl)},\ \text{woman(conny)}. \end{aligned}$$

Are the following consequences valid? **Justify** your answers.

1. $\mathcal{K}_T \cup \mathcal{K}_A \models \text{grandparent(conny)}$

   Solution

   Due to grandparent = parent $\sqcap \exists$child : parent

   grandparent(conny) follows from: parent(conny) and $(\exists Y)(\text{child}(\text{conny}, Y) \wedge \text{parent}(Y))$

   (a) parent(conny) : given

   (b) $(\exists Y)(\text{child}(\text{conny}, Y) \wedge \text{parent}(Y))$:
   yes, since satisfied by carl , due to child(conny, carl) and parent(carl)

2. $\mathcal{K}_T \cup \mathcal{K}_A \models \text{father(carl)}$

   Solution

   Recall: father = man $\sqcap \exists$child : person

   Consequently, we have to show:
   $\text{man(carl)} \wedge (\exists Y)(\text{child}(\text{carl}, Y) \wedge \text{person}(Y))$ resp. $\text{man(carl)} \wedge (\exists \text{child} : \text{person})(\text{carl}))$

   Given parent(carl) it follows: (mother(carl) or father(carl))

   - From mother(carl) it follows: woman(carl) and $(\exists$child : person)(carl)
   - From father(carl) it follows: man(carl) and $(\exists$child : person)(carl)

Consequently: From (mother(carl) or father(carl))

follows: (woman(carl) or man(carl)) and $(\exists$child : person$)$(carl)

and consequently, $(\exists$child : person$)$(carl) .

On the other hand man(carl) is given.

3. $\mathcal{K}_T \cup \mathcal{K}_A \models$ father_without_son(carl)

Solution

That father(carl) holds has been shown in the subproblem above.

We obtain a counter example as follows:

We start with a Herbrand interpretation of the language given by the given formulae and which is a model of the given formulae $\mathcal{K}_T \cup \mathcal{K}_A$. (We assume this interpretation to be represented as a subset of the Herbrand base.)

Then we extend the Herbrand universe by a further constant mike (thus obtaining a new domain)

and we add the atoms child(carl, mike) and man(mike) to the interpretation.

The atom father_without_son(carl) will be deleted if present.

Then carl has a son (named mike).

Problem 1.2

Prove that $F \sqsubseteq G \;\; \equiv \;\; F \sqcap \neg G = \bot$

Solution

Let $I = (\mathcal{D}, \cdots^I)$ be an arbitrary interpretation.

$I \models F \sqsubseteq G$	iff
$F^I \subseteq G^I$	iff
$F^I \cap (\mathcal{D} \setminus G^I) = \emptyset$	iff
$F^I \cap (\neg G)^I = \emptyset$	iff
$(F \sqcap \neg G)^I = \emptyset$	iff
$(F \sqcap \neg G)^I = \bot^I$	iff
$I \models F \sqcap \neg G = \bot$	

Problem 1.3

Show that grandparent $\sqsubseteq_{\mathcal{K}_T}$ parent by reducing subsumption into concept satisfiability, where $\mathcal{K}_T$ is the T-Box from Problem 1.1.

Solution

We answer the question whether there exists a model I of the terminological knowledge base $\mathcal{K}_T$ such that (grandparent $\sqcap \neg$parent$)^I \neq \emptyset$.
If the answer is no , then grandparent $\sqsubseteq_T$ parent .

Suppose there exists a model $I = (\mathcal{D}, \cdots^I)$ of T such that $(\text{grandparent} \sqcap \neg\text{parent})^I \neq \emptyset$. Then, there is $a \in \mathcal{D}$ such that $a \in \text{grandparent}^I$ and $a \notin \text{parent}^I$.

Since $I \models T$, we conclude with $a \in \text{grandparent}^I$ and $(\text{grandparent} = \text{parent} \sqcap \exists \text{child.parent})$, that $a \in \text{parent}^I$. But this is a contradiction.

Hence, there is no model I of $\mathcal{K}_T$ such that $(\text{grandparent} \sqcap \neg\text{parent})^I \neq \emptyset$. Consequently, $\text{grandparent} \sqsubseteq_T \text{parent}$.

Problem 1.4

Is the concept $(\text{father} \sqcap \text{mother})$ satisfiable w.r.t. $\mathcal{K}_T$ from Problem 1.1?

Solution

$(\text{father} \sqcap \text{mother})$ unsatisfiable iff $(\text{father} \sqcap \text{mother}) = \bot$ for all models of the T-box T.

Therefore, we have to find an intepretation I where $(\text{father} \sqcap \text{mother})^I$ is not empty.

The anser is YES. Consider the following interpretation

$I = \{\text{father}(p), \text{mother}(p), \text{child}(p, c), \text{person}(c), \text{man}(p), \text{woman}(p), \text{person}(p), \text{parent}(p)\}$

grandparent and father_without_son are empty on I.

Problem 1.5

1. Which generalized concept axioms must be added to prevent that a person is female and male?
2. Is there a single generalized concept axiom that prevents that a person is female and male?

Solution

1. Either one of the following generalized concept axiom can be added to the T-Box to prevent that a person is female and male:
   - $\text{man} \sqsubseteq \neg\text{woman}$
   - $\text{woman} \sqsubseteq \neg\text{man}$
   - $\text{man} \sqcap \text{woman} \sqsubseteq \bot$
2. Same as part 1.

Problem 1.6

Give an equivalent concept of $(\text{woman} \sqcap \exists \text{child.person})$ without using the constructors $\sqcap$ and $\exists r.C$

Solution

e.g. $\neg(\neg$woman $\sqcup$ $\forall$child.$\neg$person$)$

(woman $\sqcap$ $\exists$child.person) means (woman$(X) \wedge (\exists Y)($child$(X,Y) \wedge$ person$(Y)))$

Which we may transform stepwise as follows:

(woman$(X) \wedge (\exists Y)($child$(X,Y) \wedge$ person$(Y)))$

$\neg\neg($woman$(X) \wedge (\exists Y)($child$(X,Y) \wedge$ person$(Y)))$

$\neg(\neg$woman$(X) \vee (\forall Y)(\neg$child$(X,Y) \vee \neg$person$(Y)))$

$\neg(\neg$woman$(X)(\forall Y)(\neg$child$(X,Y) \rightarrow \neg$person$(Y)))$

which abbreviates to

$\neg(\neg$woman $\sqcup$ $\forall$child.$\neg$person$)$

Problem 1.7

Prove the following:

If $(\forall r.C)(a) \in \mathcal{A}$, and $r(a,b) \in \mathcal{A}$, then $\mathcal{A} \models C(b)$.

Solution

Let $I \models \mathcal{A}$ and suppose $I \models (\forall r.C)(a)$ and $I \models r(a,b)$.

Then, $a^I \in (\forall r.C)^I$ and $b^I \in r^I(a^I)$.

By the definition of the universal quantifier, $(\forall r : C)^I = \{d \in D \mid r^I(d) \subseteq C^I\}$, so $r^I(a^I) \subseteq C^I$. Thus $b^I \in C^I$.

Hence, $I \models C(b)$.