Problem 1.1

In the lectures the following example from Description Logics was presented:

\[
\begin{align*}
K_T : & \text{ woman } \sqsubseteq \text{ person, } & \text{ K}_A : & \text{ parent(carl), parent(conn), } \\
& \text{ man } \sqsubseteq \text{ person, } & & \text{ child(conn, joe), child(conn, carl), } \\
& \text{ mother } = \text{ woman } \sqcap \exists \text{ child : person, } & & \text{ man(joe), man(carl), woman(conn). } \\
& \text{ father } = \text{ man } \sqcap \exists \text{ child : person, } & & \\
& \text{ parent } = \text{ mother } \sqcup \text{ father, } & & \\
& \text{ grandparent } = \text{ parent } \sqcap \exists \text{ child : parent, } & & \\
& \text{ father_{without_{son}} } = \text{ father } \sqcap \forall \text{ child : } \neg \text{ man } \\
\end{align*}
\]

Are the following consequences valid? **Justify** your answers.

1. \( K_T \cup K_A \models \text{ grandparent(conn)} \)

   **Solution**
   Due to \( \text{ grandparent } = \text{ parent } \sqcap \exists \text{ child : parent } \)
   
   \( \text{ grandparent(conn) } \) follows from: \( \text{ parent(conn) } \) and \( (\exists Y)(\text{ child(conn, Y) } \land \text{ parent(Y) }) \)

   (a) \( \text{ parent(conn) } \) : given
   (b) \( (\exists Y)(\text{ child(conn, Y) } \land \text{ parent(Y) }) \) :
       yes, since satisfied by carl, due to \( \text{ child(conn, carl) } \) and \( \text{ parent(carl) } \)

2. \( K_T \cup K_A \models \text{ father(carl)} \)

   **Solution**
   Recall: \( \text{ father } = \text{ man } \sqcap \exists \text{ child : person } \)

   Consequently, we have to show:
   \( \text{ man(carl) } \land (\exists Y)(\text{ child(carl, Y) } \land \text{ person(Y) }) \) resp. \( \text{ man(carl) } \land (\exists \text{ child : person})(\text{ carl) } \)

   Given \( \text{ parent(carl) } \) it follows: ( mother(carl) or father(carl) )

   - From mother(carl) it follows: \( \text{ woman(carl) } \) and \( (\exists \text{ child : person})(\text{ carl) } \)
   - From father(carl) it follows: \( \text{ man(carl) } \) and \( (\exists \text{ child : person})(\text{ carl) } \)
Consequently: From (mother(carl) or father(carl)) follows: (woman(carl) or man(carl)) and (∃child : person)(carl) and consequently, (∃child : person)(carl).

On the other hand, man(carl) is given.

3. $\mathcal{K}_T \cup \mathcal{K}_A \models \text{father}_\text{without}_\text{son}(\text{carl})$

**Solution**
That father(carl) holds has been shown in the subproblem above.

We obtain a counter example as follows:

We start with a Herbrand interpretation of the language given by the given formulae and which is a model of the given formulae $\mathcal{K}_T \cup \mathcal{K}_A$. (We assume this interpretation to be represented as a subset of the Herbrand base.)

Then we extend the Herbrand universe by a further constant mike (thus obtaining a new domain)
and we add the atoms child(carl, mike) and man(mike) to the interpretation.

The atom father_without_son(carl) will be deleted if present.

Then carl has a son (named mike).

**Problem 1.2**
Prove that $F \subseteq G \equiv F \cap \neg G = \bot$.

**Solution**
Let $I = (D, \cdots I)$ be an arbitrary interpretation.

$I \models F \subseteq G$ iff $F^I \subseteq G^I$ iff $F^I \cap (D \setminus G^I) = \emptyset$ iff $(F \cap \neg G)^I = \emptyset$ iff $(F \cap \neg G)^I = I^I$ iff $I \models F \cap \neg G = \bot$.

**Problem 1.3**
Show that $\text{grandparent} \sqsubseteq_{\mathcal{K}_T} \text{parent}$ by reducing subsumption into concept satisfiability, where $\mathcal{K}_T$ is the T-Box from Problem 1.1.

**Solution**
We answer the question whether there exists a model $I$ of the terminological knowledge base $\mathcal{K}_T$ such that $(\text{grandparent} \cap \neg \text{parent})^I \neq \emptyset$.
If the answer is no, then $\text{grandparent} \sqsubseteq_{\mathcal{T}} \text{parent}$.
Suppose there exists a model \( I = (\mathcal{D}, \cdots) \) of \( T \) such that \((\text{grandparent} \cap \neg \text{parent})^I \neq \emptyset\). Then, there is \( a \in \mathcal{D} \) such that \( a \in \text{grandparent}^I \) and \( a \notin \text{parent}^I \).

Since \( I \models T \), we conclude with \( a \in \text{grandparent}^I \) and \((\text{grandparent} = \text{parent} \cap \exists \text{child} \cdot \text{parent})\), that \( a \in \text{parent}^I \). But this is a contradiction.

Hence, there is no model \( I \) of \( \mathcal{K}_T \) such that \((\text{grandparent} \cap \neg \text{parent})^I \neq \emptyset\). Consequently, \text{grandparent} \sqsubseteq T \text{parent} \cdot

**Problem 1.4**

Is the concept \((\text{father} \cap \text{mother})\) satisfiable w.r.t. \( \mathcal{K}_T \) from Problem 1.1?

**Solution**

\((\text{father} \cap \text{mother})\) unsatisfiable iff \((\text{father} \cap \text{mother}) = \bot\) for all models of the T-box \( T \).

Therefore, we have to find an interpretation \( I \) where \((\text{father} \cap \text{mother})^I \) is not empty.

The answer is YES. Consider the following interpretation

\( I = \{\text{father}(p), \text{mother}(p), \text{child}(p, c), \text{person}(c), \text{man}(p), \text{woman}(p), \text{person}(p), \text{parent}(p)\} \)

\text{grandparent} and \text{father, without, son} are empty on \( I \).

**Problem 1.5**

1. Which generalized concept axioms must be added to prevent that a person is female and male?

2. Is there a single generalized concept axiom that prevents that a person is female and male?

**Solution**

1. \( \text{father} \sqsubseteq \neg \text{mother} \) and \( \text{mother} \sqsubseteq \neg \text{father} \)

2. \( \text{father} \cap \text{mother} \sqsubseteq \bot \)

**Problem 1.6**

Give an equivalent concept of \((\text{woman} \cap \exists \text{child} \cdot \text{person})\) without using the constructors \( \cap \) and \( \exists r.C \)

**Solution**

e.g. \( \neg(\neg \text{woman} \cup \forall \text{child} \cdot \neg \text{person}) \)

\((\text{woman} \cap \exists \text{child} \cdot \text{person})\) means \((\text{woman}(X) \land (\exists Y)(\text{child}(X, Y) \land \text{person}(Y)))\)

Which we may transform stepwise as follows:

\((\text{woman}(X) \land (\exists Y)(\text{child}(X, Y) \land \text{person}(Y)))\)
\[-(\text{woman}(X) \land (\exists Y)(\text{child}(X,Y) \land \text{person}(Y)))\]
\[-(\neg \text{woman}(X) \lor (\forall Y)(\neg \text{child}(X,Y) \lor \neg \text{person}(Y)))\]
\[-(\neg \text{woman}(X)(\forall Y)(\neg \text{child}(X,Y) \rightarrow \neg \text{person}(Y)))\]
which abbreviates to
\[-(\neg \text{woman} \sqcup \forall \text{child.}\neg \text{person})\]

**Problem 1.7**

Prove the following:

If \((\forall r.C)(a) \in \mathcal{A}\), and \(r(a,b) \in \mathcal{A}\), then \(\mathcal{A} \models C(b)\).

**Solution**

Let \(I \models K\) and suppose \(I \models (\forall r.C)(a)\) and \(I \models r(a,b)\).

Then, \(a^I \in (\forall r.C)^I\) and \(b^I \in r^I(a^I)\).

By the definition of the universal quantifier, \((\forall r : C)^I = \{d \in D \mid r^I(d) \subseteq C^I\}\), so \(r^I(a^I) \subseteq C^I\). Thus \(b^I \in C^I\).

Hence, \(I \models C(b)\).