COMPLEXITY THEORY

Lecture 16: Alternation

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Review
Theorem 14.18 (Baker, Gill, Solovay, 1975): The answer to $P \neq NP$ does not relativise: there are languages $A$ and $B$ such that $P^A = NP^A$ and $P^B \neq NP^B$.

In words: The $P$ vs. $NP$ problem does not relativise, and therefore cannot be solved by any techniques that do.

- Equality was shown using $A = \text{TRUE QBF}$. It is so far not known that this oracle is not in $P$, so this might be the world we are living in.
- Inequality was shown using $B$ that diagonalises against all polytime OTM to show that they cannot decide $L_B$. 
Alternation
Alternating Computations

Non-deterministic TMs:

- Accept if there is an accepting run.
- Used to define classes like NP
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Complements of non-deterministic classes:
- Accept if all runs are accepting.
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We have seen that existential and universal modes can also alternate:
• Players take turns in games
• Quantifiers may alternate in QBF

Is there a suitable Turing Machine model to capture this?
Definition 16.1: An alternating Turing machine (ATM) \( M = (Q, \Sigma, \Gamma, \delta, q_0) \) is a Turing machine with a non-deterministic transition function \( \delta : Q \times \Gamma \rightarrow 2^{Q \times \Gamma \times \{L, R\}} \).

whose set of states is partitioned into existential and universal states:

- \( Q_\exists \): set of existential states
- \( Q_\forall \): set of universal states

- Configurations of ATMs are the same as for (N)TMs:
  - tape(s) + state + head position
- A configuration can be universal or existential, depending on whether its state is universal or existential
- Possible transitions between configurations are defined as for NTMs
Acceptance is defined inductively:

**Definition 16.2:** The set of accepting configurations of an ATM $\mathcal{M}$ is the least set of configurations $C$ for which either of the following is true:

- $C$ is existential and some successor configuration of $C$ is accepting.
- $C$ is universal and all successor configurations of $C$ are accepting.

$\mathcal{M}$ accepts a word $w$ if the start configuration on $w$ is accepting.
Alternating Turing Machines: Acceptance

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**Note 1:** configurations with no successor are a base case, since we have:
- An existential configuration without any successor configurations is rejecting.
- A universal configuration without any successor configurations is accepting.

Hence we don’t need to specify accepting or rejecting states explicitly.

**Note 2:** defining this to be the least set implies that infinite runs are never enough to declare a configuration to be accepting.
Nondeterminism and Parallelism

ATMs can be seen as a generalisation of non-deterministic TMs:

An NTM is an ATM where all states are existential (besides the single accepting state, which is always universal according to our definition).
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ATMs can be seen as a model of parallel computation:

In every step, fork the current process to create sub-processes that explore each possible transition in parallel

- for universal states, combine the results of sub-processes with AND
- for existential states, combine the results of sub-processes with OR

Alternative view: an ATM accepts if its computation tree, considered as an AND-OR tree, evaluates to true
Example: Alternating Algorithm for MinFormula

**MinFormula**

**Input:** A propositional formula $\varphi$.

**Problem:** Is $\varphi$ the shortest formula that is satisfied by the same assignments as $\varphi$?
Example: Alternating Algorithm for MinFormula

\[
\text{MinFormula}
\]

Input: A propositional formula \( \varphi \).
Problem: Is \( \varphi \) the shortest formula that is satisfied by the same assignments as \( \varphi \)?

\[
\text{MinFormula} \text{ can be solved by an alternating algorithm:}
\]

\begin{verbatim}
01 MinFormula(formula \( \varphi \)):
02 universally choose \( \psi := \text{formula shorter than } \varphi \)
03 existentially guess \( I := \text{assignment for variables in } \varphi \)
04 if \( \varphi^I = \psi^I \):
05 return false
06 else :
07 return true
\end{verbatim}
Example: Alternating Algorithm for Geography

Recall the **Geography** game discussed in Lecture 10:

```python
01 ALTGEOGRAPHY(directed graph G, start node s) :
02     Visited := \{s\}  // visited nodes
03     cur := s  // current node
04     while true :
05         // existential move:
06         if all successors of cur are in Visited:
07             return false
08         existentially guess cur := unvisited successor of cur
09         Visited := Visited ∪ \{cur\}
10         // universal move:
11         if all successors of cur are in Visited:
12             return true
13         universally choose cur := unvisited successor of cur
14         Visited := Visited ∪ \{cur\}
```
As before, time and space bounds apply to any computation path in the computation tree.

**Definition 16.3:** Let $M$ be an alternating Turing machine and let $f : \mathbb{N} \rightarrow \mathbb{R}^+$ be a function.

1. $M$ is $f$-time bounded if it halts on every input $w \in \Sigma^*$ and on every computation path after $\leq f(|w|)$ steps.
2. $M$ is $f$-space bounded if it halts on every input $w \in \Sigma^*$ and on every computation path using $\leq f(|w|)$ cells on its tapes.

(Here we typically assume that Turing machines have a separate input tape that we do not count in measuring space complexity.)
Definition 16.4: Let $f : \mathbb{N} \to \mathbb{R}^+$ be a function.

(1) $\text{ATime}(f(n))$ is the class of all languages $L$ for which there is an $O(f(n))$-time bounded alternating Turing machine deciding $L$.

(2) $\text{ASpace}(f(n))$ is the class of all languages $L$ for which there is an $O(f(n))$-space bounded alternating Turing machine deciding $L$. 
Common Alternating Complexity Classes

\[
\begin{align*}
AP &= \text{APTime} = \bigcup_{d \geq 1} \text{ATime}(n^d) & \text{alternating polynomial time} \\
AExp &= \text{AExpTime} = \bigcup_{d \geq 1} \text{ATime}(2^{n^d}) & \text{alternating exponential time} \\
A2Exp &= \text{A2ExpTime} = \bigcup_{d \geq 1} \text{ATime}(2^{2^{n^d}}) & \text{alt. double-exponential time} \\
AL &= \text{ALogSpace} = \text{ASpace}(\log n) & \text{alternating logarithmic space} \\
\text{APSpace} &= \bigcup_{d \geq 1} \text{ASpace}(n^d) & \text{alternating polynomial space} \\
\text{AExpSpace} &= \bigcup_{d \geq 1} \text{ASpace}(2^{n^d}) & \text{alternating exponential space}
\end{align*}
\]

Example 16.5: Geography \(\in\) APTime.
Nondeterminism:
ATMs can do everything that the corresponding NTMs can do, e.g., $\text{NP} \subseteq \text{APTime}$.
Alternating Complexity Classes: Basic Properties

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Reductions: Polynomial many-one reductions can be used to show membership in many alternating complexity classes, e.g., if \( L \in \text{APTime} \) and \( L' \leq_p L \) then \( L' \in \text{APTime} \).
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In particular: \( PSpace \subseteq APTime \) (since \( \text{Geography} \in APTime \))
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In particular: PSpace ⊆ APTime (since Geography ∈ APTime)

Complementation: ATMs are easily complemented:
- Let M be an ATM accepting language L(M)
- Let M′ be obtained from M by swapping existential and universal states
- Then L(M′) = L(M)

For alternating algorithms this means: (1) negate all return values, (2) swap universal and existential branching points
Example: Complement of \textsc{MinFormula}

Original algorithm:

\begin{verbatim}
01 \textbf{MinFormula}(\text{formula } \varphi) :
02 universally choose \psi := \text{formula shorter than } \varphi
03 existentially guess \ I := \text{assignment for variables in } \varphi
04 if \varphi^I = \psi^I :
05 \quad return false
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Complemented algorithm:

\begin{verbatim}
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\end{verbatim}
Alternating Time vs. Deterministic Space
Theorem 16.6: For $f(n) \geq n$, we have $\text{ATime}(f) \subseteq \text{DSpace}(f^2)$. 

Proof: We simulate an ATM $M$ using a TM $S$:

- $S$ performs a depth-first search of the configuration tree of $M$.
- The acceptance status of each node is computed recursively (similar to typical PSpace algorithms we have seen before).
- $M$ accepts exactly if the root of the configuration tree is accepting.

The maximum recursion depth is $f(n)$. The maximum size of a configuration is $O(f(n))$. Hence the claim follows. □

Note: The result can be strengthened to $\text{ATime}(f) \subseteq \text{DSpace}(f)$ by not storing the whole configuration. See [Sipser, Lemma 10.22].
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Challenge: the computing paths of \( M \) might be up to \( 2^{df(n)} \) in length.
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Solution: recursively solve Yieldability problems, as in Savitch’s Theorem:

- We want to check if $M$ can go from configuration $C_1$ to $C_2$ in at most $k$ steps
- To do this, existentially guess an intermediate configuration $C'$.
- Universally check if $M$ can go from $C_1$ to $C'$ in $k/2$ steps, and from $C'$ to $C_2$ in $k/2$ steps.

Storing one intermediate configuration $C'$ takes space $O(f(n))$. Maximal recursion depth is $O(f(n))$. Hence the result follows. □
Harvest: Alternating Time = Deterministic Space

For \( f(n) \geq n \), we have shown

\[
\text{ATime}(f) \subseteq \text{DSpace}(f^2) \quad \text{and} \quad \text{DSpace}(f) \subseteq \text{NSpace}(f) \subseteq \text{ATime}(f^2).
\]

The quadratic increase is swallowed by (super)polynomial bounds:

**Corollary 16.8 ("Alternating Time = Deterministic Space"):** \( \text{APTime} = \text{PSpace} \) and \( \text{AExpTime} = \text{ExpSpace} \).

**Proof:**

- \( \text{ATime}(n^d) \subseteq \text{DSpace}(n^{2d}) \subseteq \text{PSpace} \)
  - \( \text{DSpace}(n^d) \subseteq \text{NSpace}(n^d) \subseteq \text{ATime}(n^{2d}) \subseteq \text{APTime} \)
- Second claim is left as an exercise

One can also read this as “Parallel Time = Sequential Space.”
Alternating Space vs. Deterministic Time
In this direction, the increase is exponential:

**Theorem 16.9:** For $f(n) \geq \log n$, we have $\text{ASpace}(f) \subseteq \text{DTime}(2^{O(f)})$. 

Proof: The proof is similar to the exponential deterministic simulation of space-bounded NTMs in Lecture 9 (Theorem 9.7):

- Construct configuration graph of ATM
- Iteratively compute acceptance status of each configuration
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Each step can be done in exponential time (in particular, computing the acceptance condition in each step is no more difficult than for plain NTMs). □
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From Deterministic Time To Alternating Space

The exponential blow-up can be reversed when going back to ATMs:

**Theorem 16.10:**
If $f(n) \geq \log n$ is space-constructible, then $\text{DTime}(2^{O(f)}) \subseteq \text{ASpace}(f)$. 

Proof:
We show: for any $g(n) \geq n$, we have $\text{DTime}(g) \subseteq \text{ASpace}(\log g)$.

We simulate a TM $M$ using an ATM $S$. This is not so easy:

- A computation of $M$ is exponentially longer than the space available to $S$ – we solved this before with Yieldability.
- A configuration of $M$ is exponentially longer than the space available to $S$ – this is more tricky . . .

There is a coarse proof sketch in [Sipser, Lemma 10.25]. We follow a more detailed proof from the lecture notes of Erich Grädel [Complexity Theory, WS 2009/10] (link).
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Notation: The proof is easier if we write a configuration $\sigma_1 \cdots \sigma_{i-1} q \sigma_i \sigma_{i+1} \cdots \sigma_m$ as a sequence

$$* \sigma_1 \cdots \sigma_{i-1} \langle q, \sigma_i \rangle \sigma_{i+1} \cdots \sigma_m *$$

of symbols from the set $\Omega = \{*\} \cup \Gamma \cup (Q \times \Gamma)$. 
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Then the \( \Omega \)-symbol (state and tape) at position \( i \) follows deterministically from the \( \Omega \)-symbols at positions \( i - 1, i, \) and \( i + 1 \) in the previous step. We write \( M(\omega_{i-1}, \omega_i, \omega_{i+1}) \) for this symbol.
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of symbols from the set $\Omega = \{*\} \cup \Gamma \cup (Q \times \Gamma)$.

Then the $\Omega$-symbol (state and tape) at position $i$ follows deterministically from the $\Omega$-symbols at positions $i - 1$, $i$, and $i + 1$ in the previous step.

We write $M(\omega_{i-1}, \omega_i, \omega_{i+1})$ for this symbol.

**Proof idea:**

- Only store a pointer to one cell in one configuration of $M$
- Verify the contents of current cell $i$ in step $j$ by guessing the previous cell contents $\omega_{i-1}, \omega_i, \omega_{i+1}$ in step $j$.
- Check iteratively that the guessed symbols are correct
Let $h : \mathbb{N} \to \mathbb{R}$ be a function in $O(g)$ that defines the exact time bound for $M$ (no $O$-notation), and that can be computed in space $O(\log g)$.

```plaintext
01 ATM_SIMULATE_TM(TM M, input word w, time bound h) :
02   existentially guess $s \leq h(|w|)$ // halting step
03   existentially guess $i \in \{0, ..., s\}$ // halting position
04   existentially guess $\omega \in Q \times \Gamma$ // halting cell + state
05   if $M$ would not halt in $\omega$ :
06       return false
07   for $j = s, ..., 1$ do :
08       existentially guess $\langle \omega_{-1}, \omega_0, \omega_1 \rangle \in \Omega^3$
09       if $M(\omega_{-1}, \omega_0, \omega_{+1}) \neq \omega$ :
10          return false
11   universally choose $\ell \in \{-1, 0, 1\}$
12       $\omega := \omega_\ell$
13       $i := i + \ell$
14   // after tracing back $s$ steps, check input configuration:
15   return “input configuration of $M$ on $w$ has $\omega$ at position $i”
```
Summary and Outlook

For $f(n) \geq \log n$, we have shown $\text{ASpace}(f) = \text{DTime}(2^{O(f)})$.

**Corollary 16.11 ("Alternating Space = Exponential Deterministic Time"):**

$\text{AL} = \text{P}$ and $\text{APSpace} = \text{ExpTime}$.
Summary and Outlook

For $f(n) \geq \log n$, we have shown $\text{ASpace}(f) = \text{DTime}(2^{O(f)})$.

**Corollary 16.11 ("Alternating Space = Exponential Deterministic Time"):**

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We can sum up our findings as follows:

- $L \subseteq \text{PTime} \subseteq \text{PSpace} \subseteq \text{ExpTime} \subseteq \text{ExpSpace}$
- $\text{ALogSpace} \subseteq \text{APTTime} \subseteq \text{APSpace} \subseteq \text{AExpTime}$

**What’s next?**

- Alternation as a resource that can be bounded
- A hierarchy between NP and PSpace
- End-of-year consultation