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# Normal-Form Games: Mixed Strategies

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# Previously . . .

- Games can model real-life situations, but model fidelity is important.
- Noncooperative (strategic) games in **normal form** comprise players, **strategies** for the players, and **gain functions** for all **strategy profiles**.
- Various concepts can help predict/analyse the outcome of a game:
  - **Dominant strategies**
  - **Pareto optimality**
  - (pure) **Nash equilibria**
- We have analysed a number of example games: prisoner's dilemma, battle of the partners, chicken, penalties, and two thirds of the average.
- Pure Nash equilibria need not always exist.

**Chicken**

Two people, `Eli` and `Fyn`, are racing towards each other in cars. Whoever swerves ("chickens out") loses face. If neither swerves, both get seriously injured.

| (`Eli`, `Fyn`) | `Swerve` | `RaceOn` |
|---|---|---|
| `Swerve` | (2, 2) | (1, 3) |
| `RaceOn` | (3, 1) | (0, 0) |

# Overview

Motivation

Mixed Strategies and Equilibria

Computation of Nash Equilibria

# Motivation

# Motivation

- So far we considered (Nash equilibria in) pure strategies.
- For some games, such pure equilibria did not exist . . . , e.g. penalties:

**Penalties**

Two football players face off at a (simplified) single penalty kick. The kicker can kick left or right; the goal keeper can jump left or right. The kicker scores a goal iff they choose a different side than the keeper.

| (Kicker, Keeper) | JumpL | JumpR |
|---|---|---|
| KickL | (−1, 1) | (1, −1) |
| KickR | (1, −1) | (−1, 1) |

What does that predict about how the game will be played?

# Background

## Recall

A (discrete) **probability distribution** on a countable set $S$ is a function

$$\pi\colon S \to [0,1] \quad \text{such that} \quad \sum_{s\in S} \pi(s) = 1$$

where $[0,1] \subseteq \mathbb{R}$ is the real unit interval.

$\pi(s)$ represents the probability of a random variable taking the value $s \in S$.

## Definition

A **lottery** $(S,\pi)$ consists of a countable set $S$ and a probability distribution $\pi$ on $S$.

## Assumption: Expected Utility Hypothesis

If $(S,\pi)$ is a lottery and there is a function $u\colon S \to \mathbb{R}$ assigning a utility to each outcome $s \in S$, then the expected utility of the lottery is given by

$$U(S,\pi) := \sum_{s\in S} u(s)\pi(s)$$

# Lotteries and Risk Neutrality

## Example

Consider the following events: and lotteries:

- $e_0$: "You get 0€."
- $e_1$: "You get 100€."
- $e_2$: "You get 200€."

- $L_1 = (\{e_1\}, \{e_1 \mapsto 1.0\})$
- $L_2 = (\{e_0, e_2\}, \{e_0 \mapsto 0.5, e_2 \mapsto 0.5\})$

Which of these lotteries would you prefer?

## Terminology

- A **risk neutral** player is one who is indifferent between $L_1$ and $L_2$;
- a **risk averse** player is one who prefers $L_1$ over $L_2$;
- a **risk seeking** player is one who prefers $L_2$ over $L_1$.

We assume throughout this course that players are risk neutral.

# Mixed Strategies and Equilibria

# Nash Equilibrium in Mixed Strategies

## Definition

Let $(P, \mathbf{S}, \mathbf{u})$ be a game in normal form with $S_i$ finite for all $i \in P$.

1. A **mixed strategy for player** $i \in P$ is a probability distribution $\pi_i$ on $S_i$.
   - $\pi_i(s_j)$ is the probability of the event that player $i$ chooses strategy $s_j \in S_i$.
   - $\Pi_i$ denotes the set of all probability distributions on $S_i$, for each $1 \leq i \leq n$.
   - Denote $\boldsymbol{\Pi} := \Pi_1 \times \cdots \times \Pi_n$ and $\boldsymbol{\Pi}_{-i} := \Pi_1 \times \cdots \times \Pi_{i-1} \times \Pi_{i+1} \times \cdots \times \Pi_n$.
2. The **expected utility of a mixed-strategy profile** $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n)$ for $i$ is

$$U_i(\boldsymbol{\pi}) := \sum_{\mathbf{s}=(s_1,\ldots,s_n)\in\mathcal{S}} u_i(\mathbf{s}) \cdot \prod_{j=1}^{n} \pi_j(s_j)$$

- $\boldsymbol{\Pi}$ is the set of all mixed-strategy profiles for all players.
- $\boldsymbol{\Pi}_{-i}$ is the set of all mixed-strategy profiles for all players except $i$.
- Likewise, for $\boldsymbol{\pi} \in \boldsymbol{\Pi}$ and $i \in P$, we have $\boldsymbol{\pi}_{-i} \in \boldsymbol{\Pi}_{-i}$.

# Mixed Strategies: Examples

## Notation

Let $(P, \mathbf{S}, \mathbf{u})$ be a game with $P = \{1, \ldots, n\}$ and $S_i = \{s_1, \ldots, s_{k_i}\}$ for $1 \leq i \leq n$.
We denote a mixed strategy $\pi_i$ for player $i$ as a $k_i$-tuple $\pi_i = (\pi_i(s_1), \ldots, \pi_i(s_{k_i}))$.

## Examples

- In penalties, a mixed strategy for `Kicker` is

$$\pi_{\texttt{Kicker}} = (\pi_{\texttt{Kicker}}(\texttt{KickL}), \pi_{\texttt{Kicker}}(\texttt{KickR})) = \left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

  Another mixed strategy for `Kicker` is $\pi_{\texttt{Kicker}} = \left(\tfrac{2}{3}, \tfrac{1}{3}\right)$.

- In rock-paper-scissors, a mixed strategy for player `Ann` is

$$\pi_{\texttt{Ann}} = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right)$$

# Best Responses and Mixed Nash Equilibria

## Definition

Let $(P, \mathbf{S}, \mathbf{u})$ be a game in normal form and assume that all $S_i$ are finite.

1. A mixed strategy $\pi_i \in \Pi_i$ is a **best response** of player $i$ to the mixed-strategy profile $\boldsymbol{\pi}_{-i} \in \boldsymbol{\Pi}_{-i}$ iff for all mixed strategies $\pi_i' \in \Pi_i$, we have
$$U_i(\pi_1, \ldots, \pi_{i-1}, \pi_i, \pi_{i+1}, \ldots, \pi_n) \geq U_i(\pi_1, \ldots, \pi_{i-1}, \pi_i', \pi_{i+1}, \ldots, \pi_n)$$
2. A profile $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n)$ is (in) a **Nash equilibrium in mixed strategies** iff mixed strategy $\pi_i$ is a best response to $\boldsymbol{\pi}_{-i}$ for all players $1 \leq i \leq n$.

## Examples

- In penalties, assume `Kicker` plays $\left(\frac{1}{2}, \frac{1}{2}\right)$. A best response of `Keeper` to this is $\left(\frac{1}{2}, \frac{1}{2}\right)$; other best responses are $(1, 0)$ and $(0, 1)$.
- In Rock-Paper-Scissors, a best response to $\left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right)$ is $(0, 1, 0)$.

# Mixed Nash Equilibria Generalise Pure NE

**Observation**

A pure strategy $s_j \in S_i$ is a special case of a mixed strategy $\pi_i$ with $\pi_i(s_j) = 1$ and $\pi_i(s_k) = 0$ for all $s_k \in S_i$ with $k \neq j$. We conveniently denote such $\pi_i$ by $s_j$.

**Theorem**

Every pure Nash equilibrium is also a mixed Nash equilibrium.

**Proof.**

$$
\begin{aligned}
& \mathbf{s} \text{ is a pure NE} \\
\Longrightarrow\ & \forall i \in P\text{: } s_i \text{ is a pure best response to } \mathbf{s}_{-i} \\
\Longrightarrow\ & \forall i \in P\text{: } \forall s_j \in S_i\text{: } u_i(\mathbf{s}_{-i}, s_i) \geq u_i(\mathbf{s}_{-i}, s_j) \\
\Longrightarrow\ & \forall i \in P\text{: } \forall s_j \in S_i\text{: } U_i(\mathbf{s}_{-i}, s_i) \geq U_i(\mathbf{s}_{-i}, s_j) \\
\Longrightarrow\ & \forall i \in P\text{: } s_i \text{ is a mixed best response to } \mathbf{s}_{-i} \\
\Longrightarrow\ & \mathbf{s} \text{ is a mixed NE}
\end{aligned}
$$

□

The converse is not the case: there are “proper” mixed Nash equilibria.

# Mixed Nash Equilibria: Characterisation (1)

**Definition**

The **support** of mixed strategy $\pi_i$ for player $i$ is the set $S^+_{\pi_i} := \{s \in S_i \mid \pi_i(s) > 0\}$.

**Theorem**

Let $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n)$ be a mixed-strategy profile in a game in normal form. A mixed strategy $\pi_i$ is a best response to mixed-strategy profile $\boldsymbol{\pi}_{-i}$ if and only if all pure strategies in the support of $\pi_i$ are best responses to $\boldsymbol{\pi}_{-i}$.

**Proof (Sketch).**

"if": For any two $s, t \in S^+_{\pi_i}$, we get $U_i(\boldsymbol{\pi}_{-i}, s) = U_i(\boldsymbol{\pi}_{-i}, t)$ and for any mix $\pi\colon S^+_{\pi_i} \to [0,1]$, we obtain $U_i(\boldsymbol{\pi}_{-i}, s) = U_i(\boldsymbol{\pi}_{-i}, t) = U_i(\boldsymbol{\pi}_{-i}, \pi) = U_i(\boldsymbol{\pi}_{-i}, \pi_i)$.

"only if": If $s \in S^+_{\pi_i}$ is in the support of $\pi_i$ but not a best response to $\boldsymbol{\pi}_{-i}$, we can remove $s$ from $\pi_i$ to obtain a better response, contradiction. □

# Mixed Nash Equilibria: Characterisation (2)

By definition, a mixed-strategy profile $\boldsymbol{\pi}$ is a Nash equilibrium iff for all $i \in P$,

$$U_i(\boldsymbol{\pi}) = \max_{\pi_i' \in \Pi_i} U_i(\boldsymbol{\pi}_{-i}, \pi_i')$$

where $U_i(\boldsymbol{\pi}_{-i}, \pi_i')$ denotes $U_i(\pi_1, \ldots, \pi_{i-1}, \pi_i', \pi_{i+1}, \ldots, \pi_n)$.

By the previous theorem it is enough to focus on the pure strategies, thus

$$\max_{\pi_i' \in \Pi_i} U_i(\boldsymbol{\pi}_{-i}, \pi_i') = \max_{s_j \in S_i} U_i(\boldsymbol{\pi}_{-i}, s_j)$$

Additionally, for all $i \in P$ and for any two $s_i, t_i \in S_{\pi_i}^+ = \{s \in S_i \mid \pi_i(s) > 0\}$:

$$U_i(\boldsymbol{\pi}) = \max_{s_j \in S_i} U(\boldsymbol{\pi}_{-i}, s_j) = U(\boldsymbol{\pi}_{-i}, s_i) = U(\boldsymbol{\pi}_{-i}, t_i)$$

Thus in a Nash equilibrium, players are indifferent between their pure strategies.

# Mixed Nash Equilibria: Examples (1)

## Battle of the Partners

- By the previous theorem, both pure Nash equilibria are also mixed ones.
- A third, mixed Nash equilibrium $\boldsymbol{\pi}$ is obtained by reasoning as follows:

$$U_{\texttt{Cat}}(\texttt{Cinema}) = 10 \cdot \pi_{\texttt{Dee}}(\texttt{Cinema}) + 2 \cdot \pi_{\texttt{Dee}}(\texttt{Dancing})$$
$$U_{\texttt{Cat}}(\texttt{Dancing}) = 0 \cdot \pi_{\texttt{Dee}}(\texttt{Cinema}) + 7 \cdot \pi_{\texttt{Dee}}(\texttt{Dancing})$$

To make Cat indifferent between the two choices, Dee must choose the values for $\pi_{\texttt{Dee}}(\texttt{Cinema})$ and $\pi_{\texttt{Dee}}(\texttt{Dancing})$ such that $U_{\texttt{Cat}}(\texttt{Cinema}) = U_{\texttt{Cat}}(\texttt{Dancing})$, i.e.

$$10 \cdot \pi_{\texttt{Dee}}(\texttt{Cinema}) + 2 \cdot \pi_{\texttt{Dee}}(\texttt{Dancing}) = 7 \cdot \pi_{\texttt{Dee}}(\texttt{Dancing})$$

With $\pi_{\texttt{Dee}}(\texttt{Cinema}) + \pi_{\texttt{Dee}}(\texttt{Dancing}) = 1$, we obtain $\pi_{\texttt{Dee}} = \left(\frac{1}{3}, \frac{2}{3}\right)$.

By symmetry, we get $\boldsymbol{\pi} = (\pi_{\texttt{Cat}}, \pi_{\texttt{Dee}}) = \left(\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right)\right)$ with $U_{\texttt{Cat}}(\boldsymbol{\pi}) = 4\frac{2}{3}$.

# Mixed Nash Equilibria: Examples (2)

## Penalties

- For the mixed strategies $\pi_{\texttt{Kicker}} = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $\pi_{\texttt{Keeper}} = \left(\frac{1}{2}, \frac{1}{2}\right)$, the mixed-strategy profile $\boldsymbol{\pi} = (\pi_{\texttt{Kicker}}, \pi_{\texttt{Keeper}})$ is a (strict) Nash equilibrium with expected utilities $U_{\texttt{Kicker}}(\boldsymbol{\pi}) = 2 \cdot \frac{1}{4} \cdot (-1) + 2 \cdot \frac{1}{4} \cdot 1 = 0 = U_{\texttt{Keeper}}(\boldsymbol{\pi})$:
- If (e.g.) `Kicker` were to deviate by (e.g.) playing $\boldsymbol{\pi}' = \left(\frac{2}{3}, \frac{1}{3}\right)$, then `Keeper` would best-respond by playing $(1, 0)$, that is, playing `JumpL`, leading to expected utilities $U_{\texttt{Kicker}}(\boldsymbol{\pi}') = \frac{2}{3} \cdot (-1) + \frac{1}{3} \cdot 1 = -\frac{1}{3} = -U_{\texttt{Keeper}}(\boldsymbol{\pi}')$.

## Rock-Paper-Scissors

- Similarly, for $\pi_{\texttt{Ann}} = \pi_{\texttt{Bob}} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, the mixed-strategy profile $\boldsymbol{\pi} = (\pi_{\texttt{Ann}}, \pi_{\texttt{Bob}})$ is a (strict) Nash equilibrium in mixed strategies.
- Note that every pure strategy is a best response to $\boldsymbol{\pi}$.

# Mixed Strategies: Discussion

**Question**

What does it mean to play a mixed strategy?

There are a (at least) four answers:

- Players may randomise to confuse their opponents.
- Players randomise because they are uncertain about actions of others.
- Mixed strategies describe what might happen in repeated play.
- Mixed strategies describe population dynamics: they describe the probability of choosing a specific pure strategy out of a population of pure strategies.

# Nash's Theorem

**Theorem (Nash, 1950)**

Let $G = (P, \mathbf{S}, \mathbf{u})$ be a noncooperative game in normal form.
If $P = \{1, \ldots, n\}$ is finite and for $\mathbf{S} = (S_1, \ldots, S_n)$ each $S_i$ is finite, then there exists a Nash equilibrium in mixed strategies.

**Proof Sketch.**

- View pure strategies $s_i \in S_i$ as unit vectors in $\mathbb{R}^{|S_i|}$; mixed strategies $\pi_i \in \Pi_i$ are then points of a simplex, a convex subset of $\mathbb{R}^{|S_i|}$; $\boldsymbol{\Pi}$ is a simplotope.
- Define functions $\varphi_{ij}(\boldsymbol{\pi}) = \max\{0, U_i(\boldsymbol{\pi}_{-i}, s_j) - U_i(\boldsymbol{\pi})\}$ for $i \in P$, $1 \leq j \leq |S_i|$.
- Define (continuous) function $f \colon \boldsymbol{\Pi} \to \boldsymbol{\Pi}$ with $\boldsymbol{\pi} \mapsto \boldsymbol{\pi}' = (\pi_1', \ldots, \pi_n')$, where

$$\pi_i'(s_j) := \frac{\pi_i(s_j) + \varphi_{ij}(\boldsymbol{\pi})}{\sum_{s_k \in S_i} (\pi_i(s_k) + \varphi_{ik(\boldsymbol{\pi})})}$$

- Use Brouwer's fixpoint theorem to deduce that $f$ has at least one fixpoint.
- Show that $f(\boldsymbol{\pi}) = \boldsymbol{\pi}$ if and only if $\boldsymbol{\pi}$ is a mixed Nash equilibrium for $G$. □

# Computation of Nash Equilibria

# Approaches to Find Equilibria for Two Players

## Lemke-Howson algorithm (1964)

- Path-finding approach with geometrical interpretation
- Needs exponentially many steps in the worst case

## Porter-Nudelman-Shoham (2004)

- Enumerates possible supports of mixed strategies, checks for equilibria
- Dominance checks and search bias for optimisation

## Mixed Integer Programming (Sandholm, Gilpin, and Conitzer, 2005)

- Encode equilibrium property for a given game as a mixed integer program, i.e., as a mathematical (numerical) feasibility problem
- "Mixed" expresses that values for some variables may be real numbers

# Mixed Integer Programming (in a Nutshell)

## Definition

- A **mixed integer (linear) program** is of the form

$$\begin{aligned} \text{maximise} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b}, \\ & \mathbf{x} \geq 0, \\ \text{and} \quad & \mathbf{x} \in \mathbb{Z}^k \times \mathbb{R}^\ell \end{aligned}$$

  where $\mathbf{x}$ is a vector of **decision variables**, and $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$ are a matrix and two vectors of real values; the expression $\mathbf{c}^T\mathbf{x}$ is the **objective function**.
- If there is no objective function the program is a **feasibility problem**.
- A **solution** is a variable-value assignment that satisfies all constraints.

E.g.: maximise $2x_1 - 3x_2$ subject to $x_1 + x_2 \leq 7$, $2x_1 - x_2 \leq 12$, $x_1, x_2 \geq 0$, and $(x_1, x_2) \in \mathbb{Z} \times \mathbb{R}$

Area of active research; used in industrial applications; solvers exist, e.g. SCIP.

# Regret

## Definition

Let $(P, \mathbf{S}, \mathbf{u})$ be a noncooperative game in normal form, $i \in P$, and $s_j \in S_i$.
The **regret** of $i$ playing $s_j$ w.r.t. opponent profile $\boldsymbol{\pi}_{-i}$ is

$$r_{\boldsymbol{\pi}_{-i}, s_j} := \left( \max_{\pi_k \in \Pi_i} U_i(\boldsymbol{\pi}_{-i}, \pi_k) \right) - U_i(\boldsymbol{\pi}_{-i}, s_j)$$

## Example: Prisoner's Dilemma

- The regret of playing `Silent` in response to `Confess` is 1.
- The regret of `Confess` in response to any opponent strategy (profile) is 0.

More generally: The regret of any best response is zero.

## Observation

A mixed strategy profile $\boldsymbol{\pi}$ is a Nash equilibrium if and only if every pure strategy is either played with probability zero or has zero regret.

# Computing Nash Equilibria via MIP (1)

For every player $i \in P$ and pure strategy $s_j \in S_i$, introduce decision variables

- $b_{s_j}$ … to express that $s_j$ is *not* played by $i$, i.e.
  - $b_{s_j} = 1$ expresses that $\pi_i(s_j) = 0$, and
  - $b_{s_j} = 0$ expresses that $\pi_i(s_j) > 0$ and $r_{\boldsymbol{\pi}_{-i}, s_j} = 0$;
- $u_i$ … to express the maximal utility of $i$ given $\boldsymbol{\pi}_{-i}$;
- $p_{s_j}$ … to express the probability $\pi_i(s_j)$ with which $s_j$ is played;
- $u_{s_j}$ … to express the expected utility $U(\boldsymbol{\pi}_{-i}, s_j)$ from playing $s_j$;
- $r_{s_j}$ … to express the regret $r_{\boldsymbol{\pi}_{-i}, s_j}$ from playing $s_j$.

The formulation also uses the constants $v_i$, denoting the maximally possible difference between two payoffs for player $i$:

$$v_i := \max_{\substack{s_u^{(i)}, s_\ell^{(i)} \in S_i, \\ s_u^{(3-i)}, s_\ell^{(3-i)} \in S_{3-i}}} \left\{ u_i(s_u^{(1)}, s_u^{(2)}) - u_i(s_\ell^{(1)}, s_\ell^{(2)}) \right\}$$

(Note that $3 - i$ for $i \in P = \{1, 2\}$ just refers to the player other than $i$.)

# Computing Nash Equilibria via MIP (2)

## Definition

Let $G = (P, \mathbf{S}, \mathbf{u})$ be a strategic game with $P = \{1, 2\}$. Its MIP formulation is

$$\text{find} \quad p_{s_j}, u_i, u_{s_j}, r_{s_j}, b_{s_j} \quad \text{such that}$$

$$\forall i : \quad \sum_{s_j \in S_i} p_{s_j} = 1 \tag{1}$$

$$\forall i : \forall s_j \in S_i : \quad u_{s_j} = \sum_{s_k \in S_{3-i}} u_i(s_k, s_j) p_{s_k} \tag{2}$$

$$\forall i : \forall s_j \in S_i : \quad r_{s_j} = u_i - u_{s_j} \tag{3}$$

$$\forall i : \forall s_j \in S_i : \quad p_{s_j} \leq 1 - b_{s_j} \tag{4}$$

$$\forall i : \forall s_j \in S_i : \quad r_{s_j} \leq v_i b_{s_j} \tag{5}$$

$$\text{where} \quad p_{s_j}, u_i, u_{s_j}, r_{s_j} \geq 0 \quad \text{and} \quad b_{s_j} \in \{0, 1\}$$

# Computing Nash Equilibria via MIP (3)

The intuition behind the constraints is as follows:

(1) Guarantees that the values of the $p_{s_j}$ constitute a valid probability distribution for each player.

(2) Guarantees the correct utility value for playing $s_j$ (where $u_i(s_k, s_j)$ denotes the constant $u_1(s_j, s_k)$ for $i = 1$ and the constant $u_2(s_k, s_j)$ otherwise).

(3) Guarantees the correct regret value for playing $s_j$.

(4) Guarantees that the probability of playing $s_j$ is indeed zero whenever the strategy is chosen not to be played (otherwise the constraint is vacuous).

(5) Guarantees that the regret for playing $s_j$ is indeed zero whenever the strategy is chosen to be played (otherwise the constraint is vacuous).

**Proposition**

For any two-player strategic game, the solutions of the MIP formulation correspond one-to-one to the mixed Nash equilibria of the game.

# Computing Nash Equilibria via MIP: Example

Consider Battle of the Partners with $P = \{\texttt{Cat}, \texttt{Dee}\}$ and the payoffs on the right.

| (Cat, Dee) | Cinema | Dancing |
|---|---|---|
| Cinema | (10, 7) | (2, 2) |
| Dancing | (0, 0) | (7, 10) |

We obtain the following feasibility problem:

$$\text{find} \quad p_s^i, u_i, u_s^i, r_s^i, b_s^i \quad \text{for} \quad i \in P, s \in \{\texttt{Cinema}, \texttt{Dancing}\} \quad \text{such that}$$

$$p_{\texttt{Cinema}}^{\texttt{Cat}} + p_{\texttt{Dancing}}^{\texttt{Cat}} = 1 \qquad \text{and} \qquad p_{\texttt{Cinema}}^{\texttt{Dee}} + p_{\texttt{Dancing}}^{\texttt{Dee}} = 1$$

$$u_{\texttt{Cinema}}^{\texttt{Cat}} = 10 \cdot p_{\texttt{Cinema}}^{\texttt{Dee}} + 2 \cdot p_{\texttt{Dancing}}^{\texttt{Dee}} \qquad \text{and} \qquad u_{\texttt{Cinema}}^{\texttt{Dee}} = 7 \cdot p_{\texttt{Cinema}}^{\texttt{Cat}} + 0 \cdot p_{\texttt{Dancing}}^{\texttt{Cat}}$$

$$u_{\texttt{Dancing}}^{\texttt{Cat}} = 0 \cdot p_{\texttt{Cinema}}^{\texttt{Dee}} + 7 \cdot p_{\texttt{Dancing}}^{\texttt{Dee}} \qquad \text{and} \qquad u_{\texttt{Dancing}}^{\texttt{Dee}} = 2 \cdot p_{\texttt{Cinema}}^{\texttt{Cat}} + 10 \cdot p_{\texttt{Dancing}}^{\texttt{Cat}}$$

$$\forall i \in P: \quad r_{\texttt{Cinema}}^i = u_i - u_{\texttt{Cinema}}^i \qquad \text{and} \qquad r_{\texttt{Dancing}}^i = u_i - u_{\texttt{Dancing}}^i$$

$$\forall i \in P: \quad p_{\texttt{Cinema}}^i \leq 1 - b_{\texttt{Cinema}}^i \qquad \text{and} \qquad p_{\texttt{Dancing}}^i \leq 1 - b_{\texttt{Dancing}}^i$$

$$\forall i \in P: \quad r_{\texttt{Cinema}}^i \leq 10 \cdot b_{\texttt{Cinema}}^i \qquad \text{and} \qquad r_{\texttt{Dancing}}^i \leq 10 \cdot b_{\texttt{Dancing}}^i$$

$$\text{where} \quad p_s, u_i, u_s, r_s \geq 0 \quad \text{and} \quad b_s^i \in \{0, 1\}$$

# Conclusion

## Summary

- A **mixed strategy** is a probability distribution on pure strategies.
- In a **mixed Nash equilibrium**, all players play best responses.
- **Nash's Theorem:** Mixed Nash equilibria always exist (for finite games).
- Nash equilibria for concrete games can be obtained via a translation to a **mixed integer program**:
  - Binary variables model the choices of pure strategies to put in the support;
  - real-valued variables model probabilities, utilities, and regret.

## Action Points

- Obtain all (mixed) Nash equilibria for chicken and interpret them.