Review
Recall our earlier definitions of space complexities:

**Definition 9.1:** Let $f : \mathbb{N} \rightarrow \mathbb{R}^+$ be a function.

1. $\text{DSpace}(f(n))$ is the class of all languages $L$ for which there is an $O(f(n))$-space bounded Turing machine deciding $L$.
2. $\text{NSpace}(f(n))$ is the class of all languages $L$ for which there is an $O(f(n))$-space bounded nondeterministic Turing machine deciding $L$.

Being $O(f(n))$-space bounded requires a (nondeterministic) TM

- to halt on every input and
- to use $\leq f(|w|)$ tape cells on every computation path.
Some important space complexity classes:

\[
\begin{align*}
L &= \text{LogSpace} = \text{DSpace}(\log n) & \text{logarithmic space} \\
\text{PSpace} &= \bigcup_{d \geq 1} \text{DSpace}(n^d) & \text{polynomial space} \\
\text{ExpSpace} &= \bigcup_{d \geq 1} \text{DSpace}(2^{n^d}) & \text{exponential space} \\
\text{NL} &= \text{NLogSpace} = \text{NSpace}(\log n) & \text{nondet. logarithmic space} \\
\text{NPSpace} &= \bigcup_{d \geq 1} \text{NSpace}(n^d) & \text{nondet. polynomial space} \\
\text{NExpSpace} &= \bigcup_{d \geq 1} \text{NSpace}(2^{n^d}) & \text{nondet. exponential space}
\end{align*}
\]
The Power of Space

Space seems to be more powerful than time because space can be reused.
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Example 9.2: **SAT** can be solved in linear space:
Just iterate over all possible truth assignments (each linear in size) and check if one satisfies the formula.
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**Example 9.2:** SAT can be solved in linear space:
Just iterate over all possible truth assignments (each linear in size) and check if one satisfies the formula.

**Example 9.3:** Tautology can be solved in linear space:
Just iterate over all possible truth assignments (each linear in size) and check if all satisfy the formula.
The Power of Space

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**Example 9.2:** \textsc{Sat} can be solved in linear space:
Just iterate over all possible truth assignments (each linear in size) and check if one satisfies the formula.

**Example 9.3:** \textsc{Tautology} can be solved in linear space:
Just iterate over all possible truth assignments (each linear in size) and check if all satisfy the formula.

More generally: $\text{NP} \subseteq \text{PSPACE}$ and $\text{coNP} \subseteq \text{PSPACE}$
Theorem 9.4: For every function $f : \mathbb{N} \rightarrow \mathbb{R}^+$, for all $c \in \mathbb{N}$, and for every $f$-space bounded (deterministic/nondeterministic) Turing machine $M$: there is a $\max\{1, \frac{1}{c} f(n)\}$-space bounded (deterministic/nondeterministic) Turing machine $M'$ that accepts the same language as $M$. 

Proof idea: Similar to (but much simpler than) linear speed-up. □
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This justifies using \( O \)-notation for defining space classes.
Theorem 9.5: For every function \( f : \mathbb{N} \rightarrow \mathbb{R}^+ \) all \( k \geq 1 \) and \( L \subseteq \Sigma^* \):

If \( L \) can be decided by an \( f \)-space bounded \( k \)-tape Turing-machine, then it can also be decided by an \( f \)-space bounded 1-tape Turing-machine.
Theorem 9.5: For every function $f : \mathbb{N} \rightarrow \mathbb{R}^+$ all $k \geq 1$ and $L \subseteq \Sigma^*$:

If $L$ can be decided by an $f$-space bounded $k$-tape Turing-machine, then it can also be decided by an $f$-space bounded 1-tape Turing-machine.

Proof idea: Combine tapes with a similar reduction as for time. Compress space to avoid linear increase.

Note: We still use a separate read-only input tape to define some space complexities, such as LogSpace.
**Theorem 9.6:** For all functions \( f : \mathbb{N} \to \mathbb{R}^+ \):

\[
\text{DTime}(f) \subseteq \text{DSpace}(f) \quad \text{and} \quad \text{NTime}(f) \subseteq \text{NSpace}(f)
\]

**Proof:** Visiting a cell takes at least one time step. \( \Box \)
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Theorem 9.7: For all functions $f : \mathbb{N} \to \mathbb{R}^+$ with $f(n) \geq \log n$:

$$\text{DSpace}(f) \subseteq \text{DTime}(2^{O(f)}) \quad \text{and} \quad \text{NSpace}(f) \subseteq \text{DTime}(2^{O(f)})$$
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Proof: Based on configuration graphs and a bound on the number of possible configurations.
Number of Possible Configurations

Let $M := (Q, \Sigma, \Gamma, q_0, \delta, q_{\text{start}})$ be a 2-tape Turing machine
(1 read-only input tape + 1 work tape)

Recall: A configuration of $M$ is a quadruple $(q, p_1, p_2, x)$ where

- $q \in Q$ is the current state,
- $p_i \in \mathbb{N}$ is the head position on tape $i$, and
- $x \in \Gamma^*$ is the tape content.

Let $w \in \Sigma^*$ be an input to $M$ and $n := |w|$.

- Then also $p_1 \leq n$.
- If $M$ is $f(n)$-space bounded we can assume $p_2 \leq f(n)$ and $|x| \leq f(n)$
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- Then also \( p_1 \leq n \).
- If \( \mathcal{M} \) is \( f(n) \)-space bounded we can assume \( p_2 \leq f(n) \) and \( |x| \leq f(n) \)

Hence, there are at most

\[
|Q| \cdot n \cdot f(n) \cdot |\Gamma|^{f(n)} = n \cdot 2^{O(f(n))} = 2^{O(f(n))}
\]

different configurations on inputs of length \( n \) (the last equality requires \( f(n) \geq \log n \)).
Configuration Graphs

The possible computations of a TM $M$ (on input $w$) form a directed graph:

- Vertices: configurations that $M$ can reach (on input $w$)
- Edges: there is an edge from $C_1$ to $C_2$ if $C_1 \vdash_M C_2$ ($C_2$ reachable from $C_1$ in a single step)

This yields the configuration graph:

- Could be infinite in general.
- For $f(n)$-space bounded 2-tape TMs, there can be at most $2^{O(f(n))}$ vertices and $(2^{O(f(n))})^2 = 2^{O(f(n))}$ edges
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A computation of $M$ on input $w$ corresponds to a path in the configuration graph from the start configuration to a stop configuration.

Hence, to test if $M$ accepts input $w$,

- construct the configuration graph and
- find a path from the start to an accepting stop configuration.
Theorem 9.6: For all functions $f : \mathbb{N} \rightarrow \mathbb{R}^+$:

$$\text{DTime}(f) \subseteq \text{DSpace}(f) \quad \text{and} \quad \text{NTime}(f) \subseteq \text{NSpace}(f)$$

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Theorem 9.7: For all functions $f : \mathbb{N} \rightarrow \mathbb{R}^+$ with $f(n) \geq \log n$:

$$\text{DSpace}(f) \subseteq \text{DTime}(2^{O(f)}) \quad \text{and} \quad \text{NSpace}(f) \subseteq \text{DTime}(2^{O(f)})$$

Proof: Build the configuration graph (time $2^{O(f(n))}$) and find a path from the start to an accepting stop configuration (time $2^{O(f(n))}$). \qed
Applying the results of the previous slides, we get the following relations:

\[ L \subseteq NL \subseteq P \subseteq NP \subseteq \mbox{PSpace} \subseteq \mbox{NPSpace} \subseteq \mbox{ExpTime} \subseteq \mbox{NExpTime} \]

We also noted \( P \subseteq \mbox{coNP} \subseteq \mbox{PSpace} \).

Open questions:

- What is the relationship between space classes and their co-classes?
- What is the relationship between deterministic and non-deterministic space classes?
Most experts think that nondeterministic TMs can solve strictly more problems when given the same amount of time than a deterministic TM:

Most believe that $P \subset NP$

How about nondeterminism in space-bounded TMs?
Most experts think that nondeterministic TMs can solve strictly more problems when given the same amount of time than a deterministic TM:

Most believe that $P \not\subseteq NP$

How about nondeterminism in space-bounded TMs?

**Theorem 9.8 (Savitch’s Theorem, 1970):** For any function $f : \mathbb{N} \rightarrow \mathbb{R}^+$ with $f(n) \geq \log n$:

\[
\text{NSpace}(f(n)) \subseteq \text{DSpace}(f^2(n)).
\]

That is: nondeterminism adds almost no power to space-bounded TMs!
Consequences of Savitch’s Theorem

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**Corollary 9.9:** \( \text{PSpace} = \text{NPSpace} \).

**Proof:** \( \text{PSPACE} \subseteq \text{NPSpace} \) is clear. The converse follows since the square of a polynomial is still a polynomial.

Similarly for “bigger” classes, e.g., \( \text{ExpSpace} = \text{NExpSpace} \).
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Similarly for “bigger” classes, e.g., $\text{ExpSpace} = \text{NExpSpace}$.

Corollary 9.10: $\text{NL} \subseteq \text{DSpace}(O(\log^2 n))$.

Note that $\log^2(n) \not\in O(\log n)$, so we do not obtain $\text{NL} = \text{L}$ from this.
Simulating nondeterminism with more space:

- Use configuration graph of nondeterministic space-bounded TM
- Check if an accepting configuration can be reached
- Store only one computation path at a time (depth-first search)

This still requires exponential space. We want quadratic space!

What to do?

Things we can do:

- Store one configuration:
  - one configuration requires $\log n + O(f(n))$ space
  - if $f(n) \geq \log n$, then this is $O(f(n))$ space
- Store $f(n)$ configurations (remember we have $f^2(n)$ space)
- Iterate over all configurations (one by one)
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Proving Savitch’s Theorem

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Proving Savitch’s Theorem: Key Idea

To find out if we can reach an accepting configuration, we solve a slightly more general question:

**YIELDABILITY**

Input: TM configurations $C_1$ and $C_2$, integer $k$

Problem: Can TM get from $C_1$ to $C_2$ in at most $k$ steps?
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Input: TM configurations $C_1$ and $C_2$, integer $k$

Problem: Can TM get from $C_1$ to $C_2$ in at most $k$ steps?

**Approach:** check if there is an intermediate configuration $C'$ such that

1. $C_1$ can reach $C'$ in $k/2$ steps and
2. $C'$ can reach $C_2$ in $k/2$ steps

$\sim$ **Deterministic:** we can try all $C'$ (iteration)

$\sim$ **Space-efficient:** we can reuse the same space for both steps
An Algorithm for Yieldability

```plaintext
01 CanYield(C_1, C_2, k) {
02     if k = 1:
03         return (C_1 = C_2) or (C_1 ⊢_M C_2)
04     else if k > 1:
05         for each configuration C of M for input size n:
06             if CanYield(C_1, C, k/2) and
07                 CanYield(C, C_2, k/2) :
08                 return true
09         // eventually, if no success:
10         return false
11 }
```

- We only call CanYield only with \( k \) a power of 2, so \( k/2 \in \mathbb{N} \)
Space Requirement for the Algorithm

```c
01 CanYield(C_1, C_2, k) {
02    if k = 1:
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• During iteration (line 05), we store one $C$ in $O(f(n))$
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Overall space usage: $O(f(n) \cdot \log k)$
Simulating Nondeterministic Space-Bounded TMs

Input: TM $\mathcal{M}$ that runs in $\text{NSpace}(f(n))$; input word $w$ of length $n$

Algorithm:

- Modify $\mathcal{M}$ to have a unique accepting configuration $C_{\text{accept}}$: when accepting, erase tape and move head to the very left
- Select $d$ such that $2^{df(n)} \geq |Q| \cdot n \cdot f(n) \cdot |\Gamma|^f(n)$
- Return $\text{CanYield}(C_{\text{start}}, C_{\text{accept}}, k)$ with $k = 2^{df(n)}$
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**Space requirements:**
$\text{CanYield}$ runs in space

$$O(f(n) \cdot \log k) = O(f(n) \cdot \log 2^{df(n)}) = O(f(n) \cdot df(n)) = O(f^2(n))$$
Did We Really Do It?

"Select $d$ such that $2df(n) \geq |Q| \cdot n \cdot f(n) \cdot |\Gamma| f(n)\)"

How does the algorithm actually do this?

• $f(n)$ was not part of the input!
• Even if we knew $f(n)$, it might not be easy to compute!

Solution: replace $f(n)$ by a parameter $\ell$ and probe its value

1. Start with $\ell = 1$
2. Check if $M$ can reach any configuration with more than $\ell$ tape cells (iterate over all configurations of size $\ell + 1$; use CanYield on each)
3. If yes, increase $\ell$ by 1; goto (2)
4. Run algorithm as before, with $f(n)$ replaced by $\ell$

Therefore: we don't need to know $f$ at all. This finishes the proof. □
“Select \( d \) such that \( 2^{df(n)} \geq |Q| \cdot n \cdot f(n) \cdot |\Gamma|^f(n) \)”

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\( \square \)
Summary: Relationships of Space and Time

Summing up, we get the following relations:

\[ L \subseteq NL \subseteq P \subseteq NP \subseteq \text{PSpace} = \text{NPSpace} \subseteq \text{ExpTime} \subseteq \text{NExpTime} \]

We also noted \( P \subseteq \text{coNP} \subseteq \text{PSpace} \).

Open questions:

- Is Savitch’s Theorem tight?
- Are there any interesting problems in these space classes?
- We have PSpace = NPSpace = coNPSpace.
  But what about L, NL, and coNL?
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- We have \( \text{PSPACE} = \text{NPSpace} = \text{coNPSpace} \).
  But what about \( L \), \( NL \), and \( \text{coNL} \)?

\[ \rightarrow \text{the first: nobody knows (YCTBF); the others: see upcoming lectures} \]