Description Logics

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- Alphabet
- Terms, Role and Concept Formulas
- Concept Axioms and the T-Box
- Semantics
- Assertions and the A-Box
- Subsumption and Unsatisfiability
- Taxonomies

"Logic is everywhere ..."
We consider an alphabet with

- constant symbols
- unary and binary relation symbols
- the variables $X, Y, \ldots$
- the connectives $\neg, \land, \lor, \rightarrow, \leftrightarrow$
- the quantifiers $\forall, \exists$ and
- the usual special symbols

**Notation**

- $C$ denotes a unary relation symbol
- $R$ denotes a binary relation symbol
Terms, Role and Concept Formulas

► The set of **terms** is the set of variables and constant symbols

► The set of **role formulas** consists of all strings of the form $R(X, Y)$, where $R/2$ is a relation symbol and $X, Y$ are variables

► The set of **atomic concept formulas** consists of all strings of the form $C(X)$, where $C/1$ is a relation symbol and $X$ a variable

► The set of **concept formulas** is the smallest set $C$ satisfying the following properties:
  ▶ All atomic concept formulas are in $C$
  ▶ If $F(X)$ is in $C$ then $\neg F(X)$ is in $C$
  ▶ If $F(X)$ and $G(X)$ are in $C$ then $(F(X) \land G(X))$ and $(F(X) \lor G(X))$ are in $C$
  ▶ If $R(X, Y)$ is a role formula and if $F(Y)$ is in $C$
    then $(\exists Y) (R(X, Y) \land F(Y))$ and $(\forall Y) (R(X, Y) \rightarrow F(Y))$ are in $C$

► **Observe** Each concept formula contains precisely one free variable
Concept Axioms and the T-Box

▶ Notation $C(X)$ denotes an atomic concept formula $F(X), G(X)$ denote concept formulas

▶ The set of concept axioms consists of all strings of the form $(\forall X)(C(X) \rightarrow F(X))$ and $(\forall X)(C(X) \leftrightarrow F(X))$

▶ A terminology or T-Box $\mathcal{K}_T$ is a finite set of concept axioms such that
  ▶ each $C$ occurs at most once as left-hand side of an axiom and
  ▶ it does not contain any cycles

▶ The set of generalized concept axioms consists of all strings of the form $(\forall X)(F(X) \rightarrow G(X))$ and $(\forall X)(F(X) \leftrightarrow G(X))$
A Simple Terminology

▶ Example

\[(\forall X) \ (\text{woman}(X) \rightarrow \text{person}(X))\]
\[(\forall X) \ (\text{man}(X) \rightarrow \text{person}(X))\]
\[(\forall X) \ (\text{mother}(X) \leftrightarrow (\text{woman}(X) \land (\exists Y) \ (\text{child}(X, Y) \land \text{person}(Y))))\]
\[(\forall X) \ (\text{father}(X) \leftrightarrow (\text{man}(X) \land (\exists Y) \ (\text{child}(X, Y) \land \text{person}(Y))))\]
\[(\forall X) \ (\text{parent}(X) \leftrightarrow (\text{mother}(X) \lor \text{father}(X)))\]
\[(\forall X) \ (\text{grandparent}(X) \leftrightarrow (\text{parent}(X) \land (\exists Y) \ (\text{child}(X, Y) \land \text{parent}(Y))))\]
\[(\forall X) \ (\text{father\_without\_son}(X) \leftrightarrow (\text{father}(X) \land (\forall Y) \ (\text{child}(X, Y) \rightarrow \neg \text{man}(Y))))\]

▶ Abbreviations

\begin{align*}
\text{woman} & \sqsubseteq \text{person} \\
\text{man} & \sqsubseteq \text{person} \\
\text{mother} & = \text{woman} \sqcap \exists \text{child} : \text{person} \\
\text{father} & = \text{man} \sqcap \exists \text{child} : \text{person} \\
\text{parent} & = \text{mother} \sqcup \text{father} \\
\text{grandparent} & = \text{parent} \sqcap \exists \text{child} : \text{parent} \\
\text{father\_without\_son} & = \text{father} \sqcap \forall \text{child} : \neg \text{man}
\end{align*}
Semantics

- Let $I = (\mathcal{D}, \cdot^I)$ be an interpretation

- Concept formulas

  \[
  C^I \subseteq \mathcal{D} \\
  (\neg F)^I = \mathcal{D} \setminus F^I \\
  (F \sqcup G)^I = F^I \cup G^I \\
  (F \sqcap G)^I = F^I \cap G^I \\
  \\
  R^I(d) := \{d' \in \mathcal{D} \mid (d, d') \in R^I\} \\
  (\exists R : F)^I = \{d \in \mathcal{D} \mid R^I(d) \cap F^I \neq \emptyset\} \\
  (\forall R : F)^I = \{d \in \mathcal{D} \mid R^I(d) \subseteq F^I\}
  \]

- Concept axioms

  \[
  I \vDash F \sqsubseteq G \iff F^I \subseteq G^I \\
  I \vDash F = G \iff F^I = G^I
  \]

- Remark

  Sometimes the language is extended by $\top$ and $\bot$ with $\top^I = \mathcal{D}$ and $\bot^I = \emptyset$
Assertions and the A-Box

- The set of assertions consists of all ground instances of $C(X)$ and $R(X,Y)$
- An A-Box is a finite set $\mathcal{K}_A$ of assertions
- Semantics
  \[
  I \models C(a) \iff a^I \in C^I \\
  I \models R(a, b) \iff b^I \in R^I(a^I)
  \]
- $I \models \mathcal{K}_A$ iff $I \models A$ for all $A \in \mathcal{K}_A$
A Simple A-Box

\[ \mathcal{K}_T \]

\begin{align*}
\text{woman} & \sqsubseteq \text{person} \\
\text{man} & \sqsubseteq \text{person} \\
\text{mother} & = \text{woman} \sqcap \exists \text{child} : \text{person} \\
\text{father} & = \text{man} \sqcap \exists \text{child} : \text{person} \\
\text{parent} & = \text{mother} \sqcup \text{father} \\
\text{grandparent} & = \text{parent} \sqcap \exists \text{child} : \text{parent} \\
\text{father\_without\_son} & = \text{father} \sqcap \forall \text{child} : \neg \text{man}
\end{align*}

\[ \mathcal{K}_A \]

\begin{align*}
\text{parent(carl)} \\
\text{parent(conny)} \\
\text{child(conny, joe)} \\
\text{child(conny, carl)} \\
\text{man(joe)} \\
\text{man(carl)} \\
\text{woman(conny)}
\end{align*}
Subsumption

► Some Relations

G subsumes F wrt $\mathcal{K}_T$ iff $\mathcal{K}_T \models F \sqsubseteq G$

G and F are equivalent wrt $\mathcal{K}_T$ iff $\mathcal{K}_T \models F = G$

G and F are disjoint wrt $\mathcal{K}_T$ iff $\mathcal{K}_T \models F \cap G = \bot$

F is unsatisfiable wrt $\mathcal{K}_T$ iff $\mathcal{K}_T \models F = \bot$

► Observations

$\triangleright F \sqsubseteq G \equiv F \cap \neg G = \bot$

$\triangleright$ Equivalence, disjointness and unsatisfiability can be reduced to subsumption
We define

\[\begin{align*}
\triangledown F \sqsubseteq T G & \text{ iff } \mathcal{K}_T \models F \sqsubseteq G \\
\triangledown F \equiv_T G & \text{ iff } \mathcal{K}_T \models F = G
\end{align*} \]

Observation Let \( C \) be a set of concept formulas

\[\begin{align*}
\triangledown \equiv_T \text{ is an equivalence relation on } C \\
\triangledown \sqsubseteq_T \text{ is a partial ordering on } C|\equiv_T \\
\text{There is a unique, minimal and binary relation } \triangledown_T \subseteq C \times C \text{ with } \ast_T = \sqsubseteq_T
\end{align*} \]

The restriction of \( \triangledown_T \) to the set of atomic concept formulas is called taxonomy
Taxonomy – Example

- person
  - woman
  - mother
- parent
  - man
- grandparent
- father
- father_without_son
Unsatisfiability

- Logical consequences wrt an A-box like

\[ \mathcal{K}_T \cup \mathcal{K}_A \models C(a) \]

are equivalent to the question whether

\[ \mathcal{K}_T \cup \mathcal{K}_A \cup \{\neg C(a)\} \text{ is unsatisfiable} \]

- Many other questions can be reduced to satisfiability testing
Some Remarks

- Subsumption and satisfiability are decidable, but intractable in the presented description logic.
- Description logics may be extended to include:
  - role restrictions
  - complex and/or transitive roles
  - cyclic concept definitions or
  - concrete domains like the reals

But sometimes they are more restricted.

- There are many applications like, for example, within the semantic web, bioinformatics, or medicine.