TECHNISCHE UNIVERSITÄT DRESDEN

Computational Logic Group

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Description Logics – Syntax and Semantics I

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Motivation

Undecidable	FOL sat
|	|
Non-Elementary decidable	...
|	|
...	...
|	|
ExpTime	Datalog sat
|	|
PSpace	...
|	|
NP	PL sat
|	|
P	Horn PL sat

Motivation

Many KR applications do not require full power of FOL

What can we leave out?

- Key reasoning problems should become decidable
- Sufficient expressive power to model application domain

Description Logics are a family of FOL fragments that meet these requirements for many applications:

- Underlying formalisms of modern ontology languages
- Widely used in bio-medical information systems
- Core component of the Semantic Web

Motivation

Recall our arthritis example:

- A juvenile disease affects only children or teenagers
- Children and teenagers are not adults
- A person is either a child, a teenager, or an adult
- Juvenile arthritis is a kind of arthritis and a juvenile disease
- Every kind of arthritis damages some joint

The important types of objects are given by unary FOL predicates:

juvenile disease, child, teenager, adult, . . .

The types of relationships are given by binary FOL predicates:

affects, damages, . . .

Motivation

The vocabulary of a Description Logic is composed of

- Unary FOL predicates

  Arthritis, Child, ...

- Binary FOL predicates

  Affects, Damages, ...

- FOL constants

  JohnSmith, *MaryJones*, *JRA*, ...

We are already restricting the expressive power of FOL

- No function symbols (of positive arity)
- No predicates of arity greater than 2

Motivation

Let us take a closer look at the FOL formulas for our example:

$$\forall x.(\mathit{JuvDis}(x) \to \forall y.(\mathit{Affects}(x,y) \to \mathit{Child}(y) \lor \mathit{Teen}(y)))$$
$$\forall x.(\mathit{Child}(x) \lor \mathit{Teen}(x) \to \neg \mathit{Adult}(x))$$
$$\forall x.(\mathit{Person}(x) \to \mathit{Child}(x) \lor \mathit{Teen}(x) \lor \mathit{Adult}(x))$$
$$\forall x.(\mathit{JuvArthritis}(x) \to \mathit{Arthritis}(x) \land \mathit{JuvDis}(x))$$
$$\forall x.(\mathit{Arthritis}(x) \to \exists y.(\mathit{Damages}(x,y) \land \mathit{Joint}(y)))$$

We can find several regularities in these formulas:

- There is an outermost universal quantifier on a single variable x
- The formulas can be split into two parts by the implication symbol
  - Each part is a formula with one free variable
- Atomic formulas involving a binary predicate occur only quantified in a syntactically restricted way.

Complexity

Undecidable	FOL-3 sat
Non-Elementary decidable	. . .
. . .	. . .
NExpTime	FOL-2 sat
ExpTime	Datalog sat
PSpace	. . .
NP	PL sat
P	Horn PL sat

Motivation

Consider as an example one of our formulas:

$$\forall x.(\mathit{Child}(x) \vee \mathit{Teen}(x) \rightarrow \neg \mathit{Adult}(x))$$

Let us look at all its sub-formulas at each side of the implication

$\mathit{Child}(x)$	Set of all children
$\mathit{Teen}(x)$	Set of all teenagers
$\mathit{Child}(x) \vee \mathit{Teen}(x)$	Set of all objects that are children or teenagers
$\mathit{Adult}(x)$	Set of all adults
$\neg \mathit{Adult}(x)$	Set of all objects that are not adults

Important observations concerning formulas with one free variable:

- Some are atomic (e.g., $\mathit{Child}(x)$)
  do not contain other formulas as subformulas
- Others are complex (e.g., $\mathit{Child}(x) \vee \mathit{Teen}(x)$)

Basic Definitions

Idea: Define operators for constructing complex formulas with one free variable out of simple building blocks

Atomic Concept: Represents an atomic formula with one free variable

$$\mathit{Child} \quad \rightsquigarrow \quad \mathit{Child}(x)$$

Complex concepts (part 1):

- Concept Union ($\sqcup$): applies to two concepts

$$\mathit{Child} \sqcup \mathit{Teen} \quad \rightsquigarrow \quad \mathit{Child}(x) \vee \mathit{Teen}(x)$$

- Concept Intersection ($\sqcap$): applies to two concepts

$$\mathit{Arthritis} \sqcap \mathit{JuvDis} \quad \rightsquigarrow \quad \mathit{Arthritis}(x) \wedge \mathit{JuvDis}(x)$$

- Concept Negation ($\neg$): applies to one concept

$$\neg\mathit{Adult} \quad \rightsquigarrow \quad \neg\mathit{Adult}(x)$$

Motivation

Consider examples with binary predicates:

$$\forall x.(\mathit{Arthritis}(x) \to \exists y.(\mathit{Damages}(x,y) \land \mathit{Joint}(y))$$
$$\forall x.(\mathit{JuvDis}(x) \to \forall y.(\mathit{Affects}(x,y) \to \mathit{Child}(y) \lor \mathit{Teen}(y)))$$

- We have a concept and a binary predicate (called a role) mentioning the concept's free variable
- The role and the concept are connected via conjunction (existential quantification) or implication (universal quantification)
- Nested sub-concepts use a fresh (existentially/universally quantified) variable, and are connected to the surrounding concept by exactly one role atom (often called a guard)

Basic Definitions

Atomic Role: Represents an atom with two free variables

$$\mathit{Affects} \quad \rightsquigarrow \quad \mathit{Affects}(x, y)$$

Complex concepts (part 2): apply to an atomic role and a concept

- Existential Restriction:

$$\exists \mathit{Damages}.\mathit{Joint} \quad \rightsquigarrow \quad \exists y.(\mathit{Damages}(x, y) \land \mathit{Joint}(y))$$

- Universal Restriction:

$$\forall \mathit{Affects}.(\mathit{Child} \sqcup \mathit{Teen}) \quad \rightsquigarrow \quad \forall y.(\mathit{Affects}(x, y) \to \mathit{Child}(y) \lor \mathit{Teen}(y))$$

$\mathcal{ALC}$ Concepts

$\mathcal{ALC}$ is the basic description logic

$\mathcal{ALC}$ concepts are inductively defined from atomic concepts and roles:

- Every atomic concept is a concept
- $\top$ and $\bot$ are concepts
- If C is a concept, then $\neg C$ is a concept
- If C and D are concepts, then so are $C \sqcap D$ and $C \sqcup D$
- If C a concept and R a role, $\forall R.C$ and $\exists R.C$ are concepts.

Concepts describe sets of objects with certain common features:

$\mathit{Woman} \sqcap \exists \mathit{hasChild}.(\exists \mathit{hasChild}.\mathit{Person})$	Women with a grandchild
$\mathit{Disease} \sqcap \forall \mathit{Affects}.\mathit{Child}$	Diseases affecting only children
$\mathit{Person} \sqcap \neg\exists \mathit{owns}.\mathit{DetHouse}$	People not owning a detached house
$\mathit{Man} \sqcap \exists \mathit{hasChild}.\top \sqcap \forall \mathit{hasChild}.\mathit{Man}$	Fathers having only sons

$\rightsquigarrow$ Very useful idea for Knowledge Representation

General Concept Inclusion Axioms

Recall our example formulas:

$$\forall x.(JuvDis(x) \to \forall y.(Affects(x,y) \to Child(y) \lor Teen(y)))$$
$$\forall x.(Child(x) \lor Teen(x) \to \neg Adult(x))$$
$$\forall x.(Person(x) \to Child(x) \lor Teen(x) \lor Adult(x))$$
$$\forall x.(JuvArthritis(x) \to Arthritis(x) \land JuvDis(x))$$
$$\forall x.(Arthritis(x) \to \exists y.(Damages(x,y) \land Joint(y))$$

They are of the following form, with $\alpha_C(x)$ and $\alpha_D(x)$ corresponding to $\mathcal{ALC}$ concepts C and D

$$\forall x.(\alpha_C(x) \to \alpha_D(x))$$

Such sentences are $\mathcal{ALC}$ General Concept Inclusion Axioms (GCIs)

$$C \sqsubseteq D$$

where C and D are $\mathcal{ALC}$-concepts

General Concept Inclusion Axioms

$$\forall x.(\mathit{JuvDis}(x) \rightarrow$$
$$\forall y.(\mathit{Affects}(x,y) \rightarrow \mathit{Child}(y) \vee \mathit{Teen}(y))) \quad \rightsquigarrow$$
$$\forall x.(\mathit{Child}(x) \vee \mathit{Teen}(x) \rightarrow \neg \mathit{Adult}(x)) \quad \rightsquigarrow$$
$$\forall x.(\mathit{Person}(x) \rightarrow \mathit{Child}(x) \vee \mathit{Teen}(x) \vee \mathit{Adult}(x)) \quad \rightsquigarrow$$
$$\forall x.(\mathit{JuvArth}(x) \rightarrow \mathit{Arth}(x) \wedge \mathit{JuvDis}(x)) \quad \rightsquigarrow$$
$$\forall x.(\mathit{Arth}(x) \rightarrow \exists y.(\mathit{Damages}(x,y) \wedge \mathit{Joint}(y))) \quad \rightsquigarrow$$

Note that we often use $C \equiv D$ as an abbreviation for a symmetrical pair of GCIs $C \sqsubseteq D$ and $D \sqsubseteq C$, e.g.:

$$\left.\begin{array}{r} \mathit{Arth} \sqcap \mathit{JuvDis} \sqsubseteq \mathit{JuvArth} \\ \mathit{JuvArth} \sqsubseteq \mathit{Arth} \sqcap \mathit{JuvDis} \end{array}\right\} \rightsquigarrow \quad \mathit{JuvArth} \equiv \mathit{Arth} \sqcap \mathit{JuvDis}$$

General Concept Inclusion Axioms

$$
\begin{aligned}
\forall x.(\mathit{JuvDis}(x) \rightarrow \\
\forall y.(\mathit{Affects}(x,y) \rightarrow \mathit{Child}(y) \vee \mathit{Teen}(y))) &\quad\rightsquigarrow\quad \mathit{JuvDis} \sqsubseteq \forall \mathit{Affects}.(\mathit{Child} \sqcup \mathit{Teen}) \\
\forall x.(\mathit{Child}(x) \vee \mathit{Teen}(x) \rightarrow \neg \mathit{Adult}(x)) &\quad\rightsquigarrow \\
\forall x.(\mathit{Person}(x) \rightarrow \mathit{Child}(x) \vee \mathit{Teen}(x) \vee \mathit{Adult}(x)) &\quad\rightsquigarrow \\
\forall x.(\mathit{JuvArth}(x) \rightarrow \mathit{Arth}(x) \wedge \mathit{JuvDis}(x)) &\quad\rightsquigarrow \\
\forall x.(\mathit{Arth}(x) \rightarrow \exists y.(\mathit{Damages}(x,y) \wedge \mathit{Joint}(y))) &\quad\rightsquigarrow
\end{aligned}
$$

Note that we often use $C \equiv D$ as an abbreviation for a symmetrical pair of GCIs $C \sqsubseteq D$ and $D \sqsubseteq C$, e.g.:

$$
\left.\begin{array}{r} \mathit{Arth} \sqcap \mathit{JuvDis} \sqsubseteq \mathit{JuvArth} \\ \mathit{JuvArth} \sqsubseteq \mathit{Arth} \sqcap \mathit{JuvDis} \end{array}\right\} \quad\rightsquigarrow\quad \mathit{JuvArth} \equiv \mathit{Arth} \sqcap \mathit{JuvDis}
$$

General Concept Inclusion Axioms

$$
\begin{array}{rcl}
\forall x.(\mathit{JuvDis}(x) \rightarrow & & \\
\forall y.(\mathit{Affects}(x,y) \rightarrow \mathit{Child}(y) \vee \mathit{Teen}(y))) & \rightsquigarrow & \mathit{JuvDis} \sqsubseteq \forall \mathit{Affects}.(\mathit{Child} \sqcup \mathit{Teen}) \\
\forall x.(\mathit{Child}(x) \vee \mathit{Teen}(x) \rightarrow \neg \mathit{Adult}(x)) & \rightsquigarrow & \mathit{Child} \sqcup \mathit{Teen} \sqsubseteq \neg \mathit{Adult} \\
\forall x.(\mathit{Person}(x) \rightarrow \mathit{Child}(x) \vee \mathit{Teen}(x) \vee \mathit{Adult}(x)) & \rightsquigarrow & \\
\forall x.(\mathit{JuvArth}(x) \rightarrow \mathit{Arth}(x) \wedge \mathit{JuvDis}(x)) & \rightsquigarrow & \\
\forall x.(\mathit{Arth}(x) \rightarrow \exists y.(\mathit{Damages}(x,y) \wedge \mathit{Joint}(y))) & \rightsquigarrow &
\end{array}
$$

Note that we often use $C \equiv D$ as an abbreviation for a symmetrical pair of GCIs $C \sqsubseteq D$ and $D \sqsubseteq C$, e.g.:

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\left.\begin{array}{r}
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\mathit{JuvArth} \sqsubseteq \mathit{Arth} \sqcap \mathit{JuvDis}
\end{array}\right\} \rightsquigarrow \mathit{JuvArth} \equiv \mathit{Arth} \sqcap \mathit{JuvDis}
$$

General Concept Inclusion Axioms

$$
\begin{array}{rcl}
\forall x.(\mathit{JuvDis}(x) \rightarrow & & \\
\forall y.(\mathit{Affects}(x,y) \rightarrow \mathit{Child}(y) \vee \mathit{Teen}(y))) & \rightsquigarrow & \mathit{JuvDis} \sqsubseteq \forall \mathit{Affects}.(\mathit{Child} \sqcup \mathit{Teen}) \\
\forall x.(\mathit{Child}(x) \vee \mathit{Teen}(x) \rightarrow \neg \mathit{Adult}(x)) & \rightsquigarrow & \mathit{Child} \sqcup \mathit{Teen} \sqsubseteq \neg \mathit{Adult} \\
\forall x.(\mathit{Person}(x) \rightarrow \mathit{Child}(x) \vee \mathit{Teen}(x) \vee \mathit{Adult}(x)) & \rightsquigarrow & \mathit{Person} \sqsubseteq \mathit{Child} \sqcup \mathit{Teen} \sqcup \mathit{Adult} \\
\forall x.(\mathit{JuvArth}(x) \rightarrow \mathit{Arth}(x) \wedge \mathit{JuvDis}(x)) & \rightsquigarrow & \\
\forall x.(\mathit{Arth}(x) \rightarrow \exists y.(\mathit{Damages}(x,y) \wedge \mathit{Joint}(y))) & \rightsquigarrow &
\end{array}
$$

Note that we often use $C \equiv D$ as an abbreviation for a symmetrical pair of GCIs $C \sqsubseteq D$ and $D \sqsubseteq C$, e.g.:

$$
\left.\begin{array}{r}
\mathit{Arth} \sqcap \mathit{JuvDis} \sqsubseteq \mathit{JuvArth} \\
\mathit{JuvArth} \sqsubseteq \mathit{Arth} \sqcap \mathit{JuvDis}
\end{array}\right\} \rightsquigarrow \mathit{JuvArth} \equiv \mathit{Arth} \sqcap \mathit{JuvDis}
$$

General Concept Inclusion Axioms

$$
\begin{array}{rcl}
\forall x.(\mathit{JuvDis}(x) \rightarrow & & \\
\forall y.(\mathit{Affects}(x,y) \rightarrow \mathit{Child}(y) \vee \mathit{Teen}(y))) & \rightsquigarrow & \mathit{JuvDis} \sqsubseteq \forall \mathit{Affects}.(\mathit{Child} \sqcup \mathit{Teen}) \\
\forall x.(\mathit{Child}(x) \vee \mathit{Teen}(x) \rightarrow \neg \mathit{Adult}(x)) & \rightsquigarrow & \mathit{Child} \sqcup \mathit{Teen} \sqsubseteq \neg \mathit{Adult} \\
\forall x.(\mathit{Person}(x) \rightarrow \mathit{Child}(x) \vee \mathit{Teen}(x) \vee \mathit{Adult}(x)) & \rightsquigarrow & \mathit{Person} \sqsubseteq \mathit{Child} \sqcup \mathit{Teen} \sqcup \mathit{Adult} \\
\forall x.(\mathit{JuvArth}(x) \rightarrow \mathit{Arth}(x) \wedge \mathit{JuvDis}(x)) & \rightsquigarrow & \mathit{JuvArth} \sqsubseteq \mathit{Arth} \sqcap \mathit{JuvDis} \\
\forall x.(\mathit{Arth}(x) \rightarrow \exists y.(\mathit{Damages}(x,y) \wedge \mathit{Joint}(y))) & \rightsquigarrow &
\end{array}
$$

Note that we often use $C \equiv D$ as an abbreviation for a symmetrical pair of GCIs $C \sqsubseteq D$ and $D \sqsubseteq C$, e.g.:

$$
\left.\begin{array}{r}
\mathit{Arth} \sqcap \mathit{JuvDis} \sqsubseteq \mathit{JuvArth} \\
\mathit{JuvArth} \sqsubseteq \mathit{Arth} \sqcap \mathit{JuvDis}
\end{array}\right\} \rightsquigarrow \mathit{JuvArth} \equiv \mathit{Arth} \sqcap \mathit{JuvDis}
$$

General Concept Inclusion Axioms

$$\forall x.(JuvDis(x) \rightarrow$$

$$\forall y.(Affects(x,y) \rightarrow Child(y) \vee Teen(y))) \quad \rightsquigarrow \quad JuvDis \sqsubseteq \forall Affects.(Child \sqcup Teen)$$

$$\forall x.(Child(x) \vee Teen(x) \rightarrow \neg Adult(x)) \quad \rightsquigarrow \quad Child \sqcup Teen \sqsubseteq \neg Adult$$

$$\forall x.(Person(x) \rightarrow Child(x) \vee Teen(x) \vee Adult(x)) \quad \rightsquigarrow \quad Person \sqsubseteq Child \sqcup Teen \sqcup Adult$$

$$\forall x.(JuvArth(x) \rightarrow Arth(x) \wedge JuvDis(x)) \quad \rightsquigarrow \quad JuvArth \sqsubseteq Arth \sqcap JuvDis$$

$$\forall x.(Arth(x) \rightarrow \exists y.(Damages(x,y) \wedge Joint(y))) \quad \rightsquigarrow \quad Arth \sqsubseteq \exists Damages.Joint$$

Note that we often use $C \equiv D$ as an abbreviation for a symmetrical pair of GCIs $C \sqsubseteq D$ and $D \sqsubseteq C$, e.g.:

$$\left.\begin{array}{r} Arth \sqcap JuvDis \sqsubseteq JuvArth \\ JuvArth \sqsubseteq Arth \sqcap JuvDis \end{array}\right\} \quad \rightsquigarrow \quad JuvArth \equiv Arth \sqcap JuvDis$$

Terminological Statements

GCIs allow us to represent a surprising variety of terminological statements:

- Sub-type statements

$$\forall x.(JuvArth(x) \rightarrow Arth(x)) \quad \rightsquigarrow \quad JuvArth \sqsubseteq Arth$$

- Full definitions:

$$\forall x.(JuvArth(x) \leftrightarrow Arth(x) \wedge JuvDis(x)) \quad \rightsquigarrow \quad JuvArth \equiv Arth \sqcap JuvDis$$

- Disjointness statements:

$$\forall x.(Child(x) \rightarrow \neg Adult(x)) \quad \rightsquigarrow \quad Child \sqsubseteq \neg Adult$$

- Covering statements:

$$\forall x.(Person(x) \rightarrow Adult(x) \vee Child(x)) \quad \rightsquigarrow \quad Person \sqsubseteq Adult \sqcup Child$$

- Type (domain and range) restrictions:

$$\forall x.(\forall y.(Affects(x,y) \rightarrow Arth(x) \wedge Person(y))) \quad \rightsquigarrow \quad \begin{array}{l} \exists Affects.\top \sqsubseteq Arth \\ \top \sqsubseteq \forall Affects.Person \end{array}$$

Concept Inclusion Axioms & Definitions

Why call $C \sqsubseteq D$ a concept inclusion axiom?

- Intuitively, every object belonging to C should belong also to D
- States that C is more specific than D

Why call it a general concept inclusion axiom?

- It may be interesting to consider restricted forms of inclusion
- E.g., axioms where l.h.s. is atomic are sometimes called definitions
  - A concept definition specifies necessary and sufficient conditions for instances, e.g.:

$$\mathit{JuvArth} \equiv \mathit{Arth} \sqcap \mathit{JuvDis}$$

  - A primitive concept definition specifies only necessary conditions for instances, e.g.:

$$\mathit{Arth} \sqsubseteq \exists \mathit{Damages}.\mathit{Joint}$$

Data Assertions

In description logics, we can also represent data:

$\mathit{Child}(\mathit{JohnSmith})$	John Smith is a child
$\mathit{JuvenileArthritis}(\mathit{JRA})$	JRA is a juvenile arthritis
$\mathit{Affects}(\mathit{JRA}, \mathit{MaryJones})$	Mary Jones is affected by JRA

Usually data assertions correspond to FOL ground atoms.

Alternative notation: $\mathit{JohnSmith} : \mathit{Child}$, $(\mathit{JRA}, \mathit{MaryJones}) : \mathit{Affects}$

In $\mathcal{ALC}$, we have two types of data assertions, for a, b individuals:

$$C(a) \quad \rightsquigarrow \quad C \text{ is an } \mathcal{ALC} \text{ concept}$$

$$R(a, b) \quad \rightsquigarrow \quad R \text{ is an atomic role}$$

Examples of acceptable data assertions in $\mathcal{ALC}$:

$$\exists \mathit{hasChild}.\mathit{Teacher}(\mathit{John}) \quad \rightsquigarrow \exists y.(\mathit{hasChild}(\mathit{John}, y) \land \mathit{Teacher}(y))$$

$$\mathit{HistorySt} \sqcup \mathit{ClassicsSt}(\mathit{John}) \quad \rightsquigarrow \mathit{HistorySt}(\mathit{John}) \lor \mathit{ClassicsSt}(\mathit{John})$$

DL Knowledge Base: TBox + ABox

An $\mathcal{ALC}$ knowledge base $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ is composed of:

- A TBox $\mathcal{T}$ (Terminological Component):
  Finite set of GCIs
- An ABox $\mathcal{A}$ (Assertional Component):
  Finite set of assertions

TBox:

$$\mathit{JuvArthritis} \sqsubseteq \mathit{Arthritis} \sqcap \mathit{JuvDisease}$$
$$\mathit{Arthritis} \sqcap \mathit{JuvDisease} \sqsubseteq \mathit{JuvArthritis}$$
$$\mathit{Arthritis} \sqsubseteq \exists \mathit{Damages}.\mathit{Joint}$$
$$\mathit{JuvDisease} \sqsubseteq \forall \mathit{Affects}.(\mathit{Child} \sqcup \mathit{Teen})$$
$$\mathit{Child} \sqcup \mathit{Teen} \sqsubseteq \neg \mathit{Adult}$$

ABox:

$$\mathit{Child}(\mathit{JohnSmith})$$
$$\mathit{JuvArthritis}(\mathit{JRA})$$
$$\mathit{Affects}(\mathit{JRA}, \mathit{MaryJones})$$
$$\mathit{Child} \sqcup \mathit{Teen}(\mathit{MaryJones})$$

Semantics via FOL Translation

- Concepts translated as formulas with one free variable (except $\top$ and $\bot$ which are mapped to themselves):

$$
\begin{aligned}
\pi_x(A) &= A(x) & \pi_y(A) &= A(y)\\
\pi_x(\neg C) &= \neg\pi_x(C) & \pi_y(\neg C) &= \neg\pi_y(C)\\
\pi_x(C \sqcap D) &= \pi_x(C) \wedge \pi_x(D) & \pi_y(C \sqcap D) &= \pi_y(C) \wedge \pi_y(D)\\
\pi_x(C \sqcup D) &= \pi_x(C) \vee \pi_x(D) & \pi_y(C \sqcup D) &= \pi_y(C) \vee \pi_y(D)\\
\pi_x(\exists R.C) &= \exists y.(R(x,y) \wedge \pi_y(C)) & \pi_y(\exists R.C) &= \exists x.(R(y,x) \wedge \pi_x(C))\\
\pi_x(\forall R.C) &= \forall y.(R(x,y) \rightarrow \pi_y(C)) & \pi_y(\forall R.C) &= \forall x.(R(y,x) \rightarrow \pi_x(C))
\end{aligned}
$$

- GCIs and assertions translated as sentences

$$
\begin{aligned}
\pi(C \sqsubseteq D) &= \forall x.(\pi_x(C) \rightarrow \pi_x(D))\\
\pi(R(a,b)) &= R(a,b)\\
\pi(C(a)) &= \pi_x(C)[x/a]
\end{aligned}
$$

- TBoxes, ABoxes and KBs are translated in the obvious way.

Semantics via FOL Translation

Note redundancy in concept-forming operators:

$$
\begin{array}{rcl}
\bot & \rightsquigarrow & \neg\top \\
C \sqcup D & \rightsquigarrow & \neg(\neg C \sqcap \neg D) \\
\forall R.C & \rightsquigarrow & \neg(\exists R.\neg C)
\end{array}
$$

These equivalences can be proved using FOL semantics:

$$
\begin{array}{rcl}
\pi_x(\neg\exists R.\neg C) & = & \neg\exists y.(R(x,y) \land \neg\pi_y(C)) \\
& \equiv & \forall y.(\neg(R(x,y) \land \neg\pi_y(C))) \\
& \equiv & \forall y.(\neg R(x,y) \lor \pi_y(C)) \\
& \equiv & \forall y.(R(x,y) \to \pi_y(C)) \\
& = & \pi_x(\forall R.C)
\end{array}
$$

We can define syntax of $\mathcal{ALC}$ using only conjunction and negation operators and the existential role operator, considering all other operators as abbreviations.

Direct (Model-Theoretic) Semantics

Direct semantics: An alternative (and convenient) way of specifying semantics

DL interpretation $\mathcal{I} = \langle \Delta^\mathcal{I}, \cdot^\mathcal{I} \rangle$ is a FOL interpretation over the DL vocabulary:

- Each individual a interpreted as an object $a^\mathcal{I} \in \Delta^\mathcal{I}$.
- Each atomic concept A interpreted as a set $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$.
- Each atomic role R interpreted as a binary relation $R^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$.

The mapping $\cdot^\mathcal{I}$ is extended to $\top$, $\bot$ and compound concepts as follows:

$$\begin{aligned}
\top^\mathcal{I} &= \Delta^\mathcal{I} \\
\bot^\mathcal{I} &= \emptyset \\
(\neg C)^\mathcal{I} &= \Delta^\mathcal{I} \setminus C^\mathcal{I} \\
(C \sqcap D)^\mathcal{I} &= C^\mathcal{I} \cap D^\mathcal{I} \\
(C \sqcup D)^\mathcal{I} &= C^\mathcal{I} \cup D^\mathcal{I} \\
(\exists R.C)^\mathcal{I} &= \{u \in \Delta^\mathcal{I} \mid \exists w \in \Delta^\mathcal{I} \text{ s.t. } \langle u, w \rangle \in R^\mathcal{I} \text{ and } w \in C^\mathcal{I}\} \\
(\forall R.C)^\mathcal{I} &= \{u \in \Delta^\mathcal{I} \mid \forall w \in \Delta^\mathcal{I},\ \langle u, w \rangle \in R^\mathcal{I} \text{ implies } w \in C^\mathcal{I}\}
\end{aligned}$$

Direct (Model-Theoretic) Semantics

Consider the interpretation $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$

$$
\begin{aligned}
\Delta^{\mathcal{I}} &= \{u, v, w\} \\
\mathit{JuvDis}^{\mathcal{I}} &= \{u\} \\
\mathit{Child}^{\mathcal{I}} &= \{w\} \\
\mathit{Teen}^{\mathcal{I}} &= \emptyset \\
\mathit{Affects}^{\mathcal{I}} &= \{\langle u, w \rangle\}
\end{aligned}
$$

We can then interpret any concept as a subset of $\Delta^{\mathcal{I}}$:

$$
\begin{aligned}
(\mathit{JuvDis} \sqcap \mathit{Child})^{\mathcal{I}} &= \\
(\mathit{Child} \sqcup \mathit{Teen})^{\mathcal{I}} &= \\
(\exists \mathit{Affects}.(\mathit{Child} \sqcup \mathit{Teen}))^{\mathcal{I}} &= \\
(\neg \mathit{Child})^{\mathcal{I}} &= \\
(\forall \mathit{Affects}.\mathit{Teen})^{\mathcal{I}} &=
\end{aligned}
$$

Direct (Model-Theoretic) Semantics

Consider the interpretation $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$

$$\begin{aligned}
\Delta^{\mathcal{I}} &= \{u, v, w\} \\
\mathit{JuvDis}^{\mathcal{I}} &= \{u\} \\
\mathit{Child}^{\mathcal{I}} &= \{w\} \\
\mathit{Teen}^{\mathcal{I}} &= \emptyset \\
\mathit{Affects}^{\mathcal{I}} &= \{\langle u, w \rangle\}
\end{aligned}$$

We can then interpret any concept as a subset of $\Delta^{\mathcal{I}}$:

$$\begin{aligned}
(\mathit{JuvDis} \sqcap \mathit{Child})^{\mathcal{I}} &= \emptyset \\
(\mathit{Child} \sqcup \mathit{Teen})^{\mathcal{I}} &= \\
(\exists \mathit{Affects}.(\mathit{Child} \sqcup \mathit{Teen}))^{\mathcal{I}} &= \\
(\neg \mathit{Child})^{\mathcal{I}} &= \\
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\end{aligned}$$

Direct (Model-Theoretic) Semantics

Consider the interpretation $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$

$$
\begin{array}{rcl}
\Delta^{\mathcal{I}} & = & \{u, v, w\} \\
\mathit{JuvDis}^{\mathcal{I}} & = & \{u\} \\
\mathit{Child}^{\mathcal{I}} & = & \{w\} \\
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\mathit{Affects}^{\mathcal{I}} & = & \{\langle u, w \rangle\}
\end{array}
$$

We can then interpret any concept as a subset of $\Delta^{\mathcal{I}}$:

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\begin{array}{rcl}
(\mathit{JuvDis} \sqcap \mathit{Child})^{\mathcal{I}} & = & \emptyset \\
(\mathit{Child} \sqcup \mathit{Teen})^{\mathcal{I}} & = & \{w\} \\
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Direct (Model-Theoretic) Semantics

We can now determine whether $\mathcal{I}$ is a model of ...

- A General Concept Inclusion Axiom $C \sqsubseteq D$:

$$\mathcal{I} \models (C \sqsubseteq D) \quad \text{iff} \quad C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$$

- An assertion $C(a)$:

$$\mathcal{I} \models C(a) \quad \text{iff} \quad a^{\mathcal{I}} \in C^{\mathcal{I}}$$

- An assertion $R(a, b)$:

$$\mathcal{I} \models R(a, b) \quad \text{iff} \quad \langle a^{\mathcal{I}}, b^{\mathcal{I}} \rangle \in R^{\mathcal{I}}$$

- A TBox $\mathcal{T}$, ABox $\mathcal{A}$, and knowledge base:

$$\begin{aligned}
\mathcal{I} \models \mathcal{T} &\quad \text{iff} \quad \mathcal{I} \models \alpha \text{ for each } \alpha \in \mathcal{T} \\
\mathcal{I} \models \mathcal{A} &\quad \text{iff} \quad \mathcal{I} \models \alpha \text{ for each } \alpha \in \mathcal{A} \\
\mathcal{I} \models \mathcal{K} &\quad \text{iff} \quad \mathcal{I} \models \mathcal{T} \text{ and } \mathcal{I} \models \mathcal{A}
\end{aligned}$$

Direct (Model-Theoretic) Semantics: Examples

Consider our previous example interpretation:

$$\Delta^{\mathcal{I}} = \{u, v, w\} \quad Affects^{\mathcal{I}} = \{\langle u, w\rangle\}$$
$$JuvDis^{\mathcal{I}} = \{u\} \qquad Child^{\mathcal{I}} = \{w\} \qquad Teen^{\mathcal{I}} = \emptyset$$

$\mathcal{I}$ is a model of the following axioms:

$$JuvDis \sqsubseteq \exists Affects.Child \quad \rightsquigarrow$$
$$Child \sqsubseteq \neg Teen \quad \rightsquigarrow$$
$$JuvDis \sqsubseteq \forall Affects.Child \quad \rightsquigarrow$$

However $\mathcal{I}$ is not a model of the following axioms:

$$JuvDis \sqsubseteq \exists Affects.(Child \sqcap Teen) \quad \rightsquigarrow$$
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Conclusion

- Description Logics are a family of knowledge representation languages
- They can be seen as syntactic fragments of first-order predicate logic
- Only unary and binary predicate symbols, no function symbols (of positive arity)
- Use of quantification is restricted by guards
- $\mathcal{ALC}$ is the basic description logic
- Syntax of DLs: concepts (atomic/complex), general concept inclusions
- DL knowledge bases: consist of TBox and ABox
- Semantics of DLs: direct model-theoretic semantics (or translation to FOL)