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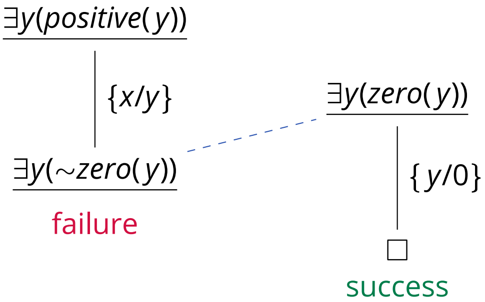
Soundness and “Completeness” of SLDNF Resolution

Lecture 8, 1st Dec 2025 // Foundations of Logic Programming, WS 2025/26

Previously ...

- **Normal logic programs** allow for “negation” in queries (clause bodies).
- The **negation as failure** rule treats negated atoms $\sim A$ in queries by asking the query A in a **subsidiary tree** and negating the answer.
- A proof theory for normal logic programs is given by **SLDNF resolution**.
- Care must be taken not to let **non-ground negative literals** get selected.
- A clause is **safe** iff each of its variables occurs in a positive body literal.

$$zero(0) \leftarrow$$
$$positive(x) \leftarrow \sim zero(x)$$



Overview

First-Order Formulas and Logical Truth

Completion of Programs

Soundness of SLDNF Resolution

Restricted Completeness of SLDNF Resolution

First-Order Formulas and Logical Truth

First-Order Formulas

Definition

Let Π and F be ranked alphabets of predicate symbols and function symbols, respectively, and V be a set of variables.

The set of **(first-order) formulas** (over Π, F, and V) is inductively defined as follows:

- if atom $A \in TB_{\Pi,F,V}$, then A is a formula;
- if G_1 and G_2 are formulas, then $\neg G_1$, $G_1 \wedge G_2$ (also written G_1, G_2), $G_1 \vee G_2$, $G_1 \leftarrow G_2$, and $G_1 \leftrightarrow G_2$ are formulas;
- if G is a formula and $x \in V$, then $\forall x G$ and $\exists x G$ are formulas.

Note

Whenever we interpret normal queries/clauses as first-order formulas, we (for now) take $\sim$ to be $\neg$.

Extended Notion of Logical Truth (1)

Definition

Let G be a formula, I be an interpretation with domain D, and $\sigma\colon V \to D$ be a state.

Formula G **is true in** I **under** σ, written $I, \sigma \Vdash G$, based on the structure of G:

- $I, \sigma \Vdash p(t_1, \ldots, t_n) \quad :\Longleftrightarrow \quad (\sigma(t_1), \ldots, \sigma(t_n)) \in p_I$
- $I, \sigma \Vdash \neg G \quad :\Longleftrightarrow \quad I, \sigma \nVdash G$
- $I, \sigma \Vdash G_1 \land G_2 \quad :\Longleftrightarrow \quad I, \sigma \Vdash G_1$ and $I, \sigma \Vdash G_2$
- $I, \sigma \Vdash G_1 \lor G_2 \quad :\Longleftrightarrow \quad I, \sigma \Vdash G_1$ or $I, \sigma \Vdash G_2$
- $I, \sigma \Vdash G_1 \leftarrow G_2 \quad :\Longleftrightarrow \quad$ if $I, \sigma \Vdash G_2$ then $I, \sigma \Vdash G_1$
- $I, \sigma \Vdash G_1 \leftrightarrow G_2 \quad :\Longleftrightarrow \quad I, \sigma \Vdash G_1$ iff $I, \sigma \Vdash G_2$
- $I, \sigma \Vdash \forall x G \quad :\Longleftrightarrow \quad$ for every $d \in D$: $I, \sigma[x \mapsto d] \Vdash G$
- $I, \sigma \Vdash \exists x G \quad :\Longleftrightarrow \quad$ for some $d \in D$: $I, \sigma[x \mapsto d] \Vdash G$

where $\sigma[x \mapsto d]$ denotes $\sigma'\colon V \to D$ with $\sigma'(x) = d$ and $\sigma'(y) = \sigma(y)$ for $y \in V \setminus \{x\}$.

Extended Notion of Logical Truth (2)

Definition

Let G be a formula, S and T be sets of formulas, and I be an interpretation. Furthermore, let $x_1, \ldots, x_k$ be the variables occurring in G.

- $\forall x_1, \ldots, \forall x_k G$ is the **universal closure** of G (abbreviated $\forall G$).
- $I \Vdash \forall G \;\; :\Longleftrightarrow \;\; I, \sigma \Vdash G$ for every state σ
- **G is true in I** (or: **I is a model of G**), written: $I \Vdash G \;\; :\Longleftrightarrow \;\; I \Vdash \forall G$
- **I is a model of S**, written: $I \Vdash S \;\; :\Longleftrightarrow \;\; I \Vdash G$ for every $G \in S$
- **T is a semantic (or: logical) consequence of S**, written: $S \models T$
  $:\Longleftrightarrow$ every model of S is a model of T

Negative Consequences of Logic Programs (1)

Consider program P_{mem}:

$$member(x, [x \mid y]) \leftarrow$$
$$member(x, [y \mid z]) \leftarrow member(x, z)$$

Then, e.g. $P_{mem} \models member(a, [a, b])$ and $P_{mem} \not\models member(a, [\,])$.

But also $P_{mem} \not\models \neg member(a, [\,])$, since $HB_{\{member\},\{\mid,[\,],a\}} \Vdash P_{mem}$ and $HB_{\{member\},\{\mid,[\,],a\}} \nVdash \neg member(a, [\,])$.

Nevertheless the SLDNF tree of $P_{mem} \cup \{\sim member(a, [\,])\}$ is successful:

$$\underline{\sim member(a, [\,])}$$
success

$$\underline{member(a, [\,])}$$
failure

Negative Consequences of Logic Programs (2)

Observation

For every normal logic program P over vocabulary Π, F, the Herbrand base $HB_{\Pi,F}$ is a model of P. (All implications are satisfied.)

Corollary

There is no negative ground literal $\neg A$ that is a logical consequence of P.

But: The SLDNF tree of $P \cup \{\sim A\}$ may be successful!

$\rightsquigarrow$ Taking $\sim$ to be $\neg$, SLDNF resolution is not sound w.r.t. $P \models$.

Solution:

Avoid Herbrand models that are "too large".

$\rightsquigarrow$ Formalise "the information in the program is all there is".

$\rightsquigarrow$ Strengthen P to its completion $comp(P)$.

Completion of Programs

Completed Definitions (Example 1)

$$P: \quad \begin{array}{lcl} \mathit{happy} & \leftarrow & \mathit{sun}, \mathit{holidays} \\ \mathit{happy} & \leftarrow & \mathit{snow}, \mathit{holidays} \\ \mathit{snow} & \leftarrow & \mathit{cold}, \mathit{precipitation} \\ \mathit{cold} & \leftarrow & \mathit{winter} \\ \mathit{precipitation} & \leftarrow & \mathit{holidays} \\ \mathit{winter} & \leftarrow & \\ \mathit{holidays} & \leftarrow & \end{array}$$

$$\mathit{comp}(P): \quad \begin{array}{lcl} \mathit{happy} & \leftrightarrow & (\mathit{sun} \land \mathit{holidays}) \lor (\mathit{snow} \land \mathit{holidays}) \\ \mathit{snow} & \leftrightarrow & \mathit{cold} \land \mathit{precipitation} \\ \mathit{cold} & \leftrightarrow & \mathit{winter} \\ \mathit{precipitation} & \leftrightarrow & \mathit{holidays} \\ \mathit{winter} & \leftrightarrow & \mathit{true} \\ \mathit{holidays} & \leftrightarrow & \mathit{true} \\ \mathit{sun} & \leftrightarrow & \mathit{false} \end{array}$$

Then, $\mathit{comp}(P) \models \mathit{happy}, \mathit{snow}, \mathit{cold}, \mathit{precipitation}, \mathit{winter}, \mathit{holidays}, \neg \mathit{sun}$.

Completed Definitions (Example 2)

$$P: \quad \begin{array}{lcl} member(x,[x\,|\,y]) & \leftarrow & \\ member(x,[y\,|\,z]) & \leftarrow & member(x,z) \\ disjoint([\,],x) & \leftarrow & \\ disjoint([x\,|\,y],z) & \leftarrow & \sim member(x,z),\ disjoint(y,z) \end{array}$$

$$comp(P): \quad \begin{array}{lcl} \forall x_1,x_2 \Big(member(x_1,x_2) & \leftrightarrow & (\exists x,y\,(x_1 = x \wedge x_2 = [x\,|\,y]) \vee \\ & & \exists x,y,z\,(x_1 = x \wedge x_2 = [y\,|\,z] \wedge member(x,z))) \Big) \\ \forall x_1,x_2 \Big(disjoint(x_1,x_2) & \leftrightarrow & (\exists x\,(x_1 = [\,] \wedge x_2 = x) \vee \\ & & \exists x,y,z\,(x_1 = [x\,|\,y] \wedge x_2 = z \wedge \\ & & \quad \neg member(x,z) \wedge disjoint(y,z))) \Big) \end{array}$$

plus standard equality and inequality axioms

Then, e.g. $comp(P) \models member(a,[a,b])$, $\neg member(a,[\,])$, $\neg disjoint([a],[a])$.

Completion (1)

Definition (Clark, 1978)

Let P be a normal logic program. The **completion** of P (denoted by $comp(P)$) is the set of formulas constructed from P by the following 6 steps:

1. Associate with every n-ary predicate symbol p (from P) a sequence of pairwise distinct variables $x_1, \ldots, x_n$ that do not occur in P.

2. Transform each clause $c = p(t_1, \ldots, t_n) \leftarrow B_1, \ldots, B_k$ into
$$p(x_1, \ldots, x_n) \leftarrow x_1 = t_1 \wedge \ldots \wedge x_n = t_n \wedge B_1 \wedge \ldots \wedge B_k$$
Any empty conjunction (for $n = 0$ or $k = 0$) is replaced by *true*.

3. Transform each resulting formula $p(x_1, \ldots, x_n) \leftarrow G$ into
$$p(x_1, \ldots, x_n) \leftarrow \exists \vec{z}\, G$$
where $\vec{z}$ is a sequence of the elements of $Var(c)$.

Completion (2)

Definition (Clark, 1978, continued)

4. For every n-ary predicate symbol p, let

$$p(x_1,\ldots,x_n) \leftarrow \exists\vec{z}_1 G_1, \quad \ldots, \quad p(x_1,\ldots,x_n) \leftarrow \exists\vec{z}_m G_m$$

be all implications obtained in Step 3 ($m \geq 0$).

If $m > 0$, then replace these by the formula

$$\forall x_1,\ldots,x_n \Big(p(x_1,\ldots,x_n) \leftrightarrow (\exists\vec{z}_1 G_1 \vee \ldots \vee \exists\vec{z}_m G_m) \Big)$$

If $m = 0$, then add the formula

$$\forall x_1,\ldots,x_n (p(x_1,\ldots,x_n) \leftrightarrow \mathit{false})$$

Completion (3)

Definition (Clark, 1978, continued)

5. Add the **standard axioms of equality**:

$$\begin{array}{ll} \forall\ [\ x = x &] \\ \forall\ [\ x = y\ \rightarrow\ y = x &] \\ \forall\ [\ x = y \land y = z\ \rightarrow\ x = z &] \\ \forall\ [\ x_i = y\ \rightarrow\ f(x_1, \ldots, x_i, \ldots, x_n) = f(x_1, \ldots, y, \ldots, x_n) &] \\ \forall\ [\ x_i = y\ \rightarrow\ (p(x_1, \ldots, x_i, \ldots, x_n) \leftrightarrow p(x_1, \ldots, y, \ldots, x_n)) &] \end{array}$$

6. Add the **standard axioms of inequality**:

$$\forall\ [\ x_1 \neq y_1 \lor \ldots \lor x_n \neq y_n\ \rightarrow\ f(x_1, \ldots, x_n) \neq f(y_1, \ldots, y_n)\ ]$$

$$\forall\ [\ f(x_1, \ldots, x_m) \neq g(y_1, \ldots, y_n)\ ] \qquad \text{(whenever } f \neq g\text{)}$$

$$\forall\ [\ x \neq t\ ] \qquad \text{(whenever } x \text{ is a proper subterm of } t\text{)}$$

Steps 5. and 6. ensure that "$=$" must be interpreted as equality.

Quiz: Completion

Quiz

Consider the following logic program P: …

Soundness of SLDNF Resolution

Soundness of SLDNF Resolution

Definition

Let P be a normal logic program, Q be a normal query, and θ be a substitution.

- $\theta|_{Var(Q)}$ is a **correct answer substitution** of Q $:\Longleftrightarrow$ $comp(P) \models Q\theta$
- $Q\theta$ is a **correct instance** of Q $:\Longleftrightarrow$ $comp(P) \models Q\theta$

Theorem (Lloyd, 1987)

If there exists a successful SLDNF derivation of $P \cup \{Q\}$ with computed answer substitution θ, then $comp(P) \models Q\theta$.

Corollary

If there exists a successful SLDNF derivation of $P \cup \{Q\}$, then $comp(P) \models \exists Q$.

We assume here that $\sim$ in queries has been replaced by $\neg$ for entailment.

Restricted Completeness of SLDNF Resolution

Incompleteness (1): Inconsistency

$$P: \quad p \leftarrow \sim p$$

By definition, it follows that $comp(P) \supseteq \{p \leftrightarrow \neg p\} \equiv \{false\}$.

Hence, $comp(P) \models p$ and $comp(P) \models \neg p$ because $I \not\Vdash comp(P)$ for every interpretation I, i.e. $comp(P)$ has no model ($comp(P)$ is **unsatisfiable**).

But there is neither a successful SLDNF derivation of $P \cup \{p\}$ nor of $P \cup \{\sim p\}$.

Incompleteness (2): Non-Strictness

$$P: \quad \begin{array}{rcl} p & \leftarrow & q \\ p & \leftarrow & \sim q \\ q & \leftarrow & q \end{array}$$

Thus $comp(P) \supseteq \{p \leftrightarrow (q \vee \neg q), \quad q \leftrightarrow q\} \equiv \{p \leftrightarrow true\}$.

Hence, $comp(P) \models p$.

But there is no successful SLDNF derivation of $P \cup \{p\}$.

Incompleteness (3): Floundering

$$P: \quad p(x) \leftarrow {\sim}q(x)$$

Thus $comp(P) \supseteq \{\forall x_1(p(x_1) \leftrightarrow \exists x(x_1 = x \wedge \neg q(x))), \quad \forall x_1(q(x_1) \leftrightarrow \mathit{false})\}$
$\equiv \{\forall x_1(p(x_1) \leftrightarrow \mathit{true}), \quad \forall x_1(q(x_1) \leftrightarrow \mathit{false})\}.$

Hence, $comp(P) \models \forall x_1 p(x_1)$.

But there is no successful SLDNF derivation of $P \cup \{p(x_1)\}$.

Incompleteness (4): Unfairness

$$\begin{array}{llll} P: & r & \leftarrow & p, q \\ & p & \leftarrow & p \end{array}$$

$$\begin{aligned} \text{Thus } \mathit{comp}(P) &\supseteq \{r \leftrightarrow (p \wedge q), \quad p \leftrightarrow p, \quad q \leftrightarrow \mathit{false}\} \\ &\equiv \{r \leftrightarrow \mathit{false}, \quad q \leftrightarrow \mathit{false}\}. \end{aligned}$$

Hence, $\mathit{comp}(P) \models \neg r$.

But there is no successful SLDNF derivation of $P \cup \{\sim r\}$ w.r.t. leftmost selection rule.

Dependency Graphs

Definition

The **dependency graph** D_P of a normal logic program P is a directed graph with labelled edges, where

- the nodes are the predicate symbols of P
- the edges are either labelled by $+$ (positive edge) or by $-$ (negative edge)
- there is an edge $q \xrightarrow{+} p$ in D_P
  $:\iff \quad P$ contains a clause $p(s_1, \ldots, s_m) \leftarrow \vec{L}, q(t_1, \ldots, t_n), \vec{N}$
- there is an edge $q \xrightarrow{-} p$ in D_P
  $:\iff \quad P$ contains a clause $p(s_1, \ldots, s_m) \leftarrow \vec{L}, {\sim}q(t_1, \ldots, t_n), \vec{N}$

Note that the direction of the edges is sometimes reversed in other texts.

Strict, Hierarchical, Stratified Programs

Definition

Let P be a normal program with dependency graph D_P,
let p, q be predicate symbols, and Q be a normal query.

- p depends **evenly** (resp. **oddly**) on q
  $:\Longleftrightarrow$ there is a path from q to p in D_P with
  an even (resp. odd) number of *negative* edges
- P is **strict** w.r.t. Q
  $:\Longleftrightarrow$ no predicate symbol occurring in Q depends both evenly and oddly on a predicate symbol in the head of a clause in P
- P is **hierarchical** $:\Longleftrightarrow$ no cycle exists in D_P
- P is **stratified** $:\Longleftrightarrow$ no cycle with a negative edge exists in D_P

“Completeness” of SLDNF Resolution

- Recall: A query is *safe* iff each of its variables occurs in some of its positive atoms.
- Recall: A selection rule is *safe* iff it never selects a non-ground negative literal.

Theorem (Lloyd, 1987)

Fix $\mathcal{R}$ to be a safe selection rule.

Let P be a hierarchical and safe program, and Q be a safe query.

If $comp(P) \models Q\theta$ for some θ such that $Q\theta$ is ground, then there exists a successful SLDNF derivation (via $\mathcal{R}$) of $P \cup \{Q\}$ with cas θ.

Theorem (Cavedon and Lloyd, 1989)

Fix $\mathcal{R}$ to be a safe and fair selection rule.

Let P be a stratified and safe program and Q be a safe query, such that P is strict w.r.t. Q.

If $comp(P) \models Q\theta$ for some θ such that $Q\theta$ is ground, then there exists a successful SLDNF derivation of $P \cup \{Q\}$ with cas θ.

Fair Selection Rules

Definition

An (SLDNF) selection rule $\mathcal{R}$ is **fair**

$:\Longleftrightarrow$ for every SLDNF tree $\mathcal{F}$ via $\mathcal{R}$ and for every branch ξ in $\mathcal{F}$:

- either ξ is failed,
- or for every literal L occurring in a query of ξ, (some further instantiated version of) L is selected within a finite number of derivation steps.

Example

- The selection rule "select leftmost literal" is unfair.
- The selection rule "select leftmost literal to the right of the literals introduced at the previous derivation step, if it exists, otherwise select leftmost literal" is fair.

Specifics of PROLOG

- Leftmost selection rule: **LDNF**-resolution, **LDNF**-resolvent, **LDNF**-tree, . . .
- Non-ground negative literals are selected.
- A program is a sequence of clauses.
- Unification is done without occur check.
- Derivation construction via depth-first search with backtracking.

Extended Prolog Trees

Definition

Let P be a normal program and Q_0 be a normal query.

The **Extended Prolog Tree** for $P \cup \{Q_0\}$ is a forest of finitely branching, ordered trees of queries, possibly marked with “success” or “failure”, produced as follows:

- start with forest $(\{T_{Q_0}\}, T_{Q_0}, subs)$, where T_{Q_0} contains the single node Q_0 and $subs(Q_0)$ is undefined;
- repeatedly apply to current forest $\mathcal{F} = (\mathcal{T}, T, subs)$ and **leftmost** unmarked leaf Q in T_1, where $T_1 \in \mathcal{T}$ is the **leftmost**, **bottommost** (most nested subsidiary) tree with an unmarked leaf, the operation *expand*$(\mathcal{F}, Q)$.

Operation Expand

Definition

The operation *expand*$(\mathcal{F}, Q)$ for query Q in tree T_1 is defined as follows:

- if $Q = \Box$, then
  1. mark Q with "success"
  2. if $T_1 \neq T$, then remove from T_1 all edges to the right of the branch that ends with Q
- if Q has no LDNF-resolvents, then mark Q with "failure"
- else let L be the leftmost literal in Q:
  - L is positive: add for each clause that is applicable to L an LDNF-resolvent as descendant of Q (respecting the order of the clauses in the program)
  - $L = \sim A$ is negative (not necessarily ground):
    - if $subs(Q)$ is undefined, then add a new tree $T' = A$ and set $subs(Q)$ to T';
    - if $subs(Q)$ is defined and successful, then mark Q with "failure";
    - if $subs(Q)$ is defined and finitely failed, then add in T_1 the LDNF-resolvent of Q as the only descendant of Q.

Floundering is Ignored (1)

```
even(0).
even(X) :- \+ odd(X).
odd(s(X)) :- even(X).
```

```
| ?- even(X).
```

```
X = 0 ;
no
```

```
| ?- even(s(s(0))).
```

```
yes
```

Floundering is Ignored (2)

```
num(0).
num(s(X)) :- num(X).
even(X) :- num(X), \+ odd(X).
odd(s(X)) :- even(X).
```

```
| ?- even(X).
```

```
X = 0 ;

X = s(s(0)) ;

X = s(s(s(s(0)))) ;
```

⋮

Conclusion

Summary

- For every normal logic program P, its **completion** $comp(P)$ replaces the logical implications of clauses by equivalences (to disjunctions of bodies).
- SLDNF resolution w.r.t. P is **sound** for entailment w.r.t. $comp(P)$.
- SLDNF resolution is only **complete** (for entailment w.r.t. $comp(P)$) for certain combinations of classes of programs, queries, and selection rules.
- For a normal program P, its **dependency graph** D_P explicitly shows positive and negative dependencies between predicate symbols.
- A normal program P is **stratified** iff D_P has no cycle with a negative edge.

Suggested action points:

- Construct the completion of the programs on slides 31 and 32.
- Find a program that shows unfairness of the leftmost selection rule.