General Concept Inclusions with High Confidence in Finite Interpretations

Daniel Borchmann

QuantLA Summer Workshop 2013

2013-09-12

Goal

Use description logic ontologies to represent knowledge of certain domains

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Problem

How to obtain these ontologies?

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Approach

Learn ontologies from domain data

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Use description logic ontologies to represent knowledge of certain domains

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Approach

Learn first versions of ontologies from domain data

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Extract terminological knowledge from factual knowledge.

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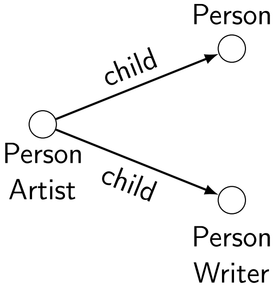

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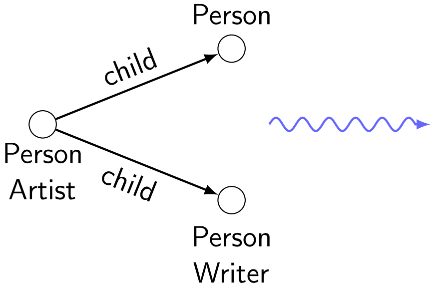
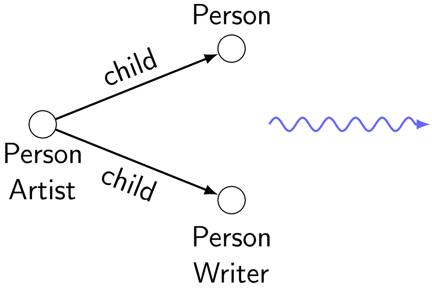


$\exists$child.Writer $\sqsubseteq$ Artist

Goal

Extract terminological knowledge from interpretations.



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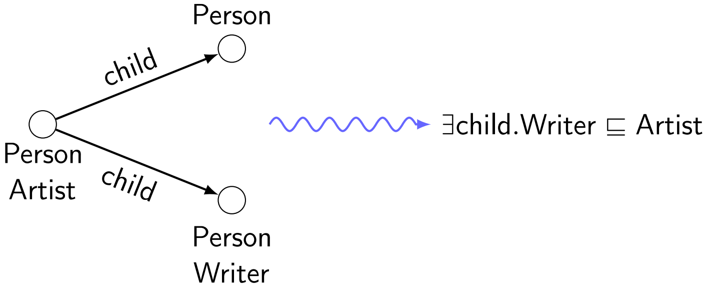
Extract finite bases of GCIs from interpretations.



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Outline

1. The Original Approach
2. Adding Confidence
3. Experiments with $\mathcal{I}_{\text{DBpedia}}$
4. Exploring Confident GCIs
5. Conclusions

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Description Logic

Use $\mathcal{EL}$ and $\mathcal{EL}^{\bot}$ with usual semantics.

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- *General Concept Inclusions* (GCIs) are of the form

$$C \sqsubseteq D$$

where C, D are $\mathcal{EL}^{\bot}$-concept descriptions.

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General Concept Inclusions

- *General Concept Inclusions* (GCIs) are of the form

$$C \sqsubseteq D$$

  where C, D are $\mathcal{EL}^\bot$-concept descriptions.
- $C \sqsubseteq D$ *holds* in $\mathcal{I}$ if and only if

$$C^\mathcal{I} \subseteq D^\mathcal{I}$$

Example (Formal Context)

	small	medium	large	inner	outer	moon	no moon
Mercury	×			×			×
Venus	×			×			×
Earth	×			×		×	
Mars	×			×		×	
Jupiter			×		×	×	
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Example (Contextual Derivation)

$$\{\, outer, small \,\}' = \{\, Pluto \,\}$$
$$\{\, Neptune, Jupiter \,\}' = \{\, outer, moon \,\}$$

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$$\{\, outer, moon \,\}'$$

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Observation

$$\{ \mathit{outer}, \mathit{moon} \}' = \{ \mathit{Jupiter}, \mathit{Saturn}, \mathit{Uranus}, \mathit{Neptune}, \mathit{Pluto} \}$$

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$$\begin{aligned} \{\, outer, moon \,\}' &= \{\, Jupiter, Saturn, Uranus, Neptune, Pluto \,\} \\ &= \{\, outer \,\}' \end{aligned}$$

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$$\{\, outer, moon \,\}' = \{\, Jupiter, Saturn, Uranus, Neptune, Pluto \,\}$$
$$= \{\, outer \,\}'$$

The *implication* $\{\, outer \,\} \to \{\, moon \,\}$ *holds.*

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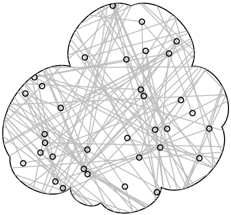
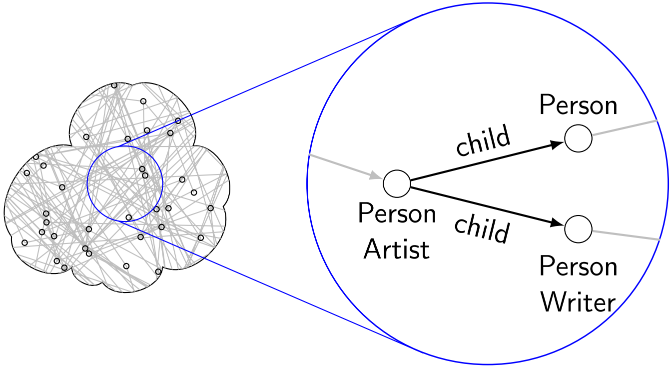
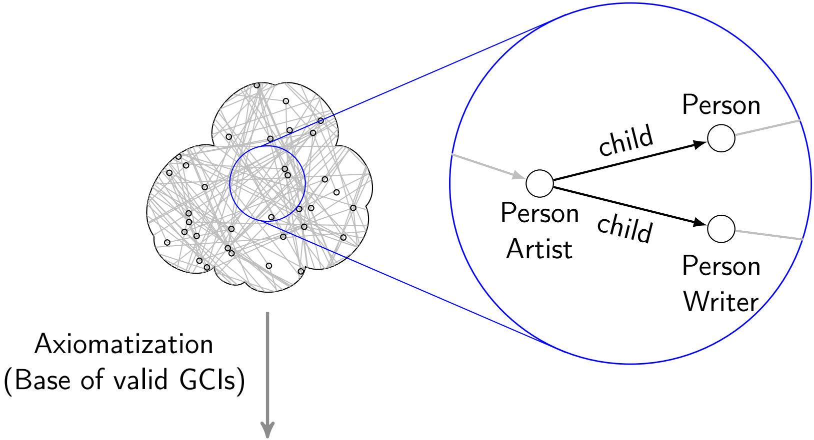
- For each $X \subseteq M$, it is true that $X \subseteq X''$ and that $X \to X''$ is valid in $\mathbb{K}$.




Person
child
Person
Artist
child
Person
Writer



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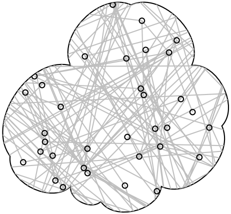

Axiomatization
(Base of valid GCIs)

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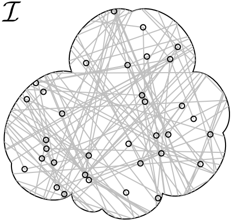
$\mathcal{I}$



$\mathbb{K}_{\mathcal{I}}$	$M_{\mathcal{I}}$		
$\Delta^{\mathcal{I}}$	×	×	.
	×	.	×
	.	.	×

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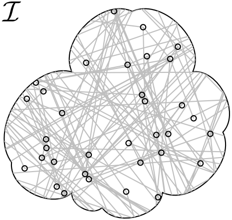

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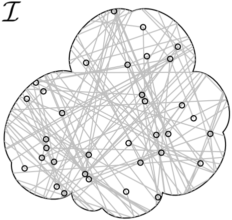

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$B', B \subseteq G$	**?**

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Need to describe sets $X \subseteq \Delta^{\mathcal{I}}$ *as good as possible*.

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Difficulty

Existence can not be guaranteed in $\mathcal{EL}^{\bot} \rightsquigarrow \mathcal{EL}^{\bot}_{\mathrm{gfp}}$

Example ($\mathcal{EL}^{\bot}_{\mathrm{gfp}}$ Concept Description)

$$C := (A, \{\, A \equiv \mathsf{Writer} \sqcap \exists \mathsf{child}.B,\ B \equiv \mathsf{Writer} \sqcap \exists \mathsf{child}.A \,\})$$

Definition

Set

$$M_{\mathcal{I}} := \{ \bot \} \cup N_C \cup \{ \exists r.X^{\mathcal{I}} \mid X \subseteq \Delta^{\mathcal{I}}, X \neq \emptyset, r \in N_R \}.$$

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and define *induced formal context* $\mathbb{K}_{\mathcal{I}}$ of $\mathcal{I}$ as $\mathbb{K}_{\mathcal{I}} = (\Delta^{\mathcal{I}}, M_{\mathcal{I}}, \nabla)$, where

$$(x, C) \in \nabla \iff x \in C^{\mathcal{I}}.$$

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Theorem

Let $\mathcal{I}$ be a finite interpretation. If $\mathcal{L}$ is a base of all confident implications of $\mathbb{K}_{\mathcal{I}}$, then the set

$$\{\, \bigsqcap U \to ((\bigsqcap U)^{\mathcal{I}})^{\mathcal{I}} \mid (U \to U'') \in \mathcal{L} \,\}$$

is a base of $\mathrm{Th}(\mathcal{I})$.

Outline

Recall

$$\exists \text{child}.\top \sqsubseteq \text{Person}$$

does not hold in $\mathcal{I}_{\text{DBpedia}}$, but there are only 4 erroneous counterexamples.

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From 2551 individuals satisfying $\exists \text{child}.\top$, 2547 also satisfy Person, i. e.

$$\text{conf}_{\mathcal{I}_{\text{DBpedia}}}(\exists \text{child}.\top \sqsubseteq \text{Person}) = \frac{2547}{2551}$$

Definition

The *confidence* of $C \sqsubseteq D$ in $\mathcal{I}$ is defined as

$$\operatorname{conf}_{\mathcal{I}}(C \sqsubseteq D) := \begin{cases} 1 & \text{if } C^{\mathcal{I}} = \emptyset, \\ \frac{|(C \sqcap D)^{\mathcal{I}}|}{|C^{\mathcal{I}}|} & \text{otherwise.} \end{cases}$$

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Let $c \in [0,1]$. Define

$$\mathrm{Th}_c(\mathcal{I}) := \{\, C \sqsubseteq D \mid \mathrm{conf}_{\mathcal{I}}(C \sqsubseteq D) \geqslant c \,\}.$$

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New Goal

Axiomatize $\mathrm{Th}_c(\mathcal{I})$, i. e. find a *finite base* of $\mathrm{Th}_c(\mathcal{I})$.

Question

How to find bases for $\mathrm{Th}_c(\mathcal{I})$?

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Related work by M. Luxenburger on *partial implications*

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Approach (Luxenburger)

- Separately axiomatize *valid* GCIs and *properly confident* GCIs

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Approach (Luxenburger)

- Separately axiomatize *valid* GCIs and *properly confident* GCIs
- Consider concept descriptions of the form $(C^{\mathcal{I}})^{\mathcal{I}}$ only

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Let $\mathcal{B}$ base of $\mathbb{K}$,

$$\mathcal{C} = \{\, X'' \to Y'' \mid 1 > \operatorname{conf}_{\mathbb{K}}(X'' \to Y'') \geqslant c \,\}$$

and $1 > \operatorname{conf}(A \to B) \geqslant c$.

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Definition

$$\mathsf{Conf}(\mathcal{I}, c) := \{\, X^{\mathcal{I}} \sqsubseteq Y^{\mathcal{I}} \mid Y, X \subseteq \Delta_{\mathcal{I}}, 1 > \mathsf{conf}_{\mathcal{I}}(X^{\mathcal{I}} \sqsubseteq Y^{\mathcal{I}}) \geqslant c \,\}.$$

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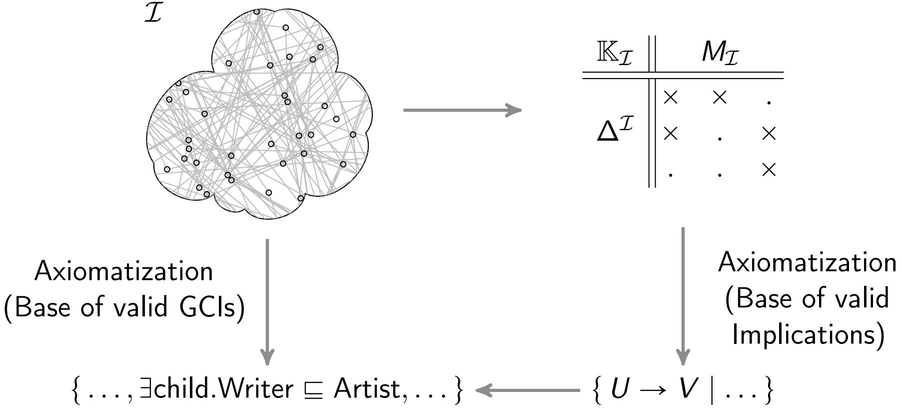
Theorem

Let $\mathcal{B}$ be a finite base of $\mathcal{I}$, and $c \in [0, 1]$. Then $\mathcal{B} \cup \mathsf{Conf}(\mathcal{I}, c)$ is a finite base of $\mathsf{Th}_c(\mathcal{I})$.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$\mathbb{K}_\mathcal{I}$
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Axiomatization (Base of valid GCIs)
Axiomatization (Base of valid Implications)
$\{ \dots, \exists \text{child.Writer} \sqsubseteq \text{Artist}, \dots \}$
$\{ U \to V \mid \dots \}$

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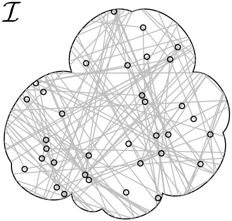

$\mathbb{K}_{\mathcal{I}}$	$M_{\mathcal{I}}$		
	×	×	.
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	.	.	×

Axiomatization
(Base of valid GCIs)

Axiomatization
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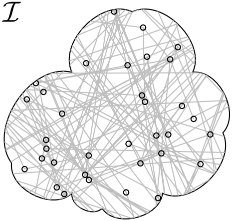

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	×	.	×
	.	.	×

Axiomatization (Base of confident GCIs)

Axiomatization (Base of confident Implications)

$\{\ldots, \exists \text{child.Writer} \sqsubseteq \text{Artist}, \ldots\}$ ← $\{U \to V \mid \ldots\}$

Definition

If $\mathbb{K}$ is a formal context, $X \to Y$ an implication of $\mathbb{K}$, then

$$\operatorname{conf}_{\mathbb{K}}(X \to Y) := \begin{cases} 1 & X' = \emptyset \\ \frac{|(X \cup Y)'|}{|X'|} & \text{otherwise} \end{cases}$$

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Goal

Transform bases of $\mathrm{Th}_c(\mathbb{K})$ to bases of $\mathrm{Th}_c(\mathcal{I})$.

Lemma

If $X, Y \subseteq M_{\mathcal{I}}$*, then*

$$\operatorname{conf}_{\mathcal{I}}(\bigsqcap X \sqsubseteq \bigsqcap Y) = \operatorname{conf}_{\mathbb{K}_{\mathcal{I}}}(X \to Y).$$

Lemma

If $X, Y \subseteq M_{\mathcal{I}}$, *then*

$$\mathrm{conf}_{\mathcal{I}}(\bigsqcap X \sqsubseteq \bigsqcap Y) = \mathrm{conf}_{\mathbb{K}_{\mathcal{I}}}(X \to Y).$$

Definition

If $\mathcal{L}$ is a set of implications of $\mathbb{K}_{\mathcal{I}}$, then

$$\bigsqcap \mathcal{L} := \{ \bigsqcap X \sqsubseteq \bigsqcap Y \mid (X \to Y) \in \mathcal{L} \}.$$

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Lemma

Let $\mathcal{L} \cup \{ X \to Y \}$ be a set of implications of $\mathbb{K}_{\mathcal{I}}$. Then

$$\mathcal{L} \models (X \to Y) \implies \bigsqcap \mathcal{L} \models \bigsqcap X \sqsubseteq \bigsqcap Y.$$

Theorem

Let $\mathcal{L}$ *be a (confident) base of* $\mathsf{Th}_c(\mathbb{K}_{\mathcal{I}})$. *Then* $\bigsqcap \mathcal{L}$ *is a (confident) base of* $\mathsf{Th}_c(\mathcal{I})$.

Theorem

Let $\mathcal{L}$ be a (confident) base of $\mathrm{Th}_c(\mathbb{K}_\mathcal{I})$. Then $\bigsqcap \mathcal{L}$ is a (confident) base of $\mathrm{Th}_c(\mathcal{I})$.

"Corollary"

We can use data mining techniques to extract complete sets of confident implications from $\mathbb{K}_\mathcal{I}$, and thus to learn confident GCIs from interpretations!

Proof (Sketch).

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□

Observation

The set

$$\mathcal{S}_{\mathcal{I}} := \{\, \{\, C \,\} \to \{\, D \,\} \mid C, D \in M_{\mathcal{I}}, C \text{ more specific than } D \,\}$$

is a set of *non-trivial* implications.

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Theorem

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Outline

"Practical" Questions

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- How large is $|\mathrm{Conf}(\mathcal{I}, c)|$?

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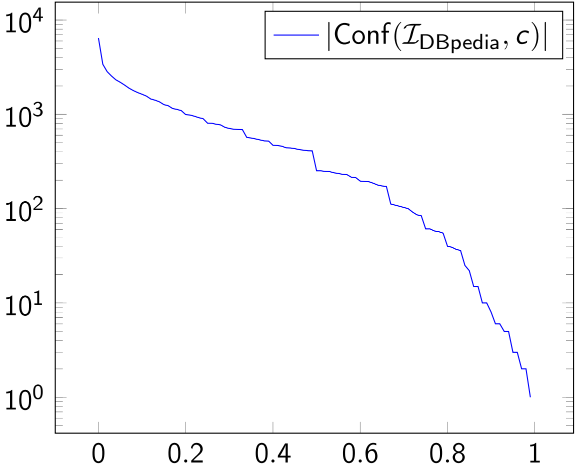
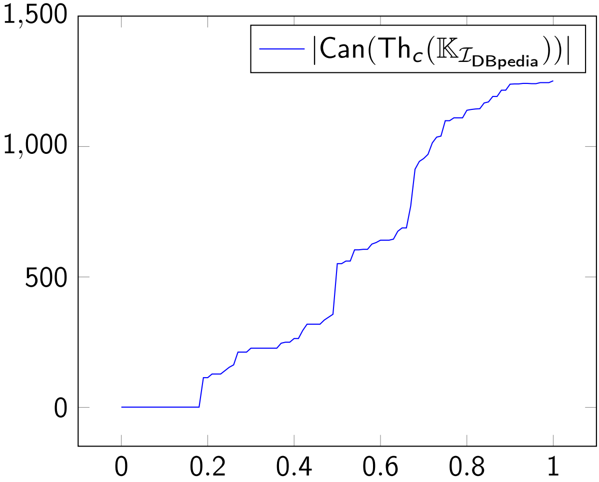
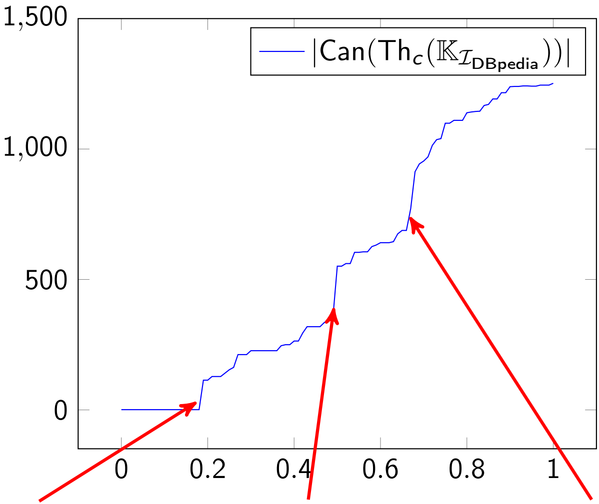
- How large is $|\mathrm{Conf}(\mathcal{I}, c)|$?
- What "kind" of GCIs are in $|\mathrm{Conf}(\mathcal{I}, c)|$?


$|\mathsf{Conf}(\mathcal{I}_{\mathsf{DBpedia}}, c)|$
10^4
10^3
10^2
10^1
10^0
0
0.2
0.4
0.6
0.8
1


1,500
1,000
500
0
0
0.2
0.4
0.6
0.8
1
$|\mathsf{Can}(\mathsf{Th}_c(\mathbb{K}_{\mathcal{I}_{\mathsf{DBpedia}}}))|$


1,500
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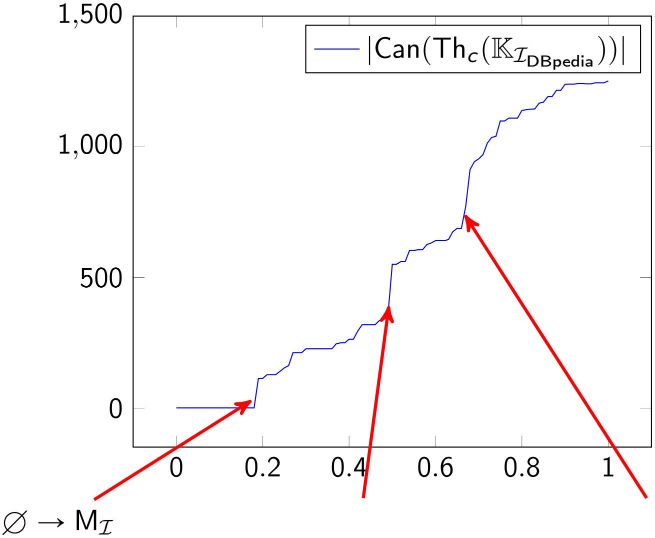
$|\mathrm{Can}(\mathrm{Th}_c(\mathbb{K}_{\mathcal{I}_{\mathsf{DBpedia}}}))|$


1,500
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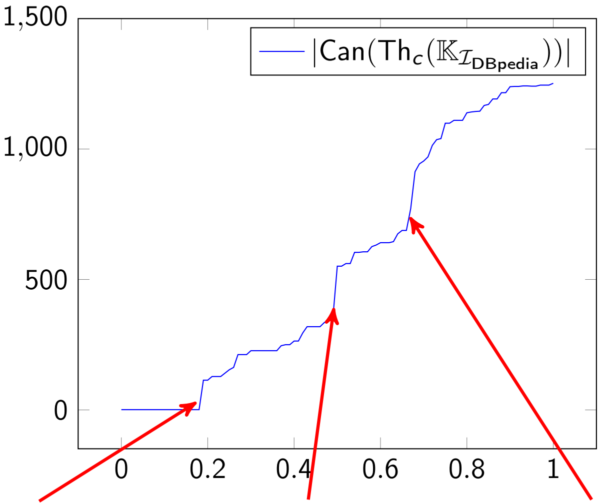
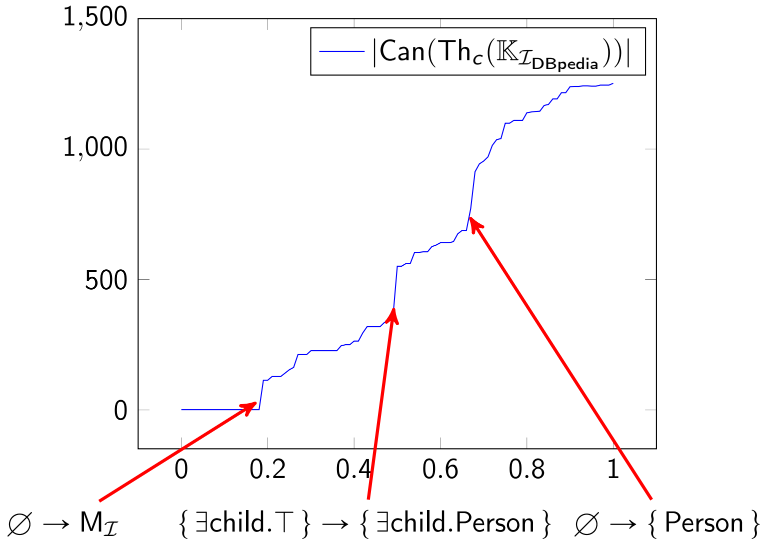
$|\mathrm{Can}(\mathrm{Th}_c(\mathbb{K}_{\mathcal{I}_{\mathsf{DBpedia}}}))|$
$\varnothing \to \mathrm{M}_{\mathcal{I}}$



$\varnothing \to M_{\mathcal{I}}$ $\{\exists \text{child}.\top\} \to \{\exists \text{child}.\text{Person}\}$


1,500
1,000
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$\varnothing \to M_{\mathcal{I}}$
$\{\,\exists\mathsf{child}.\top\,\} \to \{\,\exists\mathsf{child}.\mathsf{Person}\,\}$
$\varnothing \to \{\,\mathsf{Person}\,\}$

Experiment

Experiment

$$\mathsf{Conf}(\mathcal{I}_{\mathsf{DBpedia}}, 0.95) =$$

Experiment

$$
\begin{aligned}
\mathsf{Conf}(\mathcal{I}_{\mathsf{DBpedia}}, 0.95) = \{\, & \mathsf{Place} \sqsubseteq \mathsf{PopulatedPlace}, \\
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& \exists \mathsf{child}.\exists \mathsf{child}.\top \sqcap \exists \mathsf{child}.\mathsf{OfficeHolder} \\
& \quad \sqsubseteq \exists \mathsf{child}.(\mathsf{OfficeHolder} \sqcap \exists \mathsf{child}.\top) \,\}
\end{aligned}
$$

Experiment

$$
\begin{aligned}
\mathsf{Conf}(\mathcal{I}_{\mathsf{DBpedia}}, 0.95) = \{\, & \mathsf{Place} \sqsubseteq \mathsf{PopulatedPlace}, \\
& \exists \mathsf{child}.\top \sqsubseteq \mathsf{Person}, \qquad \checkmark \\
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$$\sqsubseteq \exists \mathsf{child}.(\mathsf{OfficeHolder} \sqcap \exists \mathsf{child}.\top) \,\}$$

Counterexample for first GCI

Greenwich_Village

Experiment

$$\mathsf{Conf}(\mathcal{I}_{\mathsf{DBpedia}}, 0.95) = \{\, \mathsf{Place} \sqsubseteq \mathsf{PopulatedPlace}, \quad \checkmark$$
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\end{aligned}
$$

Counterexample for third GCI

Pierre_Samuel_du_Pont_de_Nemours

Experiment

$$\begin{aligned}\mathsf{Conf}(\mathcal{I}_{\mathsf{DBpedia}}, 0.95) = \{\, & \mathsf{Place} \sqsubseteq \mathsf{PopulatedPlace}, \quad \checkmark \\ & \exists \mathsf{child}.\top \sqsubseteq \mathsf{Person}, \quad \checkmark \\ & \exists \mathsf{child}.\exists \mathsf{child}.\top \sqcap \exists \mathsf{child}.\mathsf{OfficeHolder} \\ & \quad \sqsubseteq \exists \mathsf{child}.(\mathsf{OfficeHolder} \sqcap \exists \mathsf{child}.\top) \,\}\end{aligned}$$

Counterexample for third GCI

Pierre_Samuel_du_Pont_de_Nemours

Two children (sons):

Experiment

$$\begin{aligned}\mathsf{Conf}(\mathcal{I}_{\mathsf{DBpedia}}, 0.95) = \{\ & \mathsf{Place} \sqsubseteq \mathsf{PopulatedPlace}, \quad \checkmark \\ & \exists \mathsf{child}.\top \sqsubseteq \mathsf{Person}, \quad \checkmark \\ & \exists \mathsf{child}.\exists \mathsf{child}.\top \sqcap \exists \mathsf{child}.\mathsf{OfficeHolder} \\ & \quad \sqsubseteq \exists \mathsf{child}.(\mathsf{OfficeHolder} \sqcap \exists \mathsf{child}.\top)\ \}\end{aligned}$$

Counterexample for third GCI

Pierre_Samuel_du_Pont_de_Nemours

Two children (sons):

- *Victor Marie du Pont*, diplomat, four children,

Experiment

$$\begin{aligned}\mathrm{Conf}(\mathcal{I}_{\mathrm{DBpedia}}, 0.95) = \{ & \mathrm{Place} \sqsubseteq \mathrm{PopulatedPlace}, \quad \checkmark \\ & \exists \mathrm{child}.\top \sqsubseteq \mathrm{Person}, \quad \checkmark \\ & \exists \mathrm{child}.\exists \mathrm{child}.\top \sqcap \exists \mathrm{child}.\mathrm{OfficeHolder} \\ & \quad \sqsubseteq \exists \mathrm{child}.(\mathrm{OfficeHolder} \sqcap \exists \mathrm{child}.\top) \}\end{aligned}$$

Counterexample for third GCI

Pierre_Samuel_du_Pont_de_Nemours

Two children (sons):

- *Victor Marie du Pont*, diplomat, four children, *but none of them were "famous."*

Experiment

$$\begin{aligned}\mathrm{Conf}(\mathcal{I}_{\mathrm{DBpedia}}, 0.95) = \{ &\mathrm{Place} \sqsubseteq \mathrm{PopulatedPlace}, \quad \checkmark \\ &\exists \mathrm{child}.\top \sqsubseteq \mathrm{Person}, \quad \checkmark \\ &\exists \mathrm{child}.\exists \mathrm{child}.\top \sqcap \exists \mathrm{child}.\mathrm{OfficeHolder} \\ &\quad \sqsubseteq \exists \mathrm{child}.(\mathrm{OfficeHolder} \sqcap \exists \mathrm{child}.\top) \}\end{aligned}$$

Counterexample for third GCI

Pierre_Samuel_du_Pont_de_Nemours

Two children (sons):

- *Victor Marie du Pont*, diplomat, four children, *but none of them were "famous."*
- *Eleuthère Irénée du Pont*, industrial (gunpowder)

Experiment

$$\begin{aligned}\mathsf{Conf}(\mathcal{I}_{\mathsf{DBpedia}}, 0.95) = \{\, & \mathsf{Place} \sqsubseteq \mathsf{PopulatedPlace}, \quad \checkmark \\ & \exists \mathsf{child}.\top \sqsubseteq \mathsf{Person}, \quad \checkmark \\ & \exists \mathsf{child}.\exists \mathsf{child}.\top \sqcap \exists \mathsf{child}.\mathsf{OfficeHolder} \quad ? \\ & \quad \sqsubseteq \exists \mathsf{child}.(\mathsf{OfficeHolder} \sqcap \exists \mathsf{child}.\top) \,\}\end{aligned}$$

Counterexample for third GCI

Pierre_Samuel_du_Pont_de_Nemours

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Counterexample for third GCI

Pierre_Samuel_du_Pont_de_Nemours

Experiment

$$
\begin{aligned}
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& \exists \mathrm{child}.\exists \mathrm{child}.\top \sqcap \exists \mathrm{child}.\mathrm{OfficeHolder} & ? \\
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\end{aligned}
$$

Counterexample for third GCI

Pierre_Samuel_du_Pont_de_Nemours

Approach

Experiment

$$\mathsf{Conf}(\mathcal{I}_{\mathsf{DBpedia}}, 0.95) = \{\, \mathsf{Place} \sqsubseteq \mathsf{PopulatedPlace}, \quad \checkmark$$
$$\exists \mathsf{child}.\top \sqsubseteq \mathsf{Person}, \quad \checkmark$$
$$\exists \mathsf{child}.\exists \mathsf{child}.\top \sqcap \exists \mathsf{child}.\mathsf{OfficeHolder} \quad ?$$
$$\sqsubseteq \exists \mathsf{child}.(\mathsf{OfficeHolder} \sqcap \exists \mathsf{child}.\top) \,\}$$

Counterexample for third GCI

Pierre_Samuel_du_Pont_de_Nemours

Approach

- Devise an exploration algorithm for confident GCIs!

Outline

Expert

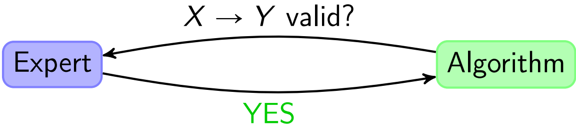
Algorithm

$\mathbb{K}$	m_1	$\dots$	m_n
g_1		$\dots$	
$\vdots$			
g_k		$\dots$	

$$\mathcal{S} = \{ A_1 \to B_1, \\ \dots \\ A_\ell \to B_\ell \}$$

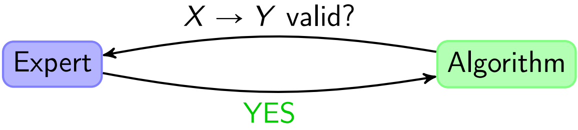
$\mathbb{K}$	m_1	$\dots$	m_n
g_1		$\dots$	
$\vdots$			
g_k		$\dots$	

$$\mathcal{S} = \{\, A_1 \to B_1, \; \dots \; A_\ell \to B_\ell \,\}$$



$\mathbb{K}$	m_1	$\ldots$	m_n
g_1		$\ldots$	
$\vdots$			
g_k		$\ldots$	

$$\mathcal{S} = \{ A_1 \to B_1, \ldots, A_\ell \to B_\ell \}$$

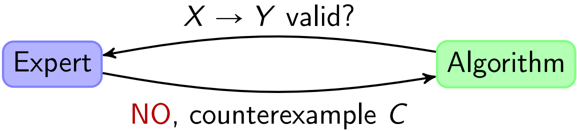


$\mathbb{K}$	m_1	$\dots$	m_n
g_1		$\dots$	
$\vdots$			
g_k		$\dots$	

$$\mathcal{S} = \{\, A_1 \to B_1,\ \dots,\ A_\ell \to B_\ell,\ X \to Y \,\}$$

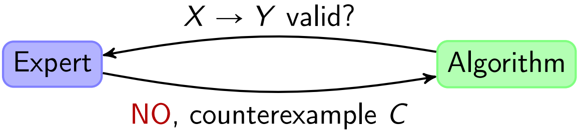
$\mathbb{K}$	m_1	$\dots$	m_n
g_1		$\dots$	
$\vdots$			
g_k		$\dots$	

$$\mathcal{S} = \{\, A_1 \to B_1, \; \dots \; A_\ell \to B_\ell \,\}$$



$\mathbb{K}$	m_1	$\ldots$	m_n
g_1		$\ldots$	
$\vdots$			
g_k		$\ldots$	

$$\mathcal{S} = \{ A_1 \rightarrow B_1, \ldots A_\ell \rightarrow B_\ell \}$$



$\mathbb{K}$	m_1	$\ldots$	m_n
g_1		$\ldots$	
$\vdots$			
g_k		$\ldots$	
g_{k+1}	——	C	——

$$\mathcal{S} = \{ A_1 \to B_1, \ldots, A_\ell \to B_\ell \}$$

Expert

Algorithm

$\mathbb{K}$	m_1	$\dots$	m_n
g_1		$\dots$	
$\vdots$			
g_k		$\dots$	

$$\mathcal{S} = \{\, A_1 \to B_1, \;\dots\; A_\ell \to B_\ell \,\}$$

Expert

Algorithm

$\mathbb{K}$	m_1	$\dots$	m_n
g_1		$\dots$	
$\vdots$			
g_k		$\dots$	

$$\mathcal{S} = \{\, A_1 \to B_1, \quad \dots \quad A_\ell \to B_\ell \,\}$$

Note

Expert

Algorithm

$\mathbb{K}$	m_1	$\dots$	m_n
g_1		$\dots$	
$\vdots$			
g_k		$\dots$	

$$\mathcal{S} = \{\, A_1 \to B_1, \;\dots\; A_\ell \to B_\ell \,\}$$

Note

- Upon termination, $\mathrm{Cn}(\mathcal{S}) = \mathrm{Th}(\mathbb{K})$ and $\mathcal{S}$ is a base of the implicational knowledge of the expert

Expert

Algorithm

$\mathbb{K}$	m_1	$\dots$	m_n
g_1		$\dots$	
$\vdots$			
g_k		$\dots$	

$$\mathcal{S} = \{\, A_1 \to B_1, \; \dots \; A_\ell \to B_\ell \,\}$$

Note

- Upon termination, $\mathrm{Cn}(\mathcal{S}) = \mathrm{Th}(\mathbb{K})$ and $\mathcal{S}$ is a base of the implicational knowledge of the expert
- The number of *confirmed* implications is as small as possible

Expert

Algorithm

$\mathbb{K}$	m_1	$\dots$	m_n
g_1		$\dots$	
$\vdots$			
g_k		$\dots$	

$$\mathcal{S} = \{ A_1 \to B_1, \dots, A_\ell \to B_\ell \}$$

Note

- Upon termination, $\mathrm{Cn}(\mathcal{S}) = \mathrm{Th}(\mathbb{K})$ and $\mathcal{S}$ is a base of the implicational knowledge of the expert
- The number of *confirmed* implications is as small as possible

Result (Distel 2008)

Attribute Exploration also possible for valid GCIs of finite interpretations

Goal

Devise an exploration algorithm for confident GCIs

Goal

Devise an exploration algorithm for confident GCIs

Challenges

Goal

Devise an exploration algorithm for confident GCIs

Challenges

- Single counterexamples given by the expert should be enough to invalidate GCIs

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Devise an exploration algorithm for confident GCIs

Challenges

- Single counterexamples given by the expert should be enough to invalidate GCIs $\rightsquigarrow$ distinguish between initial data and counterexamples provided by expert

Goal

Devise an exploration algorithm for confident GCIs

Challenges

- Single counterexamples given by the expert should be enough to invalidate GCIs $\rightsquigarrow$ distinguish between initial data and counterexamples provided by expert
- Compute candidate GCIs presented to the expert

Goal

Devise an exploration algorithm for confident GCIs

Challenges

- Single counterexamples given by the expert should be enough to invalidate GCIs $\rightsquigarrow$ distinguish between initial data and counterexamples provided by expert
- Compute candidate GCIs presented to the expert

Plan

Goal

Devise an exploration algorithm for confident GCIs

Challenges

- Single counterexamples given by the expert should be enough to invalidate GCIs $\rightsquigarrow$ distinguish between initial data and counterexamples provided by expert
- Compute candidate GCIs presented to the expert

Plan

- Extend classical attribute exploration to an *exploration by confidence*

Goal

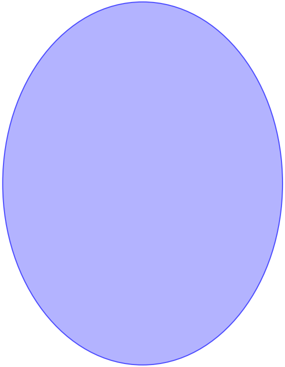
Devise an exploration algorithm for confident GCIs

Challenges

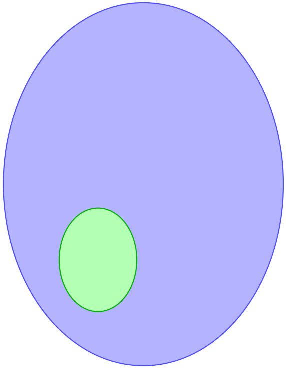
- Single counterexamples given by the expert should be enough to invalidate GCIs $\rightsquigarrow$ distinguish between initial data and counterexamples provided by expert
- Compute candidate GCIs presented to the expert

Plan

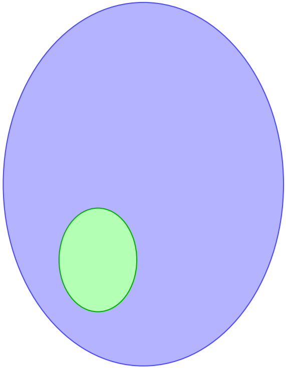
- Extend classical attribute exploration to an *exploration by confidence*
- Lift this exploration by confidence to GCIs



- *Interesting Implications* $\mathcal{L}$



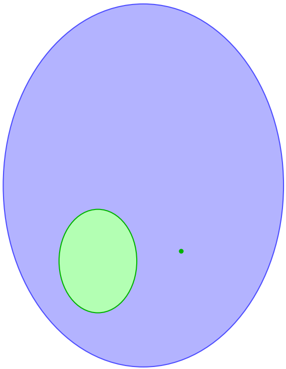
- *Interesting Implications $\mathcal{L}$*
- *Known Implications $\mathcal{S}$*



- *Interesting Implications $\mathcal{L}$*
- *Known Implications $\mathcal{S}$*

Goal

Which interesting but unknown implications are *valid* in our domain?



- *Interesting Implications $\mathcal{L}$*
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- Expert can *confirm* implications,

Goal

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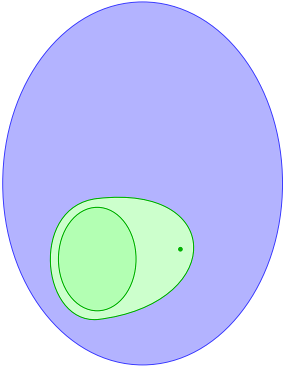
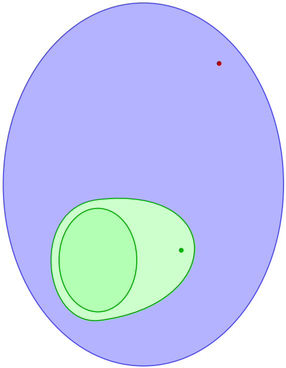


- *Interesting Implications $\mathcal{L}$*
- *Known Implications $\mathcal{S}$*
- Expert can *confirm* implications, extending $\mathcal{S}$

Goal

Which interesting but unknown implications are *valid* in our domain?



- *Interesting Implications* $\mathcal{L}$
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- Expert can *reject* implications, providing counterexamples,

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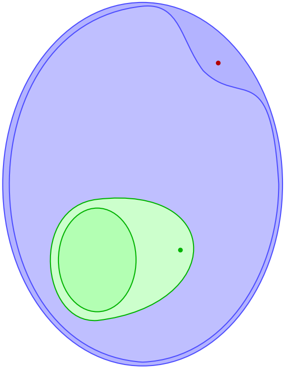
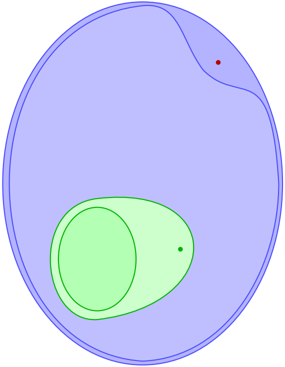


- *Interesting Implications $\mathcal{L}$*
- *Known Implications $\mathcal{S}$*
- Expert can *confirm* implications, extending $\mathcal{S}$
- Expert can *reject* implications, providing counterexamples, shrinking $\mathcal{L}$

Goal

Which interesting but unknown implications are *valid* in our domain?



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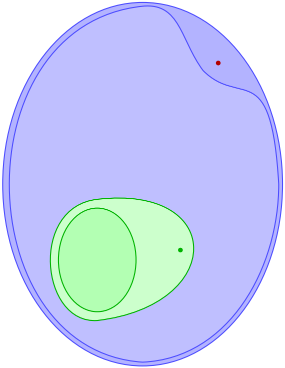
- Iterate until $\mathcal{L} = \mathcal{S}$.

Goal

Which interesting but unknown implications are *valid* in our domain?



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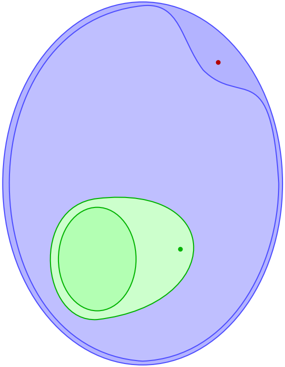
Which interesting but unknown implications are *valid* in our domain?

- Classical Attribute Exploration: $\mathcal{L} = \mathrm{Th}(\mathbb{K})$, ask special implications



- *Interesting Implications* $\mathcal{L}$
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- Expert can *confirm* implications, extending $\mathcal{S}$
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- Iterate until $\mathcal{L} = \mathcal{S}$.

Goal

Which interesting but unknown implications are *valid* in our domain?

- Classical Attribute Exploration: $\mathcal{L} = \mathrm{Th}(\mathbb{K})$, ask special implications
- Exploration by Confidence: $\mathcal{L} = \mathrm{Th}_c(\mathbb{K})$, ask special implications

Optimal Exploration by Confidence

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- Computes an implicational base of the expert knowledge present in $\mathrm{Th}_c(\mathbb{K})$

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- Allows to distinguish rare counterexamples from errors

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- Asks minimal number of questions confirmed by the expert

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Drawbacks

Optimal Exploration by Confidence

- Computes an implicational base of the expert knowledge present in $\mathrm{Th}_c(\mathbb{K})$
- Allows to distinguish rare counterexamples from errors
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Drawbacks

- Closure operator $\mathrm{Th}_c(\mathbb{K})$ hard to compute

Optimal Exploration by Confidence

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- Allows to distinguish rare counterexamples from errors
- Asks minimal number of questions confirmed by the expert

Drawbacks

- Closure operator $\mathrm{Th}_c(\mathbb{K})$ hard to compute
- Time between questions may grow exponentially

Question

Why is the Closure under $\mathrm{Th}_c(\mathbb{K})$ harder to compute?

Question

Why is the Closure under $\mathrm{Th}_c(\mathbb{K})$ harder to compute? Let $A \subseteq M$.

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Closure under $\mathrm{Th}(\mathbb{K})$:

Question

Why is the Closure under $\mathrm{Th}_c(\mathbb{K})$ harder to compute? Let $A \subseteq M$.

Closure under $\mathrm{Th}(\mathbb{K})$: Observe

$$(A \to A'') \in \mathrm{Th}(\mathbb{K}).$$

Question

Why is the Closure under $\mathrm{Th}_c(\mathbb{K})$ harder to compute? Let $A \subseteq M$.

Closure under $\mathrm{Th}(\mathbb{K})$: Observe

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- If $B \subseteq A$, $C \subseteq M$, then $(A \to C) \notin \mathrm{Th}_c(\mathbb{K})$, $(B \to C) \in \mathrm{Th}_c(\mathbb{K})$ possible

A "Faster" Exploration by Confidence

- Computes intermediate questions:

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Drawbacks

- May ask much more questions than necessary

Goal

Transfer exploration by confidence into setting of confident GCIs

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Idea

Do exploration on induced context $\mathbb{K}_{\mathcal{I}}$

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Outline

Summary

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- Motivated learning of confident GCIs from finite interpretations

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- Exploration Algorithm for confident GCIs
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- Minimal bases of confident GCIs

Thank You

Recall

So far, all bases of $\mathsf{Th}_c(\mathcal{I})$ were formulated in $\mathcal{EL}^{\bot}_{\mathsf{gfp}}$.

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$$\sqsubseteq$$

$$\begin{aligned}(A, \{\, A &\equiv \mathrm{Person} \sqcap \exists\mathrm{child}.B \sqcap \exists\mathrm{child}.C \sqcap \exists\mathrm{child}.D,\\ B &\equiv \mathrm{Person} \sqcap \exists\mathrm{child}.A \sqcap \exists\mathrm{child}.C \sqcap \exists\mathrm{child}.D,\\ C &\equiv \top,\\ D &\equiv \mathrm{Person} \sqcap \exists\mathrm{child}.C \,\})\end{aligned}$$

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Construct easier-to-understand $\mathcal{EL}^{\bot}$ bases.

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Approach

Use idea of *unravelling* $\mathcal{EL}^{\bot}_{\mathrm{gfp}}$ concept descriptions.

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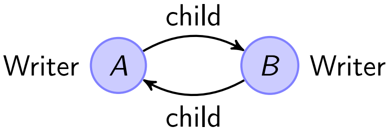

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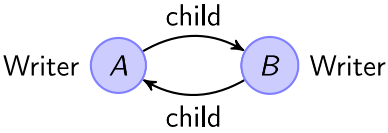

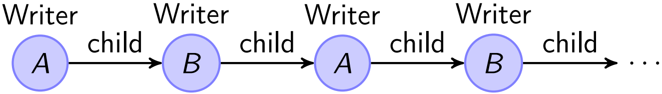

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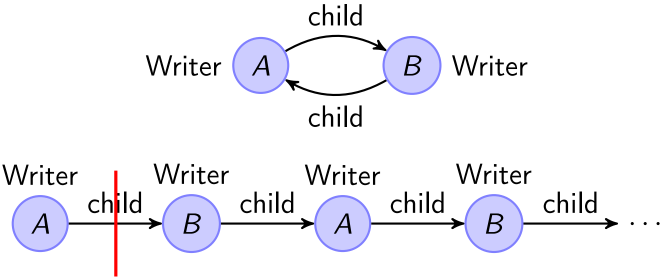

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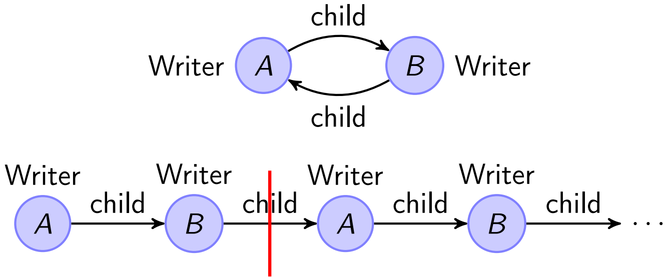

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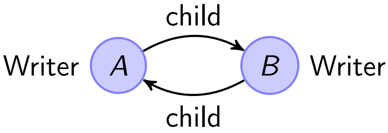

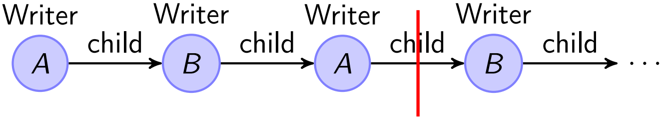

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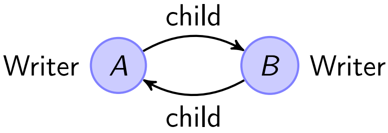

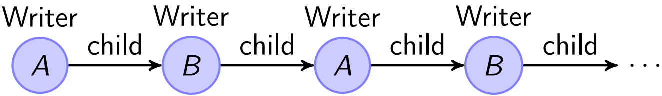

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How to keep the expressiveness of $\mathcal{EL}^{\bot}_{\mathrm{gfp}}$ concept descriptions?

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Lemma (Distel 2008)

Then there exists an effectively computable $d \in \mathbb{N}$ *such that*

$$(C_d)^{\mathcal{I}} = C^{\mathcal{I}}$$

is true for each $\mathcal{EL}^{\bot}_{gfp}$ *concept description* C.

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Lemma

For each $Y \subseteq \Delta^{\mathcal{I}}$, it is true that

$$\mathcal{X} \models ((Y^{\mathcal{I}})_d \sqsubseteq Y^{\mathcal{I}}).$$

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$$\mathcal{B}_0 \cup \mathcal{C}_0 \cup \mathcal{X} \models C \sqsubseteq C_d \sqsubseteq (C^{\mathcal{I}\mathcal{I}})_d \sqsubseteq (D^{\mathcal{I}\mathcal{I}})_d \sqsubseteq D^{\mathcal{I}\mathcal{I}}$$

Let $\mathcal{B} \cup \mathcal{C}$ be a confident base of $\mathrm{Th}_c(\mathcal{I})$ such that

- $\mathcal{B} \subseteq \mathrm{Th}(\mathcal{I})$,
- $\mathcal{C} \cap \mathrm{Th}(\mathcal{I}) = \varnothing$, and
- $\mathcal{B}$ only contains GCIs of the form $E \sqsubseteq E^{\mathcal{I}\mathcal{I}}$.

Definition

$$\mathcal{X} := \{ (X^{\mathcal{I}})_d \sqsubseteq (X^{\mathcal{I}})_{d+1} \mid X \subseteq \Delta^{\mathcal{I}}, X \neq \varnothing \}$$

$$\mathcal{B}_0 := \{ E_d \sqsubseteq (E^{\mathcal{I}\mathcal{I}})_d \mid (E \sqsubseteq E^{\mathcal{I}\mathcal{I}}) \in \mathcal{B} \} \cup \{ C_d \sqsubseteq (C^{\mathcal{I}\mathcal{I}})_d \mid (C \sqsubseteq D) \in \mathcal{C} \}$$

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$$

thus $\mathcal{B}_0 \cup \mathcal{C}_0 \cup \mathcal{X} \models \mathcal{C}$.

Let $\mathcal{B} \cup \mathcal{C}$ be a confident base of $\mathrm{Th}_c(\mathcal{I})$ such that

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\end{aligned}$$

Theorem

The set $\mathcal{B}_0 \cup \mathcal{C}_0 \cup \mathcal{X}$ is a confident base of $\mathsf{Th}_c(\mathcal{I})$ that only contains $\mathcal{EL}^{\bot}$ GCIs.

Drawback

The base $\mathcal{B}_0 \cup \mathcal{C}_0 \cup \mathcal{X}$ might be considerably larger than $\mathcal{B} \cup \mathcal{C}$ (d might be too large!)

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- Can we do better?

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Question

- Can we do better?
- Consider depth-bounded axiomatization instead?