TECHNISCHE UNIVERSITÄT DRESDEN

Computational Logic Group

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Approximation Fixpoint Theory

Lecture 12, 27th Jan 2025 // Foundations of Knowledge Representation, WS 2024/25

Motivation: Objective

Goal: Define semantics for (rule-based) KR formalisms in the presence of:

Recursion

- transitive closure
- indirect effects of actions

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Recursion Through Negation

- mutually exclusive alternatives
- non-deterministic effects of actions

Motivation: Basic Idea

Approximation Fixpoint Theory

- Framework for studying semantics of (non-monotonic) KR formalisms
- Due to Denecker, Marek, and Truszczyński [2000, 2003, 2004]
- Based on lattice theory and fixpoint theory:

$$\boxed{\text{KB}} \xrightarrow{\text{has}} \boxed{\text{Models}}$$

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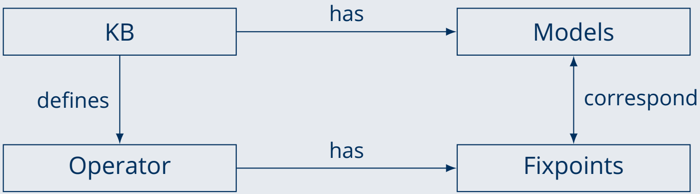

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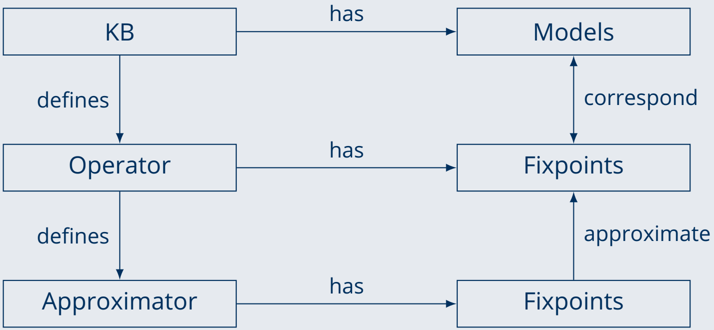

Motivation: History and Context

Approximation Fixpoint Theory

... emerged from similarities in the semantics of

- Default Logic [Reiter, 1980]
- Autoepistemic Logic [Moore, 1985]
- Logic Programs, in particular Stable Models [Gelfond and Lifschitz, 1988]

... and has since been applied to define/reconstruct semantics of ...

- Abstract Argumentation Frameworks
- Abstract Dialectical Frameworks
- Active Integrity Constraints
- Recursive SHACL

Agenda

Preliminaries

Partially Ordered Sets

Definition

A **partially ordered set** is a pair $(L, \leqslant)$ with

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- $\leqslant \subseteq L \times L$ a partial order. (reflexive, antisymmetric, transitive)

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- **bottom element** $\bot \in L$ iff $\bot \leqslant x$ for all $x \in L$,
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Examples

- $(\mathbb{N}, \leq)$: natural numbers with "usual" ordering, $\bot = 0$, no $\top$
- $(2^S, \subseteq)$: any powerset with subset relation, $\bot = \emptyset$, $\top = S$
- $(\mathbb{N}, \mid)$: natural numbers with divisibility relation, $\bot = 1$, $\top = 0$

Minimal, Maximal, Least, Greatest

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Let $(L, \leqslant)$ be a partially ordered set with $S \subseteq L$ and $x \in S$. We say that:

- x is a **minimal element** of S iff for each $y \in S$, $y \leqslant x$ implies $y = x$, dually,
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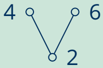
Example

In $(\mathbb{N}, |)$ (natural numbers with divisibility $a \mid b \iff (\exists k \in \mathbb{N}) a \cdot k = b$), ...

- the set $\{2, 3, 6\}$ has minimal elements 2 and 3, greatest element 6,
- the set $\{2, 4, 6\}$ has least element 2, and maximal elements 4 and 6.



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Examples

- In $(2^S, \subseteq)$, $\wedge = \cap$ and $\vee = \cup$;
- in $(\mathbb{N}, \mid)$, $\wedge = \text{gcd}$ and $\vee = \text{lcm}$, e.g. $4 \vee 6 = 12$ and $23 \wedge 42 = 1$.

(Complete) Lattices

Definition

Let $(L, \leqslant)$ be a partially ordered set.

1. $(L, \leqslant)$ is a **lattice** if and only if for all $x, y \in L$, both $x \wedge y$ and $x \vee y$ exist;

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Examples

- $(2^S, \subseteq)$ is a complete lattice for every set S.
- $(\mathbb{N}, |)$ is a complete lattice.
- $(\{M \subseteq \mathbb{N} \mid M \text{ is finite}\}, \subseteq)$ is a lattice.
- Every lattice $(L, \leqslant)$ with L finite is a complete lattice. (induction on $|S|$)

Further reading: B.A. Davey and H.A. Priestley. *Introduction to Lattices and Order*. Second Edition. Cambridge University Press, 2002

Operators and Their Properties

Definition

Let $(L, \leqslant)$ be a partially ordered set.
An operator $O \colon L \to L$ is $\leqslant$-**monotone** if and only if for all $x, y \in L$,

$$x \leqslant y \quad \text{implies} \quad O(x) \leqslant O(y)$$

Intuition: Operator application preserves ordering.

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- $O(\{2,3\}) = \{1,2,3,6\}$ and $O(\{2,3,5\}) = \{1,2,3,5,6,10,15,30\}$.

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  - Then there is a $K \subseteq M_1$ with $k = \prod K$.
  - By $K \subseteq M_1 \subseteq M_2$, we get $k \in O(M_2)$.

Fixpoints of Operators

Definition

Let $(L, \leqslant)$ be a partially ordered set and $O\colon L \to L$ be an operator.

- $x \in L$ is a **fixpoint** of O iff $O(x) = x$;
- $x \in L$ is a **prefixpoint** of O iff $O(x) \leqslant x$;
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Theorem (Knaster/Tarski)

Let $(L, \leqslant)$ be a complete lattice and $O\colon L \to L$ be a monotone operator.
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Example (Continued.)

Consider $(2^{\mathbb{N}}, \subseteq)$ with operator $O \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}, \quad M \mapsto \{\prod K \mid K \subseteq M,\ K \text{ finite}\}$.
O has least and greatest fixpoints: $O(\{1\}) = \{1\}$ and $O(\mathbb{N}) = \mathbb{N}$.

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Proof.

Define $A = \{x \in L \mid O(x) \leqslant x\}$ and $a = \bigwedge A$. $(A \neq \emptyset$ as $\top \in A.)$

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- Since a is a lower bound of A, we get $a \leqslant O(a)$, thus $O(a) = a$.
- Greatest fixpoint β is obtained dually: $B = \{x \in L \mid x \leqslant O(x)\}$, $\beta = \bigvee B$. $\square$

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Proof.

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- Furthermore, monotonicity yields $O(O(\alpha)) \leqslant O(\alpha)$, whence $O(\alpha) \in A$.
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- Greatest fixpoint β is obtained dually: $B = \{x \in L \mid x \leqslant O(x)\}$, $\beta = \bigvee B$. $\square$

$(F, \leqslant)$ is a complete lattice: for $G \subseteq F$, take $([\bigvee G, \bigvee L], \leqslant)$ and $([\bigwedge L, \bigwedge G], \leqslant)$.

Fixpoints of Operators (3)

Nice to know there is one, but how do we get there?

Theorem

Let $(L, \leqslant)$ be a complete lattice and $O\colon L \to L$ be a $\leqslant$-monotone operator. For ordinals α, β, define

$$\begin{aligned} O^0(\bot) &= \bot && \\ O^{\alpha+1}(\bot) &= O(O^\alpha(\bot)) && \text{for successor ordinals} \\ O^\beta(\bot) &= \bigvee \left\{ O^\alpha(\bot) \;\middle|\; \alpha < \beta \right\} && \text{for limit ordinals} \end{aligned}$$

Then for some ordinal α, the element $O^\alpha(\bot)$ is a fixpoint of O.

Example (Continued.)

Consider $(2^{\mathbb{N}}, \subseteq)$ with operator $O\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}, \quad M \mapsto \{\prod K \mid K \subseteq M,\ K \text{ finite}\}$.
We obtain the chain $O^0(\emptyset) = \emptyset \rightsquigarrow O^1(\emptyset) = \{1\} \rightsquigarrow O^2(\emptyset) = O(\{1\}) = \{1\}$.

Definite Logic Programs

Consider a set $\mathcal{A}$ of propositional atoms.

Definition

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Does such a least model always exist?

Semantics via Operators

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Let P be a definite logic program over atoms $\mathcal{A}$.
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Then there is a rule $a \leftarrow a_1, \ldots, a_m \in P$ with $\{a_1, \ldots, a_m\} \subseteq S_1$.
But then $\{a_1, \ldots, a_m\} \subseteq S_1 \subseteq S_2$, thus $a \in {}_P T(S_2)$. $\square$

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For any definite logic program P, the operator ${}_PT$ is $\subseteq$-monotone.

Theorem

Every definite logic program P has a least model, given by the least fixpoint of ${}_PT$ in $(2^{\mathcal{A}}, \subseteq)$.

The least model of P captures its intended meaning.

Definite Logic Programs: Example

Example

Consider $\mathcal{A} = \{a, b, c\}$ and the logic program $P = \{a \leftarrow, \quad b \leftarrow a, \quad c \leftarrow c\}$.
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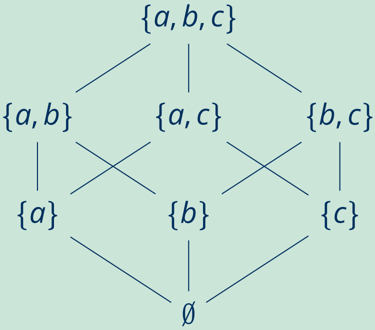

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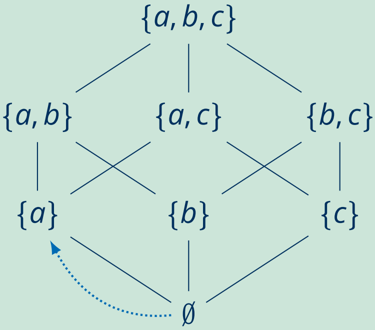

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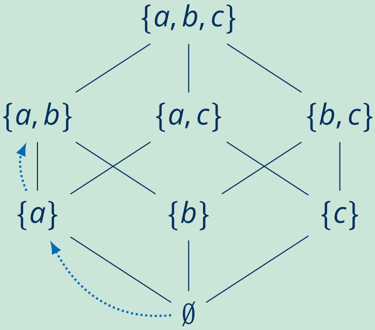

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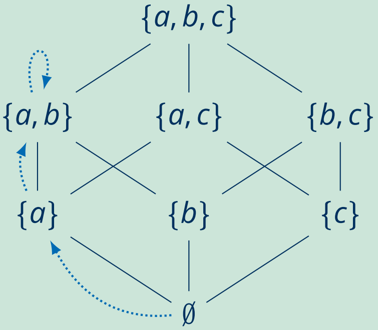

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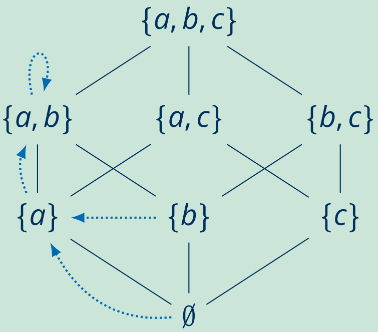

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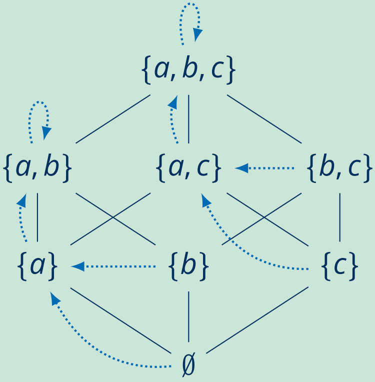

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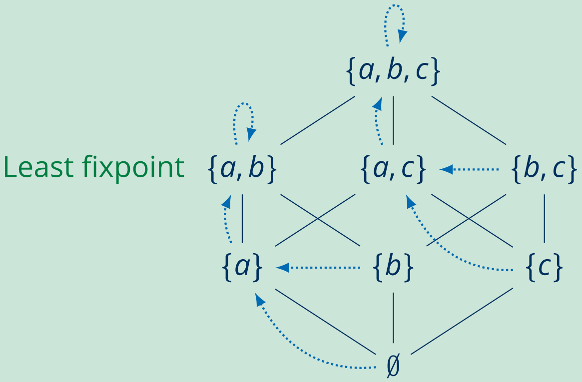

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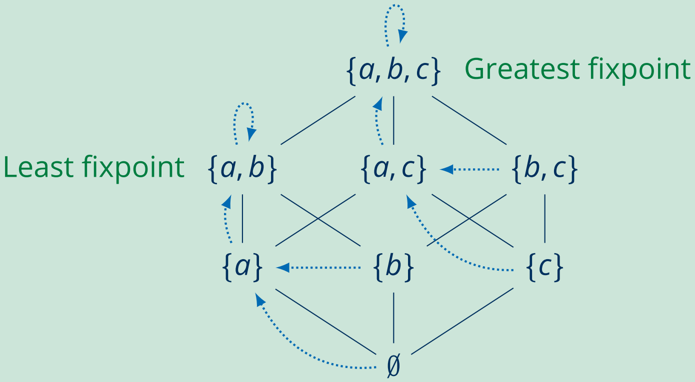

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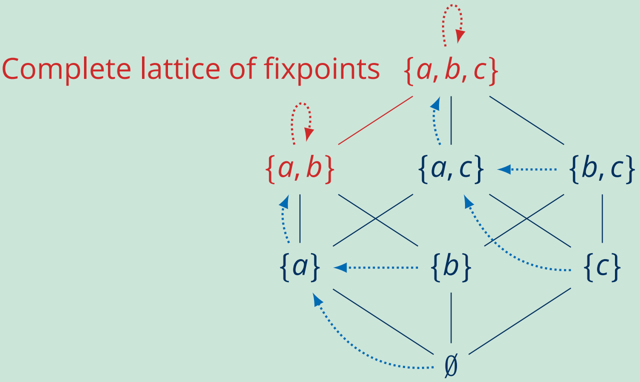

Normal Logic Programs

Definition

A **normal logic program** over $\mathcal{A}$ is a set P of rules of the form
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Allow negated atoms ${\sim}a$ in rule bodies.

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Definition

Let P be a normal logic program. The operator ${}_P T$ on $(2^{\mathcal{A}}, \subseteq)$ assigns thus:

$$S \mapsto \{a_0 \in \mathcal{A} \mid a_0 \leftarrow a_1, \ldots, a_m, \sim a_{m+1}, \ldots, \sim a_n \in P, \\ \{a_1, \ldots, a_m\} \subseteq S, \ \{a_{m+1}, \ldots, a_n\} \cap S = \emptyset\}$$

A set $S \subseteq \mathcal{A}$ is a **supported model** of P iff it is a fixpoint of ${}_P T$.

Allow to derive the rule head from S whenever the rule body is satisfied in S.

Alternative definition of supported models via Clark completion.

Normal Logic Programs: Example

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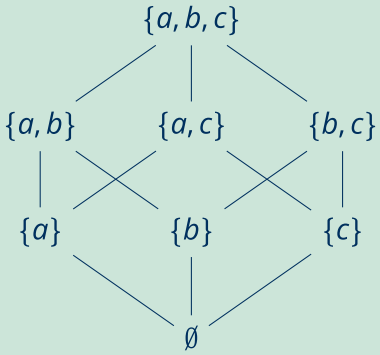
Operator ${}_PT$ visualised by



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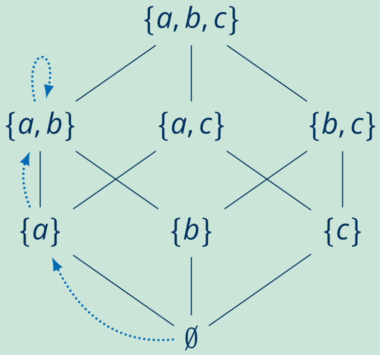

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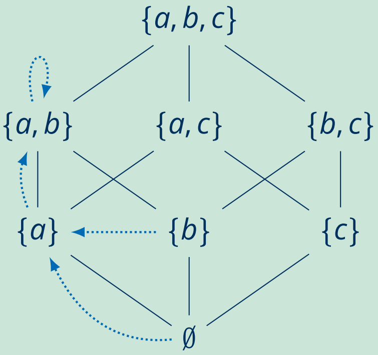

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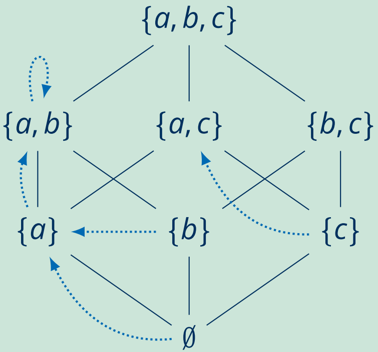

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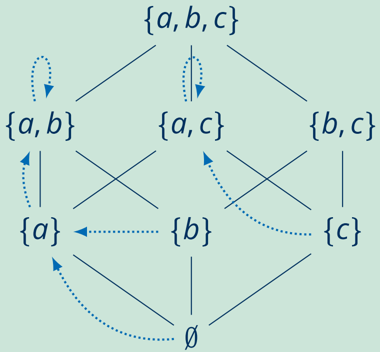

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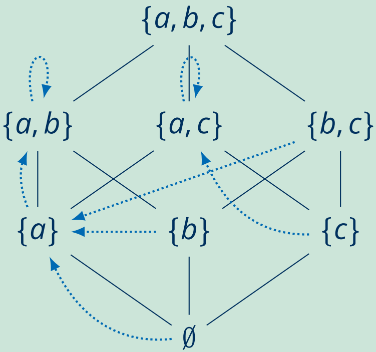

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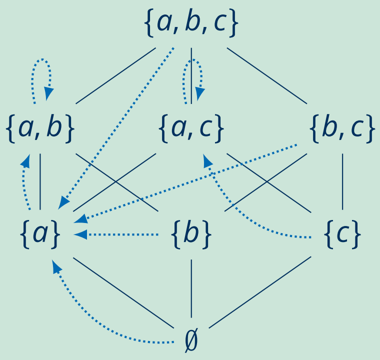

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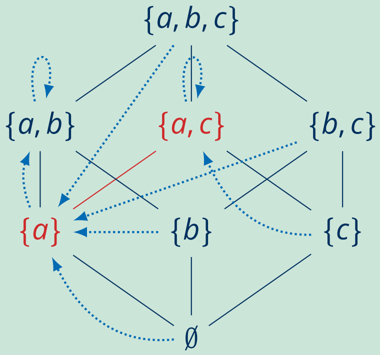

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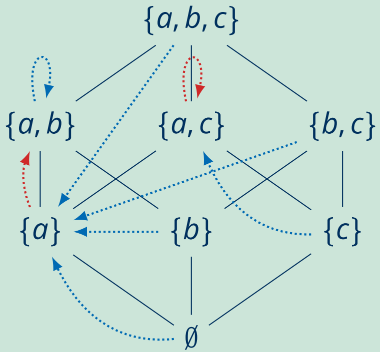

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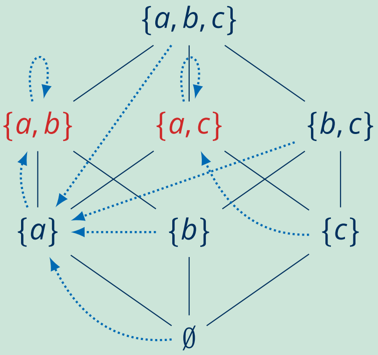

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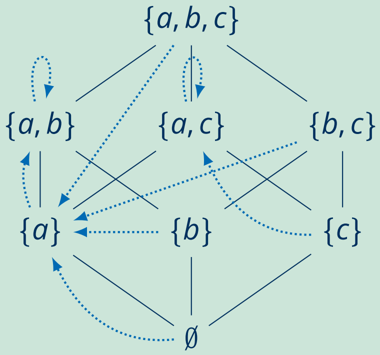

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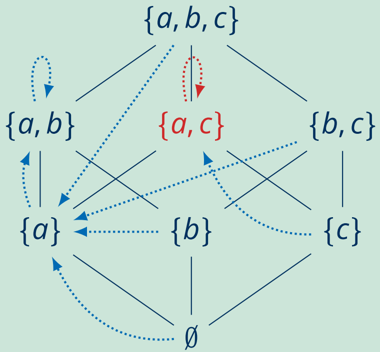

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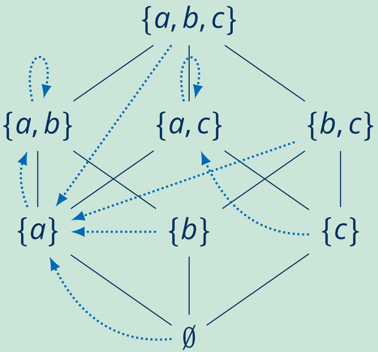

- How to avoid self-justification?
- How to obtain interpretation operators with “nice” properties?

Stable Model Semantics

Definition

Let P be a normal logic program and $S \subseteq \mathcal{A}$ be a set of atoms.
The **reduct of** P **with** S is the definite logic program P^S given by:

$$\{a \leftarrow a_1, \ldots, a_m \mid a \leftarrow a_1, \ldots, a_m, \sim a_{m+1}, \ldots, \sim a_n \in P, \{a_{m+1}, \ldots, a_n\} \cap S = \emptyset\}$$

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Reconsider logic program $P = \{a \leftarrow, \quad b \leftarrow a, {\sim}c, \quad c \leftarrow c, {\sim}b\}$ with supported models $\{a, b\}$ and $\{a, c\}$. Are they stable models?

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- $P^{\{a,b\}} = \{a \leftarrow, \quad b \leftarrow a\}$ with least model $\{a, b\}$, so $\{a, b\}$ is a stable model.

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- $P^{\{a,b\}} = \{a \leftarrow, \quad b \leftarrow a\}$ with least model $\{a, b\}$, so $\{a, b\}$ is a stable model.
- $P^{\{a,c\}} = \{a \leftarrow, \quad c \leftarrow c\}$ with least model $\{a\}$, so $\{a, c\}$ is not stable.

Stocktaking

- Monotone operators in complete lattices have (least and greatest) fixpoints.
- Operators can be associated with knowledge bases such that their fixpoints correspond to models.
- Definite logic programs lead to an operator that is monotone on $(2^{\mathcal{A}}, \subseteq)$, and thus have unique least models.
- Normal logic programs lead to a non-monotone operator; model existence and uniqueness cannot be guaranteed.
- Stable model semantics deals with self-justification.
- Can we find an operator-based version of stable model semantics?

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Main Idea

Use a more fine-grained structure to keep track of (partial) truth values.

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Approach

- Approximate sets of models by intervals.
- Use an information ordering on these approximations.
- Approximate operators by approximators – operators on intervals.
- Guarantee that fixpoints of approximators contain original fixpoints.

From Lattices to Bilattices

Definition

Let $(L, \leqslant)$ be a partially ordered set.
Its associated **information bilattice** is $(L^2, \leq_i)$ with $L^2 = L \times L$ and

$$(u, v) \leq_i (x, y) \quad \text{iff} \quad u \leqslant x \text{ and } y \leqslant v$$

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Its associated **information bilattice** is $(L^2, \leq_i)$ with $L^2 = L \times L$ and

$$(u, v) \leq_i (x, y) \quad \text{iff} \quad u \leqslant x \text{ and } y \leqslant v$$

- A pair (x, y) approximates all $z \in L$ with $x \leqslant z \leqslant y$.
- Information ordering $\triangleq$ interval inclusion: $(u, v) \leq_i (x, y)$ iff $[x, y] \subseteq [u, v]$

From Lattices to Bilattices

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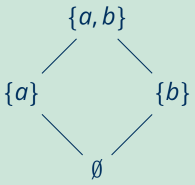
Proposition

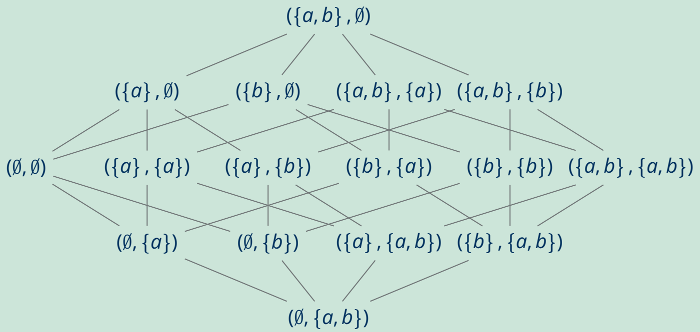
If $(L, \leqslant)$ is a complete lattice, then $(L^2, \leq_i)$ is a complete lattice. For $S \subseteq L^2$:

$$\bigwedge_i S = \left(\bigwedge S_1, \bigvee S_2\right) \quad \text{and} \quad \bigvee_i S = \left(\bigvee S_1, \bigwedge S_2\right) \qquad \begin{array}{l} S_1 = \{x \mid (x,y) \in S\} \\ S_2 = \{y \mid (x,y) \in S\} \end{array}$$

From Lattice to Bilattice: Example

Example





Original lattice $\left(2^{\{a,b\}}, \subseteq\right)$

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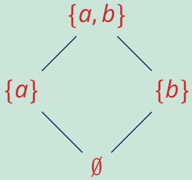

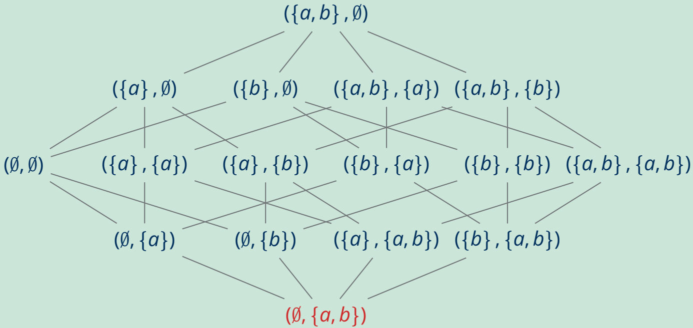

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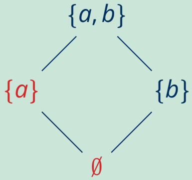

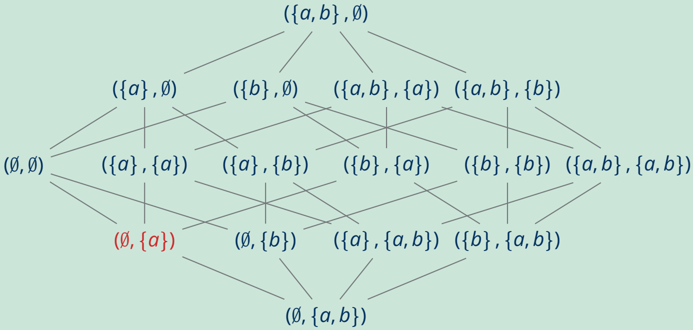

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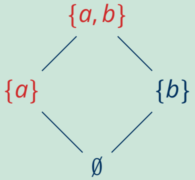

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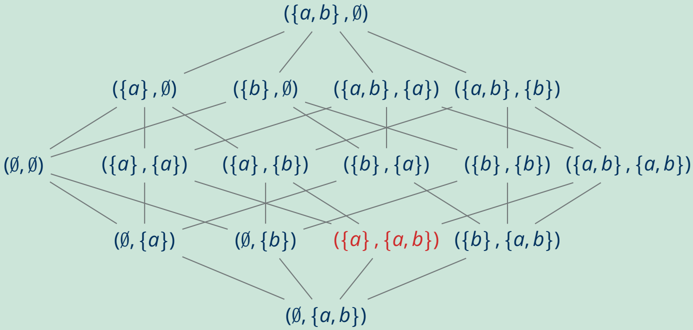

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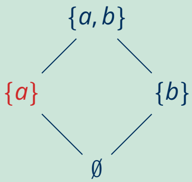

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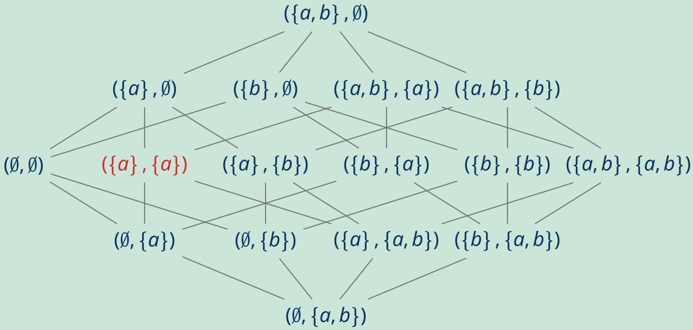

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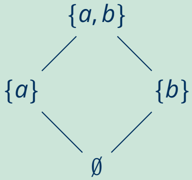

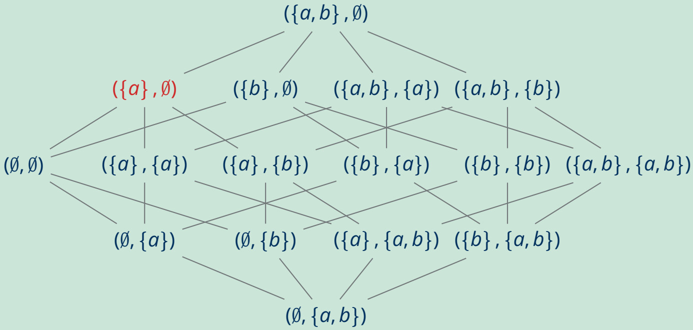

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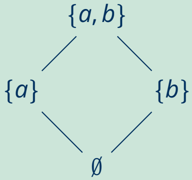

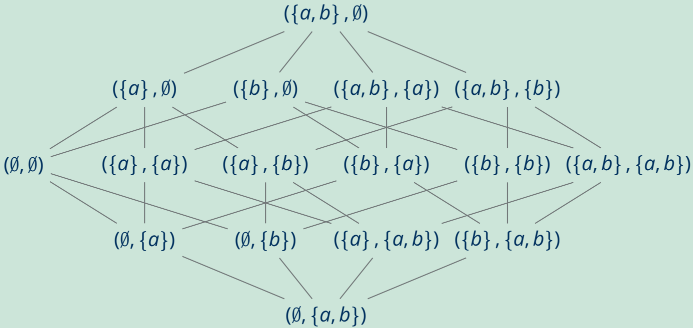

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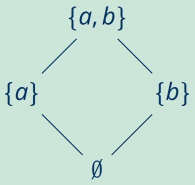

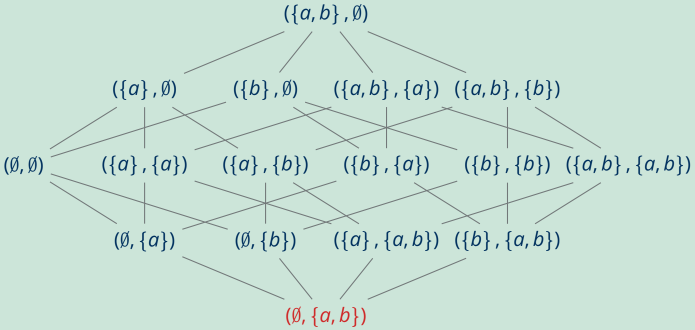

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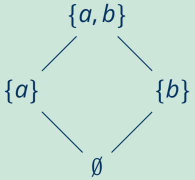

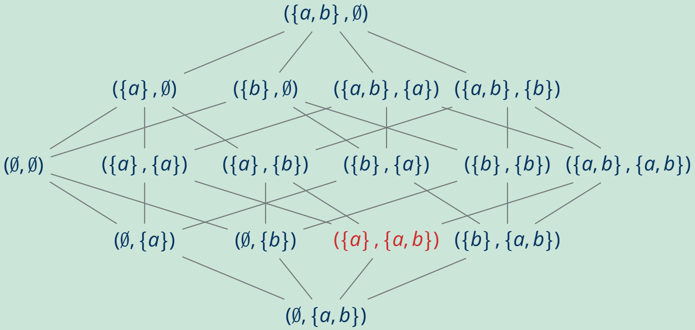

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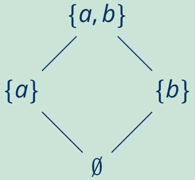

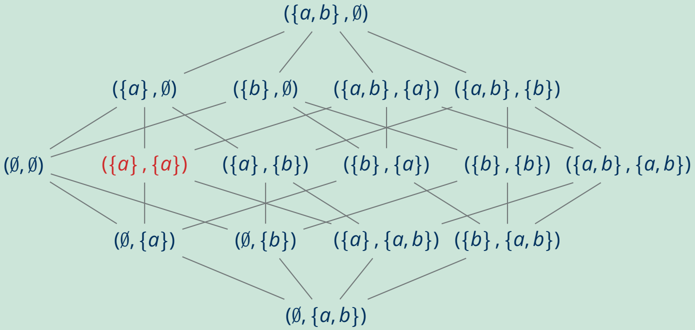

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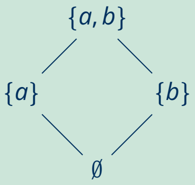

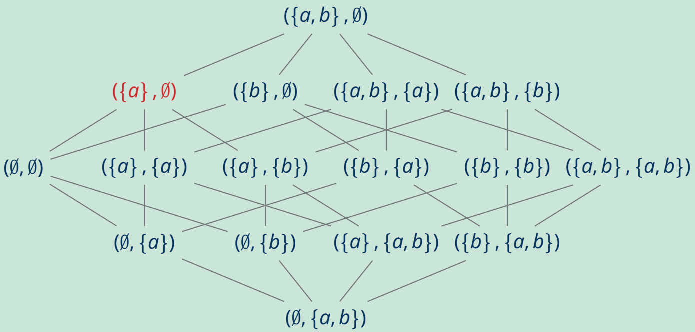

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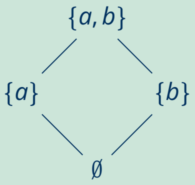

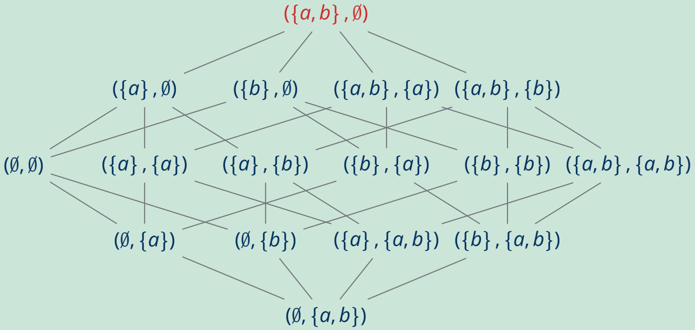

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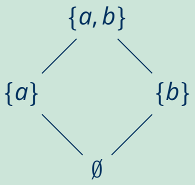

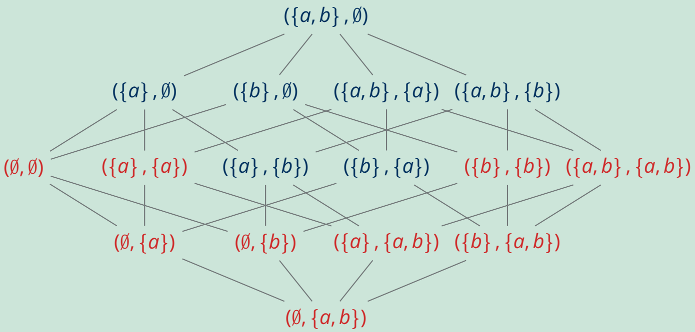

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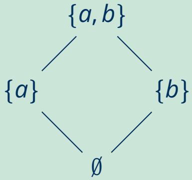

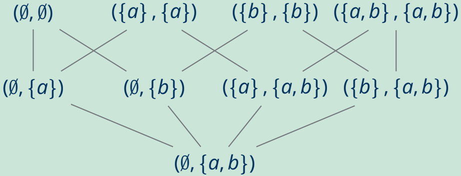

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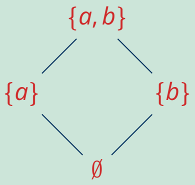

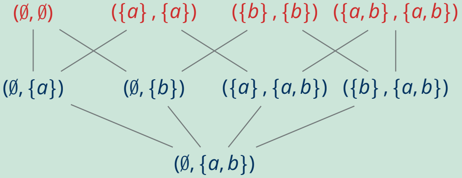

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Elements of the original lattice correspond to exact pairs.

Approximator

Recall approach: Approximate lattice operators on a richer structure.

Definition

Let $(L, \leqslant)$ be a complete lattice and $O\colon L \to L$ be an operator.
An operator $A\colon L^2 \to L^2$ **approximates** O iff for all $x \in L$, we have

$$A(x, x) = (O(x), O(x))$$

A is an **approximator** iff A approximates some O and A is $\leq_i$-monotone.

Approximator coincides with the operator on exact pairs.

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If A is symmetric, then $A(x,y) = (A_1(x,y), A_1(y,x))$, so A_1 fully specifies A.

Approximator: Example

Example

Let P be a normal logic program.
Recall its one-step consequence operator ${}_PT$, defined by

$$
\begin{aligned}
{}_PT(S) = \{a_0 \in \mathcal{A} \mid\ & a_0 \leftarrow a_1, \ldots, a_m, {\sim}a_{m+1}, \ldots, {\sim}a_n \in P, \\
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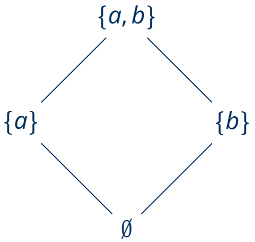
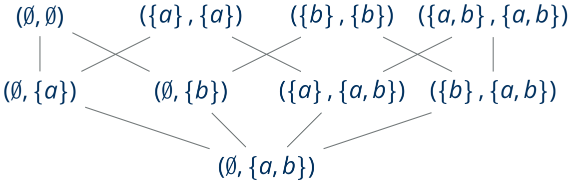
For new lower bound: check truth against lower, falsity against upper bound.

Approximator $_P\mathcal{T}$: Example





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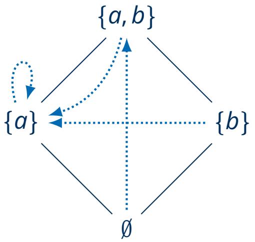

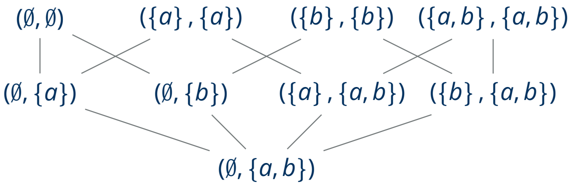

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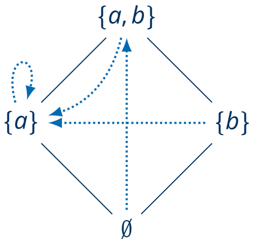

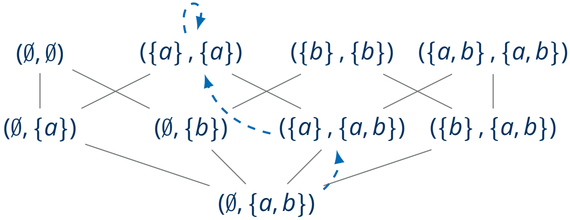

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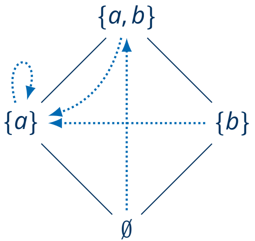

Original lattice $\left(2^{\{a,b\}}, \subseteq\right)$

Normal logic program
$P = \{a \leftarrow, \quad b \leftarrow {\sim}a, {\sim}b\}$

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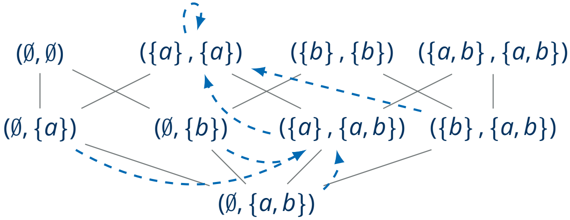
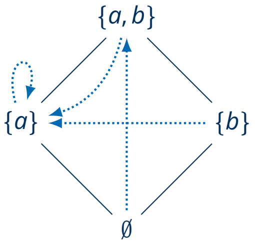


Bilattice $\left(2^{\{a,b\}} \times 2^{\{a,b\}}, \leq_i\right)$

Approximator $_P\mathcal{T}$ for $_PT$:

Approximator $_P\mathcal{T}$: Example



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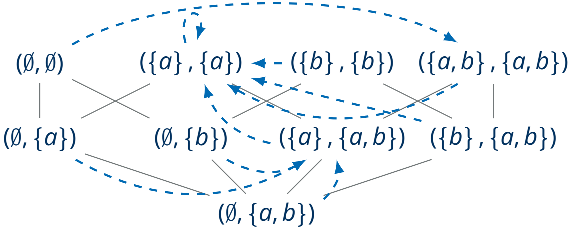

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Approximator $_P\mathcal{T}$ for $_PT$:

Quiz: Approximator ${}_P\mathcal{T}$

Recall that for $L, U \subseteq \mathcal{A}$ we defined ${}_P\mathcal{T}(L, U) = \left({}_P\mathcal{T}_1(L, U), {}_P\mathcal{T}_1(U, L)\right)$ with

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\begin{aligned}
{}_P\mathcal{T}_1(L, U) = \{a_0 \in \mathcal{A} \mid \; & a_0 \leftarrow a_1, \ldots, a_m, \sim a_{m+1}, \ldots, \sim a_n \in P, \\
& \{a_1, \ldots, a_m\} \subseteq L, \; \{a_{m+1}, \ldots, a_n\} \cap U = \emptyset\}
\end{aligned}
$$

Quiz

Consider the normal logic program P:

$$a \leftarrow \qquad\qquad b \leftarrow a, \sim c \qquad\qquad c \leftarrow c$$

What is the result of applying ${}_P\mathcal{T}$ to $(\{a\}, \{a, b\})$?

1. $(\emptyset, \{a, b\})$
2. $(\{a\}, \{a, b\})$
3. $(\{a, b\}, \{a, b\})$
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3. $(\{a,b\}, \{a,b\})$ ✓
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Approximator: Observations (1)

Lemma

Let $(L, \leqslant)$ be a complete lattice and A an approximator on $(L^2, \leq_i)$.

1. If C is a non-empty chain of consistent pairs, then $\bigvee_i C$ is consistent.
2. If (x, y) is consistent, then $A(x, y)$ is consistent.

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2. If $x \leqslant y$, then for z with $x \leqslant z \leqslant y$ we have $(x, y) \leq_i (z, z)$. A is $\leq_i$-monotone, thus $A(x, y) \leq_i A(z, z)$.

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2. If $x \leqslant y$, then for z with $x \leqslant z \leqslant y$ we have $(x, y) \leq_i (z, z)$. A is $\leq_i$-monotone, thus $A(x, y) \leq_i A(z, z)$. A approximates some O, thus $A(z, z) = (O(z), O(z))$.

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Approximators map consistent pairs to consistent pairs.

Approximator: Observations (2)

Theorem

Let $(L, \leqslant)$ be a complete lattice with $O \colon L \to L$, and A an approximator for O.

1. A has a $\leq_i$-least fixpoint (x^*, y^*) with $x^* \leqslant y^*$.
2. Every fixpoint z of O satisfies $x^* \leqslant z \leqslant y^*$.

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   Define $Q = \left\{(x, y) \in L^2 \;\middle|\; x \leqslant y \;\; \& \;\; (x, y) \leq_i A(x, y) \;\; \& \;\; (x, y) \leq_i (x^*, y^*)\right\}$.

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Let $\emptyset \neq C \subseteq Q$ be a chain. Define $d = \bigvee_i C$. (1) By the previous lemma, d is consistent. (2) For every $c \in C$ we have $c \leq_i d$ and thus $c \leq_i A(c) \leq_i A(d)$; thus $A(d)$ is an upper bound of C, whence $d \leq_i A(d)$. (3) We know that $C \subseteq Q$ whence (x^*, y^*) is an upper bound of C, thus $d \leq_i (x^*, y^*)$.

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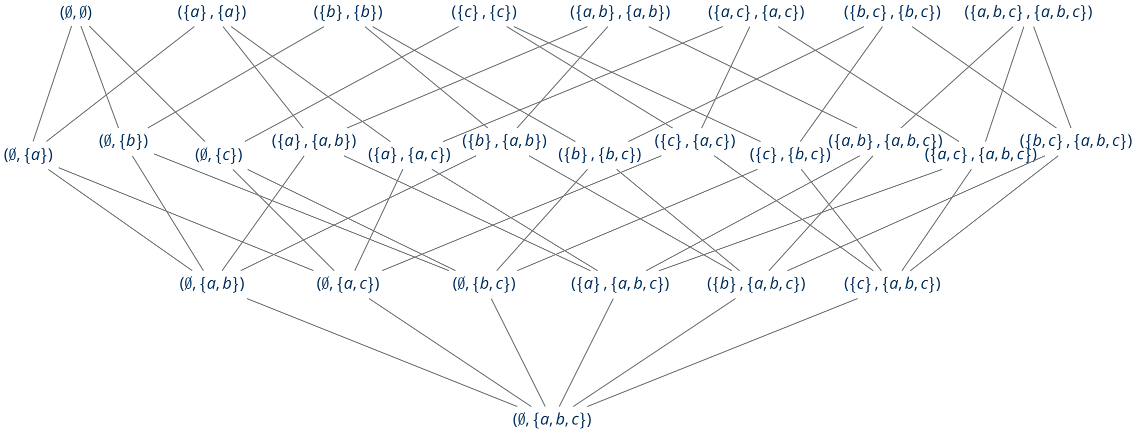
2. If $O(z) = z$ then $A(z,z) = (O(z), O(z)) = (z,z)$ and $(x^*, y^*) \leq_i (z,z)$. □

Let $\emptyset \neq C \subseteq Q$ be a chain. Define $d = \bigvee_i C$. (1) By the previous lemma, d is consistent. (2) For every $c \in C$ we have $c \leq_i d$ and thus $c \leq_i A(c) \leq_i A(d)$; thus $A(d)$ is an upper bound of C, whence $d \leq_i A(d)$. (3) We know that $C \subseteq Q$ whence (x^*, y^*) is an upper bound of C, thus $d \leq_i (x^*, y^*)$.

The least fixpoint of A is consistent and approximates all fixpoints of O.

Approximator ${}_{\mathcal{P}}\mathcal{T}$: Examples



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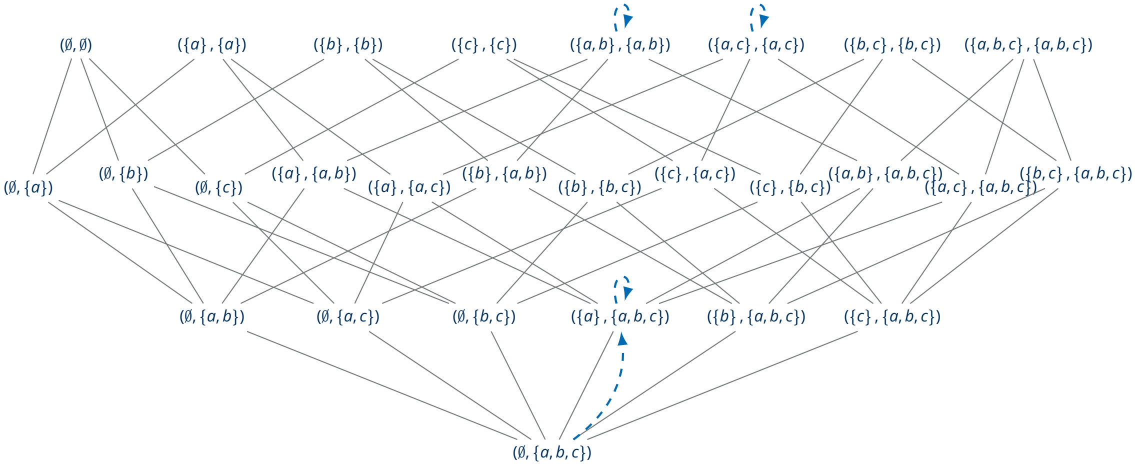

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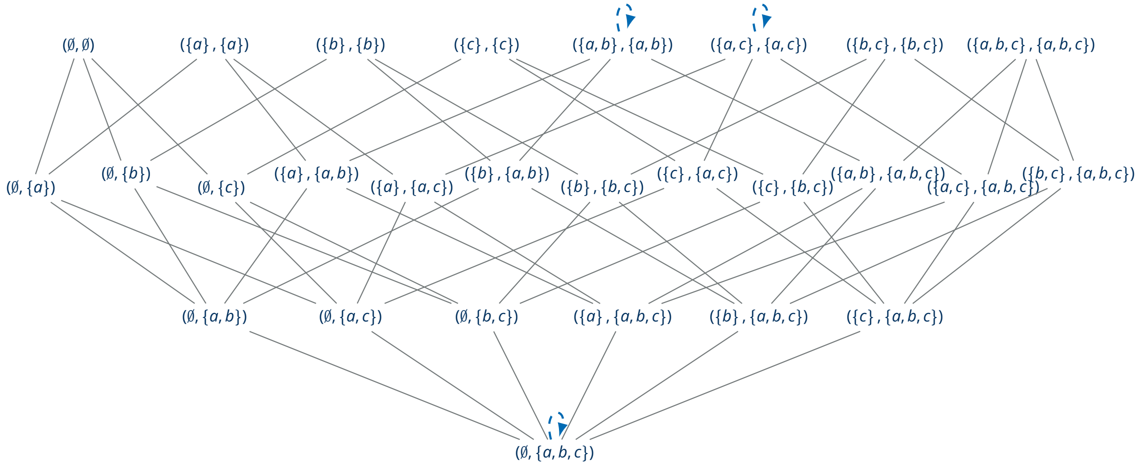

$$P_2 = \{a \leftarrow b, \quad a \leftarrow c, \quad b \leftarrow \sim c, \quad c \leftarrow \sim b\}$$

Recovering Semantics

Approximator fixpoints give rise to several semantics.

Proposition

Let P be a normal logic program over $\mathcal{A}$ with approximator ${}_P\mathfrak{T}$, $X \subseteq Y \subseteq \mathcal{A}$.

- X is a supported model of P iff ${}_P\mathfrak{T}(X, X) = (X, X)$.
- (X, Y) is a three-valued supported model of P iff ${}_P\mathfrak{T}(X, Y) = (X, Y)$.
- (X, Y) is the Kripke-Kleene semantics of P iff $(X, Y) = \mathrm{lfp}({}_P\mathfrak{T})$.

But what about stable model semantics?

Stable Operators

Stable Operator: Intuition

The Gelfond-Lifschitz Reduct of P …

- …starts out with a two-valued interpretation $S \subseteq \mathcal{A}$;
- …removes all rules requiring some $a \in S$ to be false;
- …assumes all $a \in \mathcal{A} \setminus S$ to be false in the remaining rules.

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- To obtain reduct P^S, assume all and only atoms $a \in \mathcal{A} \setminus S$ to be false.
- Using P^S, try to constructively prove all and only atoms $a \in S$ to be true.
- P^S is a definite logic program, so ${}_{P^S}T$ is a $\subseteq$-monotone operator on $(2^{\mathcal{A}}, \subseteq)$.

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Expressing the Reduct via an Operator

- For pair (X, Y), an $a \in \mathcal{A}$ is true iff $a \in X$; atom a is false iff $a \notin Y$.
- Use ${}_P\mathcal{T}_1$ to reconstruct what is true, fixing the upper bound to S:

$$ {}_P\mathcal{T}_1(\cdot, S) \colon 2^{\mathcal{A}} \to 2^{\mathcal{A}}, \quad X \mapsto {}_P\mathcal{T}_1(X, S) $$

Stable Operator: Preparation

Proposition

Let $(L, \leqslant)$ be a complete lattice and A be an approximator on $(L^2, \leq_i)$.
For every pair $(x, y) \in L^2$, the following operators are $\leqslant$-monotone:

$$A_1(\cdot, y)\colon L \to L, \quad z \mapsto A_1(z, y) \qquad \text{and} \qquad A_2(x, \cdot)\colon L \to L, \quad z \mapsto A_2(x, z)$$

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Proof.

1. Let $x_1 \leqslant x_2$ and $y \in L$.

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$$A_1(\cdot, y) \colon L \to L, \quad z \mapsto A_1(z, y) \qquad \text{and} \qquad A_2(x, \cdot) \colon L \to L, \quad z \mapsto A_2(x, z)$$

Proof.

1. Let $x_1 \leqslant x_2$ and $y \in L$.
   Then $(x_1, y) \leq_i (x_2, y)$ and $A(x_1, y) \leq_i A(x_2, y)$, thus $A_1(x_1, y) \leqslant A_1(x_2, y)$.

Stable Operator: Preparation

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2. Let $x \in L$ and $y_1 \leqslant y_2$.
   Then $(x, y_2) \leq_i (x, y_1)$ and $A(x, y_2) \leq_i A(x, y_1)$, thus $A_2(x, y_1) \leqslant A_2(x, y_2)$. $\square$

- $A_1(\cdot, y)$ has a $\leqslant$-least fixpoint, denoted $\mathrm{lfp}(A_1(\cdot, y))$;
- $A_2(x, \cdot)$ has a $\leqslant$-least fixpoint, denoted $\mathrm{lfp}(A_2(x, \cdot))$.

Stable Operator: Definition

Definition

Let $(L, \leqslant)$ be a complete lattice and A be an approximator on $(L^2, \leq_i)$.
The **stable approximator** for A is given by $A^{st}: L^2 \to L^2$ with

$$
\begin{aligned}
A_1^{st}&: L^2 \to L, && (x, y) \mapsto \operatorname{lfp}(A_1(\cdot, y)) \\
A_2^{st}&: L^2 \to L, && (x, y) \mapsto \operatorname{lfp}(A_2(x, \cdot))
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- A_1^{st}: improve lower bound for all fixpoints of O at or below upper bound;
- A_2^{st}: obtain tightmost new upper bound (eliminate non-minimal fixpoints).

Proposition

Let (x, y) be a postfixpoint of approximator A. Then

$$a \in [\bot, y] \text{ implies } A_1(a, y) \in [\bot, y] \quad \text{and} \quad b \in [x, \top] \text{ implies } A_2(x, b) \in [x, \top].$$

In particular, $\mathrm{lfp}(A_1(\cdot, y)) \leqslant y$ and $x \leqslant \mathrm{lfp}(A_2(x, \cdot))$.

Stable Operator: Observations

Theorem

Let $(L, \leqslant)$ be a complete lattice and A be an approximator on $(L^2, \leq_i)$.

1. A^{st} is $\leq_i$-monotone.
2. If (x, y) is a consistent postfixpoint of A, then $A^{\mathrm{st}}(x, y)$ is consistent.

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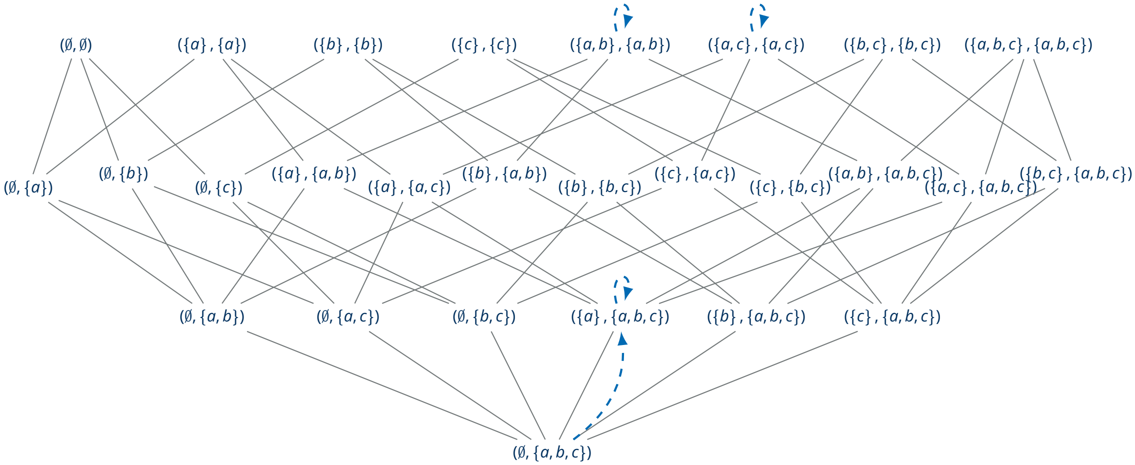
2. Let $x \leqslant y$ with $(x, y) \leq_i A(x, y)$. For every $z \in L$ with $x \leqslant z \leqslant y$, we have $A_1^{\mathrm{st}}(x, y) \leqslant A_1^{\mathrm{st}}(z, z) = \mathrm{lfp}(A_1(\cdot, z)) \leqslant z \leqslant \mathrm{lfp}(A_2(z, \cdot)) = A_2^{\mathrm{st}}(z, z) \leqslant A_2^{\mathrm{st}}(x, y)$.

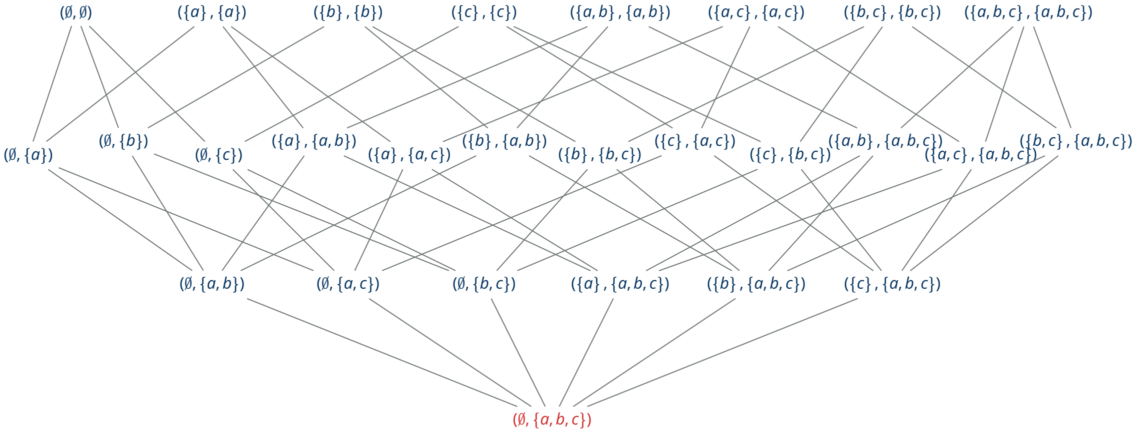
$\square$

Stable Operator $\mathcal{J}_P^{\mathrm{st}}$: Example



$$P_1 = \{a \leftarrow, \quad b \leftarrow a, {\sim}c, \quad c \leftarrow c\}$$

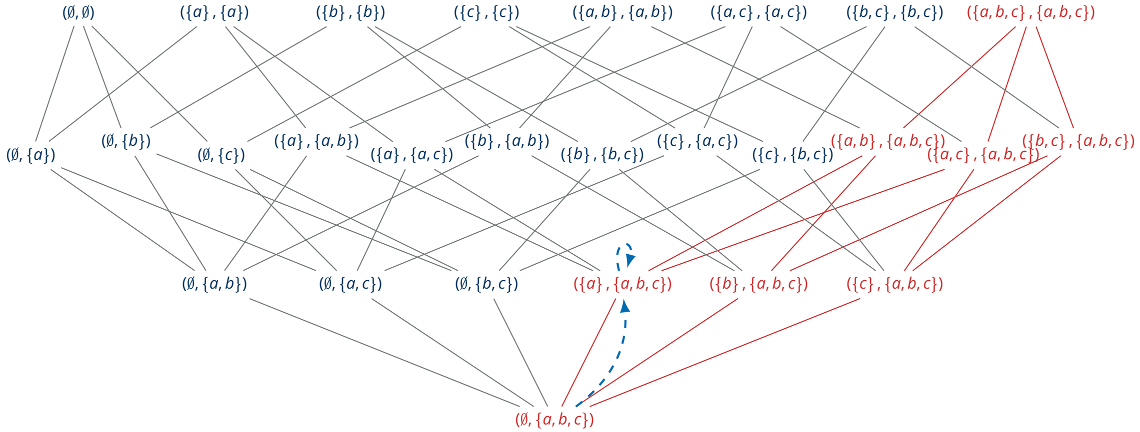
Stable Operator ${}_{P}\mathcal{T}^{\mathrm{st}}$: Example



$P_1 = \{a \leftarrow, \quad b \leftarrow a, {\sim}c, \quad c \leftarrow c\}$

${}_{P}\mathcal{T}^{\mathrm{st}}(\emptyset, \{a, b, c\}) = (\mathrm{lfp}({}_{P}\mathcal{T}_1(\cdot, \{a, b, c\})), \mathrm{lfp}({}_{P}\mathcal{T}_2(\emptyset, \cdot)))$

Stable Operator ${}_P\mathcal{T}^{\mathrm{st}}$: Example



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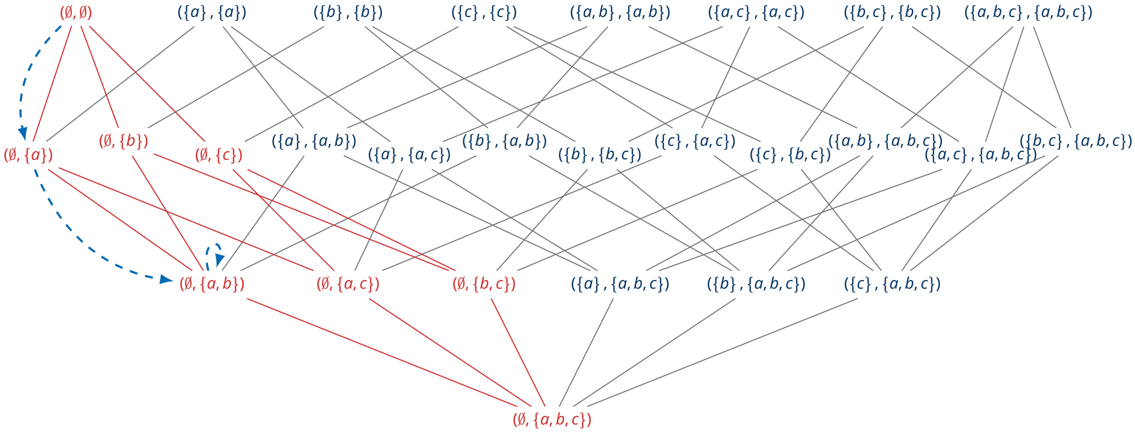

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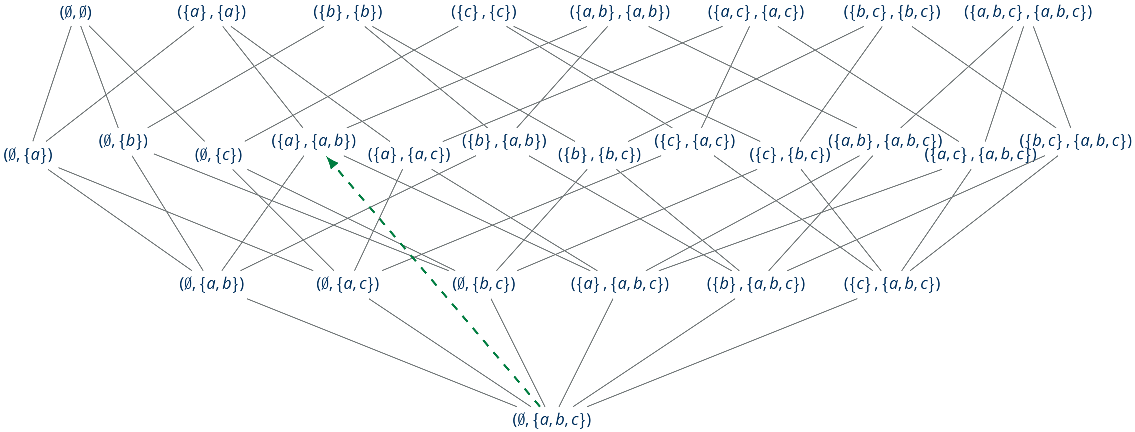

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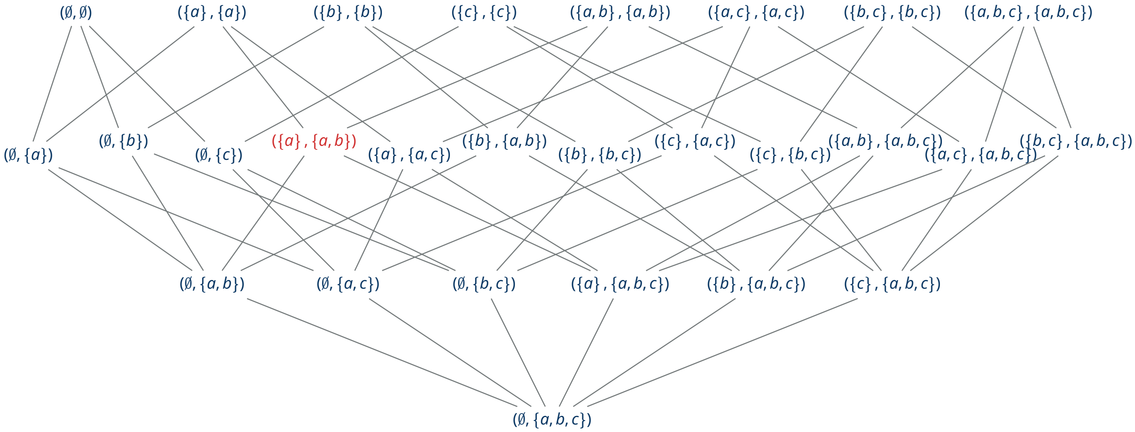

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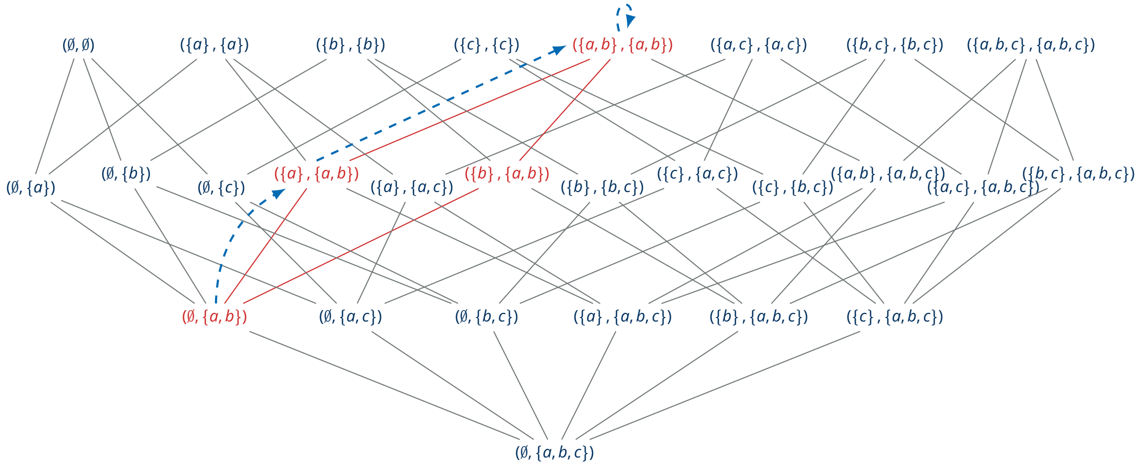

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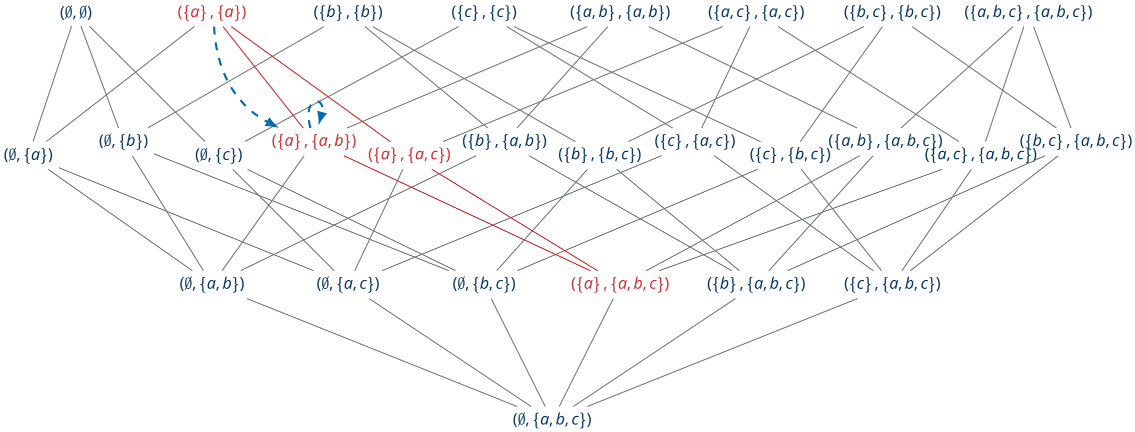
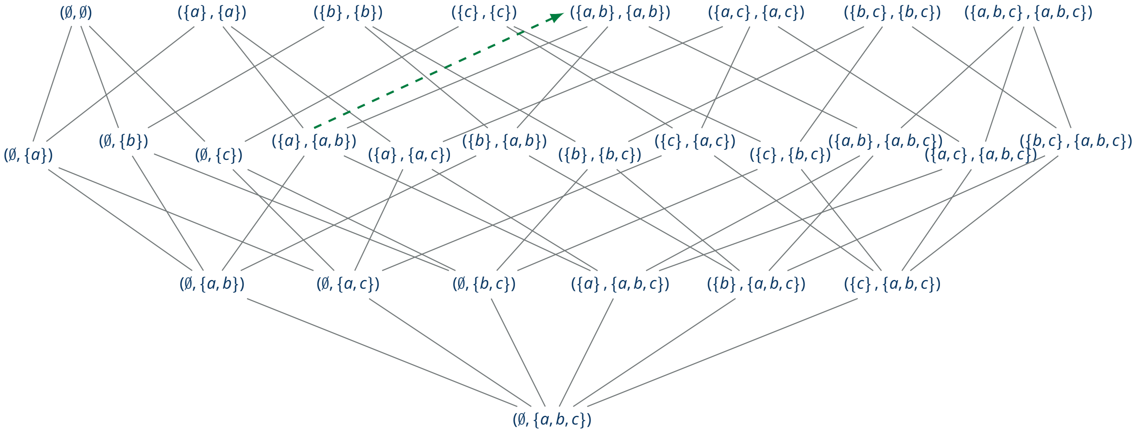
Stable Operator ${}_P\mathcal{T}^{st}$: Example



$P_1 = \{a \leftarrow, \quad b \leftarrow a, \sim c, \quad c \leftarrow c\}$

${}_P\mathcal{T}^{st}(\{a\}, \{a, b\}) = (\{a, b\}, \mathrm{lfp}({}_P\mathcal{T}_2(\{a\}, \cdot)))$

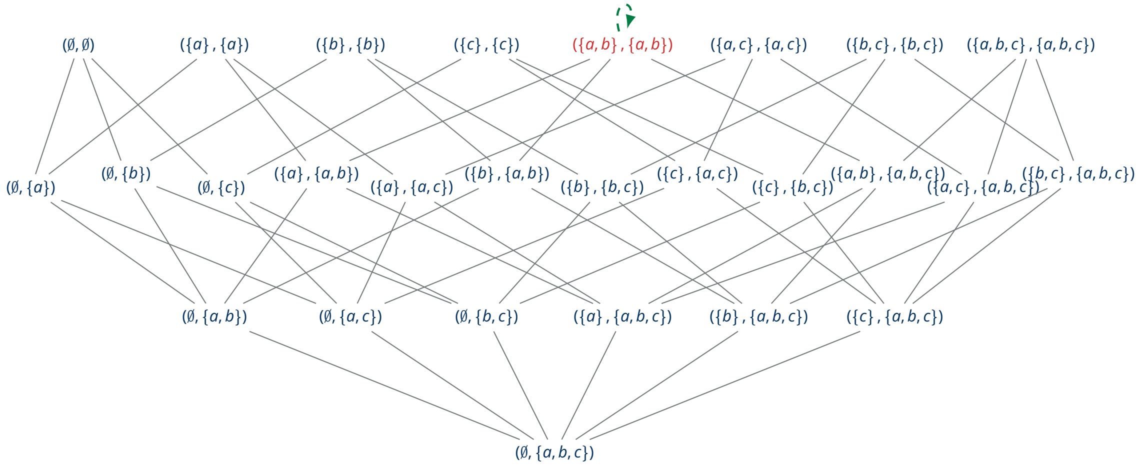
Stable Operator $_P\mathcal{T}^{st}$: Example



$P_1 = \{a \leftarrow, \quad b \leftarrow a, \sim c, \quad c \leftarrow c\}$

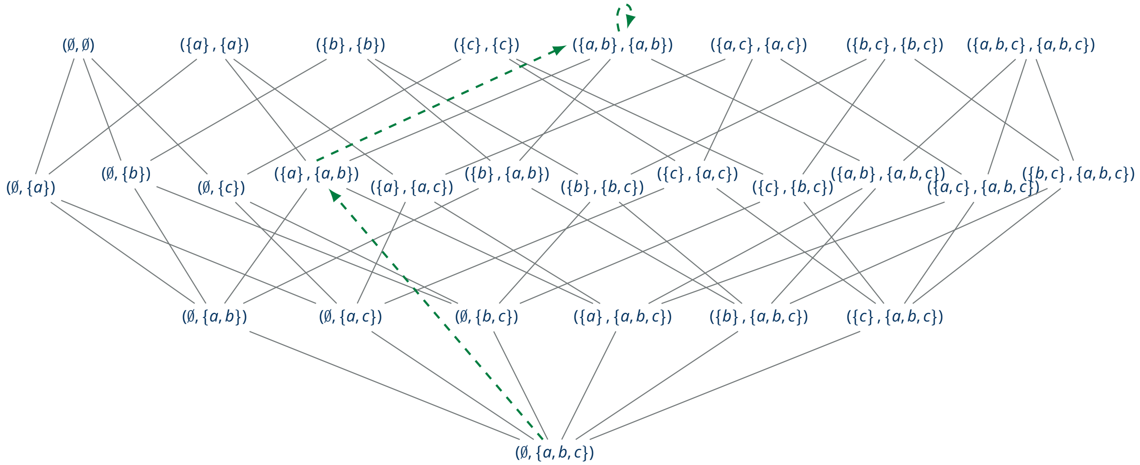
$_P\mathcal{T}^{st}(\{a\}, \{a, b\}) = (\{a, b\}, \{a, b\})$

Stable Operator ${}_{P}\mathcal{T}^{\mathrm{st}}$: Example



$P_1 = \{a \leftarrow, \quad b \leftarrow a, \sim c, \quad c \leftarrow c\}$ $\qquad$ ${}_{P}\mathcal{T}^{\mathrm{st}}(\{a,b\}, \{a,b\}) = ({}_{P}T(\{a,b\}), {}_{P}T(\{a,b\})) = (\{a,b\}, \{a,b\})$

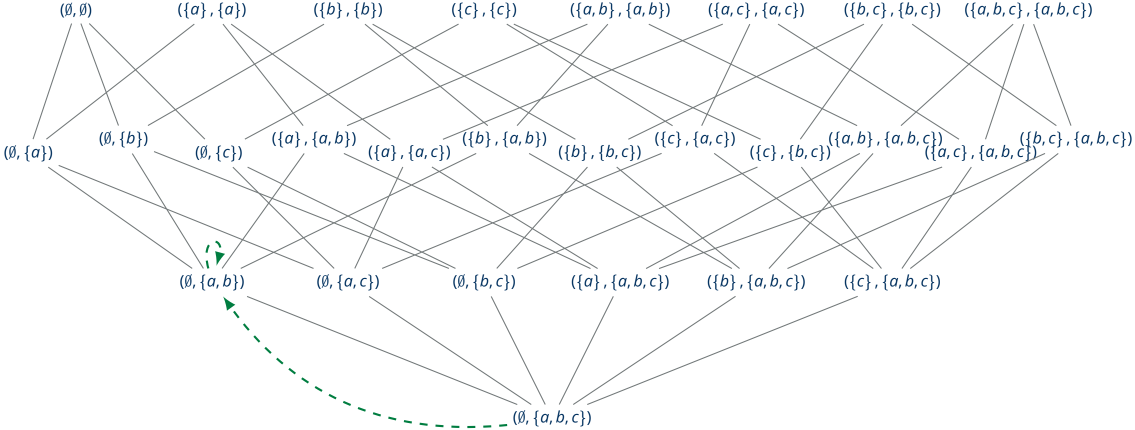
Stable Operator $_{P}\mathcal{J}^{\mathrm{st}}$: Example



$P_1 = \{a \leftarrow, \quad b \leftarrow a, {\sim}c, \quad c \leftarrow c\}$

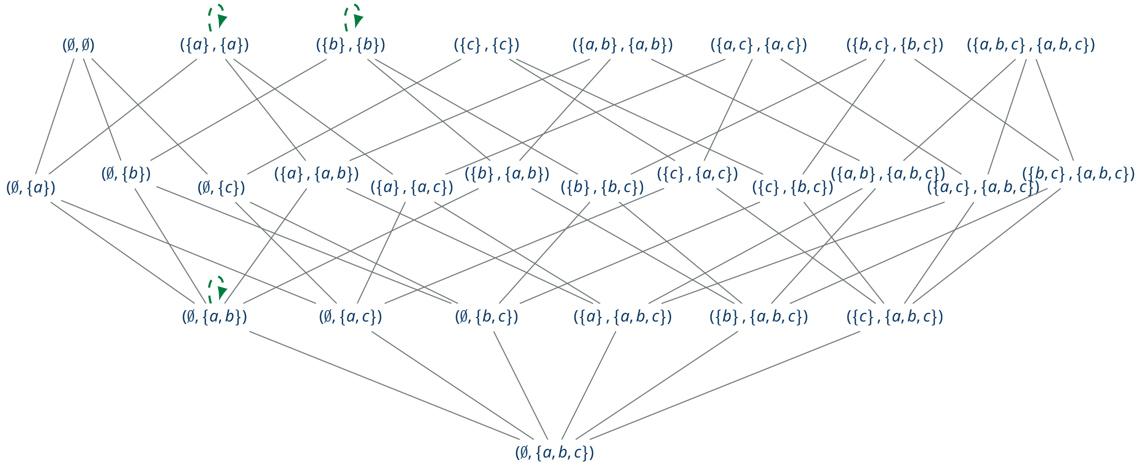
$\mathrm{lfp}({}_{P}\mathcal{J}^{\mathrm{st}}) = (\{a, b\}, \{a, b\})$: well-founded semantics of P_1

Stable Operator ${}_P\mathcal{J}^{\mathrm{st}}$: Example



$P_2 = \{a \leftarrow \sim b, \quad b \leftarrow \sim a, \quad c \leftarrow c\}$ $\quad$ $\mathrm{lfp}({}_P\mathcal{J}^{\mathrm{st}})$: well-founded semantics of P_2

Stable Operator ${}_P\mathcal{T}^{\mathrm{st}}$: Example



$P_2 = \{a \leftarrow \sim b, \quad b \leftarrow \sim a, \quad c \leftarrow c\}$ three-valued stable models of P_2

Stable Semantics: Definition via Operators

Definition

Let $(L, \leqslant)$ be a complete lattice, $O\colon L \to L$ be an operator.
Let $A\colon L^2 \to L^2$ be an approximator of O in $(L^2, \leq_i)$. A pair $(x,y) \in L^2$ is

- a **two-valued stable model of** A iff $x = y$ and $A^{\mathrm{st}}(x,y) = (x,y)$;
- a **three-valued stable model of** A iff $x \leqslant y$ and $A^{\mathrm{st}}(x,y) = (x,y)$;
- the **well-founded model of** A iff it is the least fixpoint of A^{st}.

Names inspired by notions from logic programming.

Theorem

1. $\mathrm{lfp}(A) \leq_i \mathrm{lfp}(A^{\mathrm{st}})$;
2. $A^{\mathrm{st}}(x,y) = (x,y)$ implies $A(x,y) = (x,y)$;
3. if $A^{\mathrm{st}}(x,x) = (x,x)$ then x is a $\leqslant$-minimal fixpoint of O;

Reprise: How to Find an Approximator?

Definition

Let $O\colon L \to L$ be an operator in a complete lattice $(L, \leqslant)$.
Define the **ultimate approximator of** O as follows:

$$U_O\colon L^2 \to L^2, \qquad (x,y) \mapsto \left(\bigwedge\{O(z) \mid x \leqslant z \leqslant y\}, \bigvee\{O(z) \mid x \leqslant z \leqslant y\}\right)$$

Intuition: Consider glb and lub of applying O pointwise to given interval.

Theorem

For every approximator A of O and consistent pair $(x,y) \in L^2$, we find

$$A(x,y) \leq_i U_O(x,y)$$

Ultimate approximator is most precise approximator possible.

Used e.g. for (PSP-)semantics of aggregates in logic programming.

Conclusion

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Summary

- Operators in complete lattices can be used to define semantics of KR formalisms.
- Approximation fixpoint theory provides a general account of operator-based semantics.
- Stable approximator reconstructs well-founded and stable model semantics of logic programming.

Outlook

AFT can be used to show correspondence of …

- … extensions of default theories with stable models of logic programs;
- … expansions of autoepistemic theories with supported models of LPs;
- … semantics of argumentation frameworks with semantics of LPs.