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SLD Resolution

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Previously ...

- A **substitution** replaces variables by terms, and is applied to terms.
- A **unifier** is a substitution that equates two terms when applied to them.
- The **Martelli-Montanari Algorithm** decides if a set of pairs of terms has a unifier and even outputs a (most general) unifier if one exists.
- The algorithm is **correct** (i.e., sound and complete) and **terminates**.

Example

Consider $E_0 = \{ g(x, f(y)) \doteq g(a, z),\ f(x) \doteq f(a) \}$. The algorithm yields:

$$
\begin{aligned}
E_1 &= \{ x \doteq a,\ f(y) \doteq z,\ f(x) \doteq f(a) \} && \text{(decompose)}\\
E_2 &= \{ x \doteq a,\ z \doteq f(y),\ f(x) \doteq f(a) \} && \text{(orient)}\\
E_3 &= \{ x \doteq a,\ z \doteq f(y),\ f(a) \doteq f(a) \} && \text{(apply)}\\
E_4 &= \{ x \doteq a,\ z \doteq f(y),\ a \doteq a \} && \text{(decompose)}\\
E_5 &= \{ x \doteq a,\ z \doteq f(y) \} && \text{(decompose)}
\end{aligned}
$$

Overview

The Logical Language of Programs

The Computation Mechanism: SLD Derivations

Choices and Their Impact

The Logical Language of Programs

Atoms, Term Bases, and Herbrand Bases

Definition

Let $TU_{F,V}$ be a term universe (V Variables, F function symbols) and Π be a ranked alphabet of **predicate symbols**.

The **term base** $TB_{\Pi,F,V}$ (over Π, F, and V) is the smallest set of **atoms** with

1. if $p \in \Pi^{(0)}$ then $p \in TB_{\Pi,F,V}$;
2. if $p \in \Pi^{(n)}$ with $n \geq 1$ and $t_1, \ldots, t_n \in TU_{F,V}$, then $p(t_1, \ldots, t_n) \in TB_{\Pi,F,V}$.

$\rightsquigarrow$ Usual definition of atoms of first-order predicate logic.

Definition

Let HU_F be a Herbrand universe, Π ranked alphabet of predicate symbols.

The **Herbrand base** $HB_{\Pi,F}$ (over Π and F) is given by $TB_{\Pi,F,\emptyset}$.

$\rightsquigarrow$ Herbrand base is the set of all variable-free (ground) atoms.

Queries and Programs

Definition

- A **query** is a finite sequence $B_1, \ldots, B_n$ of atoms (denoted $\vec{B}$).
- The **empty query** (empty sequence of atoms) is denoted by $\Box$.
- A **(definite) clause** is an expression $H \leftarrow \vec{B}$ where

  H is an atom (the **head** of the clause) and $\vec{B}$ is a query (the **body** of the clause).
- $H \leftarrow \vec{B}$ **unit clause** (also called: **fact**)

  $:\Longleftrightarrow$ $\vec{B}$ is empty (standard notation: $H \leftarrow$)
- **Horn clause** $:\Longleftrightarrow$ clause or negated query
- **(definite) logic program** $:\Longleftrightarrow$ finite set of (definite) clauses

We will mostly use “program” and take it to mean “definite logic program”.

Clauses and Queries: Examples

Example

Let x, y, z be variables. Then the following expressions are examples for ...

- an atom:

$$direct(maui, honolulu)$$

- a query:

$$direct(frankfurt, x), direct(x, honolulu)$$

- a fact:

$$direct(maui, honolulu) \leftarrow$$

- a (definite) clause:

$$connection(x, y) \leftarrow direct(x, z), connection(z, y)$$

Recall

In predicate logic, a **clause** is a disjunction of literals.

Logical Reading of Clauses and Queries

Clauses

A clause $H \leftarrow B_1, \ldots, B_n$ can be understood as the formula

$$\forall x_1, \ldots, x_k((B_1 \wedge \ldots \wedge B_n) \rightarrow H) \text{ (definite clause } \forall x_1, \ldots, x_k(\neg B_1 \vee \ldots \vee \neg B_n \vee H))$$

where $x_1, \ldots, x_k$ are the variables occurring in $H \leftarrow B_1, \ldots, B_n$.

(Thus a unit clause $H \leftarrow$ encodes $\forall x_1, \ldots, x_k H$.)

Queries

A query $A_1, \ldots, A_n$ can be understood as the formula

$$\exists x_1, \ldots, x_k(A_1 \wedge \ldots \wedge A_n) \qquad \text{(or: } \neg \forall x_1, \ldots, x_k(\neg A_1 \vee \ldots \vee \neg A_n))$$

where $x_1, \ldots, x_k$ are the variables occurring in $A_1, \ldots, A_n$.

(Thus the empty query $\square$ is equivalent to *true*.)

What is Being Computed?

- ▷ A program P can be interpreted as a set of axioms.
- ▷ A query Q can be interpreted as the request for finding an instance $Q\theta$ which is a logical consequence of P.
- ▷ A successful derivation provides such a substitution θ.
- ▷ In this way, the derivation is a proof of $Q\theta$ from the set of premises P.
- ▷ Thus SLD resolution provides a proof theory for programs.

$\rightsquigarrow$ To be continued in Lecture 4 (Correctness of SLD Resolution), where we introduce the corresponding *model theory*.

How Do We Compute?

- A computation is a sequence of derivation steps.
- In each step an atom A is selected in the current query and a program clause $H \leftarrow \vec{B}$ is chosen.
- If A and H are unifiable (in the sense of $A \doteq H$), then A is replaced by $\vec{B}$ and an mgu of A and H is applied to the resulting query.
- The computation is successful if it ends with the empty query.
- The resulting answer substitution θ is obtained by combining the mgus of each step.

Observation

For atoms A and H to be unifiable, they must use the same predicate $p \in \Pi^{(n)}$ and furthermore, for $A = p(s_1, \ldots, s_n)$ and $H = p(t_1, \ldots, t_n)$ the resulting set $E = \{s_1 \doteq t_1, \ldots, s_n \doteq t_n\}$ must have an mgu.

The Computation Mechanism: SLD Derivations

An SLD Derivation Step (No Variables)

Note

SLD = Selection rule driven Linear resolution for Definite clauses

Definition

Consider

- a program P
- a query $\vec{A}, B, \vec{C}$
- a clause $c = B \leftarrow \vec{B} \in P$

- B is the **selected atom**
- The resulting query $\vec{A}, \vec{B}, \vec{C}$ is called the **SLD resolvent**
- Notation: $\vec{A}, B, \vec{C} \xrightarrow[c]{} \vec{A}, \vec{B}, \vec{C}$

Example Ground Program and Query:

...

```
(1) happy :- sun, holidays.
(2) happy :- snow, holidays.
(3) snow :- cold, precipitation.
(4) cold :- winter.
(5) precipitation :- holidays.
(6) winter.
(7) holidays.
```

```
| ?- happy.
```

An SLD Derivation Step (General Case)

Definition

Consider

- a program P
- a query $\vec{A}, B, \vec{C}$
- a clause $c \in P$
- a variant $H \leftarrow \vec{B}$ of c that is variable disjoint with the query
- an mgu θ of B and H

- **SLD resolvent** of $\vec{A}, B, \vec{C}$ and c w.r.t. B with mgu θ $:\Longleftrightarrow$ $(\vec{A}, \vec{B}, \vec{C})\theta$
- **SLD derivation step** $:\Longleftrightarrow$ $\vec{A}, B, \vec{C} \xrightarrow[c]{\theta} (\vec{A}, \vec{B}, \vec{C})\theta$
- **input clause** $:\Longleftrightarrow$ variant $H \leftarrow \vec{B}$ of c

We say: “Clause c is **applicable** to atom B.”

Example Program and Query:

```
(1) add(X,0,X).
(2) add(X,s(Y),s(Z)) :- add(X,Y,Z).

(3) mul(X,0,0).
(4) mul(X,s(Y),Z) :- mul(X,Y,U), add(X,U,Z).
...
```

```
| ?- mul(s(s(0)),s(s(0)),V).

| ?- mul(V,W,s(s(0))).
```

The 4 Steps of Resolving Query and Clause

1.	**Selection**	Select an atom in the query.
2.	**Renaming**	Rename the clause (if necessary).
3.	**Instantiation**	Instantiate query and clause by an mgu of the selected atom and the head of the clause.
4.	**Replacement**	Replace the instance of the selected atom by the instance of the body of the clause.

SLD Derivations

Definition

A maximal sequence of SLD derivation steps

$$Q_0 \xrightarrow[c_1]{\theta_1} Q_1 \cdots Q_n \xrightarrow[c_{n+1}]{\theta_{n+1}} Q_{n+1} \cdots$$

is an **SLD derivation of** $P \cup \{Q_0\}$

$$:\Longleftrightarrow$$

- $Q_0, \ldots, Q_{n+1}, \ldots$ are queries, each empty or with one atom selected in it;
- $\theta_1, \ldots, \theta_{n+1}, \ldots$ are substitutions;
- $c_1, \ldots, c_{n+1}, \ldots$ are clauses of P;
- for every SLD derivation step, **standardization apart** holds.

Standardization Apart

Definition

For a sequence of SLD derivation steps as before, let $Q_{i-1} \xrightarrow[c_i]{\theta_i} Q_i$ be the i-th SLD derivation step for all $i \geq 1$ and c_i' be the input clause used in that step. Then **standardization apart** holds (for the i-th step)

$$:\Longleftrightarrow \qquad Var(c_i') \cap \left(Var(Q_0) \cup \bigcup_{j=1}^{i-1} \left(Var(\theta_j) \cup Var(c_j') \right) \right) = \emptyset$$

Intuitively: The input clause is variable disjoint from the initial query and from the substitutions and input clauses used at earlier steps.

Example

Consider program $P = \{p(f(x)) \leftarrow \}$ and query $p(x)$. Without standardizing apart, the query would fail, while the intuitively equivalent query $p(y)$ would succeed.

Result of a Derivation

Definition

Let $\xi = Q_0 \xrightarrow{\theta_1} Q_1 \cdots \xrightarrow{\theta_n} Q_n$ be a finite SLD derivation.

- ξ **successful** $:\Longleftrightarrow \quad Q_n = \square$
- ξ **failed** $:\Longleftrightarrow \quad Q_n \neq \square$ and no clause is applicable to the selected atom of Q_n

Definition

Let ξ be successful.

- **computed answer substitution (cas)** of Q_0 (w.r.t. ξ) $:= \ (\theta_1 \cdots \theta_n)|_{Var(Q_0)}$
- **computed instance** of $Q_0 \ := \ Q_0\theta_1 \cdots \theta_n$

Choices and Their Impact

Choices

In each SLD derivation step the following four choices are made:

1	Choice of the renaming
2	Choice of the most general unifier
3	Choice of the selected atom in the query
4	Choice of the program clause

How do they influence the result?

Choices

1	**Choice of the renaming**
2	**Choice of the most general unifier**
3	Choice of the selected atom in the query
4	Choice of the program clause

Resultants: What is proved after a step?

Definition

The **resultant** associated with $Q \xrightarrow{\theta} Q_1$ is the implication $Q\theta \leftarrow Q_1$.

Definition

Consider

- a program P
- a resultant $R = Q \leftarrow \vec{A}, B, \vec{C}$
- a clause c
- a variant $H \leftarrow \vec{B}$ of c that is variable disjoint with R
- an mgu θ of B and H

SLD resolvent of resultant R and c w.r.t. B with mgu θ $:=$ $(Q \leftarrow \vec{A}, \vec{B}, \vec{C})\theta$

SLD resultant step $:=$ $Q \leftarrow \vec{A}, B, \vec{C} \xrightarrow[c]{\theta} (Q \leftarrow \vec{A}, \vec{B}, \vec{C})\theta$

Resultants and SLD derivations

Definition

Consider an SLD derivation

$$\xi = Q_0 \xrightarrow[c_1]{\theta_1} Q_1 \cdots Q_n \xrightarrow[c_{n+1}]{\theta_{n+1}} Q_{n+1} \cdots$$

For $i \geq 0$,

$$R_i := Q_0\theta_1 \cdots \theta_i \leftarrow Q_i$$

is called the **resultant of level i** of ξ.

The resultant R_i describes what is proved after i derivation steps.

In particular:

- $R_0 = Q_0 \leftarrow Q_0$
- $R_n = Q_0\theta_1 \cdots \theta_n$, if $Q_n = \square$ (because $\square \mathrel{\hat{=}}$ "true")

Propagation (1)

Definition

The **selected atom** of a resultant $Q \leftarrow Q_i$ is the atom that is selected in Q_i.

Lemma 3.12

Suppose that $R \xrightarrow[c]{\theta} R_1$ and $R' \xrightarrow[c]{\theta'} R'_1$ are two SLD resultant steps where

- R is an instance of R',
- in R and R' atoms in the same positions are selected.

Then R_1 is an instance of R'_1.

Proof: [Apt97, page 55]

Propagation (2)

Corollary

Suppose that $Q \xrightarrow[c]{\theta} Q_1$ and $Q' \xrightarrow[c]{\theta'} Q'_1$ are two SLD derivation steps where

- Q is an instance of Q',
- in Q and Q' atoms in the same positions are selected.

Then Q_1 is an instance of Q'_1.

Similar SLD derivations

Definition

Consider two (initial fragments of) SLD derivations

$$\xi = Q_0 \xrightarrow[c_1]{\theta_1} Q_1 \cdots Q_n \xrightarrow[c_{n+1}]{\theta_{n+1}} Q_{n+1}$$

$$\xi' = Q'_0 \xrightarrow[c_1]{\theta'_1} Q'_1 \cdots Q'_n \xrightarrow[c_{n+1}]{\theta'_{n+1}} Q'_{n+1}$$

We say that ξ and ξ' are **similar**

$$:\Longleftrightarrow$$

- $\text{length}(\xi) = \text{length}(\xi')$,
- Q_0 and Q'_0 are variants,
- in Q_i and Q'_i atoms in the same positions are selected ($i \in [0, n]$)

A Theorem on Variants

Theorem 3.18

Consider two similar SLD derivations ξ, ξ'. For every $i \geq 0$, the resultants R_i and R'_i of level i of ξ and ξ', respectively, are variants of each other.

Proof.

By induction on i.

Base Case ($i = 0$): $R_0 = Q_0 \leftarrow Q_0$ and $R'_0 = Q'_0 \leftarrow Q'_0$ are variants of each other.

Inductive Case ($i \rightsquigarrow i + 1$): Consider $R_i \xrightarrow[c_{i+1}]{\theta_{i+1}} R_{i+1}$ and $R'_i \xrightarrow[c_{i+1}]{\theta'_{i+1}} R'_{i+1}$.

$$
\begin{array}{ll}
 & R_i \text{ variant of } R'_i \text{ (induction hypothesis)} \\
\text{implies} & R_i \text{ instance of } R'_i \text{ and vice versa} \\
\text{implies} & R_{i+1} \text{ instance of } R'_{i+1} \text{ and vice versa (Lemma 3.12)} \\
\text{implies} & R_{i+1} \text{ variant of } R'_{i+1}
\end{array}
$$

$\square$

Answer Substitutions of similar derivations

Corollary

Consider two similar successful SLD derivations of Q_0 with computed answer substitutions θ and η. Then $Q_0\theta$ and $Q_0\eta$ are variants of each other.

Proof.

By Theorem 3.18 applied to the final resultants $Q_0\theta \leftarrow \square$ and $Q_0\eta \leftarrow \square$ of these SLD derivations. $\square$

This shows that choice 1 (choice of a renaming) and choice 2 (choice of an mgu) have no influence – modulo renaming – on the statement proved by a successful SLD derivation.

Choices

1	Choice of the renaming
2	Choice of the most general unifier
3	**Choice of the selected atom in the query**
4	Choice of the program clause

Selecting Atoms in Queries

Definition

Let INIT be the set of *all* initial fragments of *all* possible SLD derivations in which the last query is non-empty.

- A **selection rule** is a function which for every $\xi^{<} \in \mathit{INIT}$ yields an occurrence of an atom in the last query of $\xi^{<}$.
- An SLD derivation ξ is **via** a selection rule $\mathcal{R}$

  $$:\Longleftrightarrow$$

  for every initial fragment $\xi^{<}$ of ξ ending with a non-empty query Q, the selected atom of Q is exactly $\mathcal{R}(\xi^{<})$.

PROLOG employs the simple selection rule "select the leftmost atom."

Switching Lemma

Lemma 3.32

Consider an SLD derivation $\xi = Q_0 \xrightarrow[c_1]{\theta_1} Q_1 \cdots Q_n \xrightarrow[c_{n+1}]{\theta_{n+1}} Q_{n+1} \xrightarrow[c_{n+2}]{\theta_{n+2}} Q_{n+2} \cdots$

where

- Q_n includes two atoms A_1 and A_2,
- A_1 is the selected atom of Q_n,
- $A_2\theta_{n+1}$ is the selected atom of Q_{n+1}.

Then the SLD derivation $\xi' = Q_0 \xrightarrow[c_1]{\theta_1} Q_1 \cdots Q_n \xrightarrow[c_{n+2}]{\theta'_{n+1}} Q'_{n+1} \xrightarrow[c_{n+1}]{\theta'_{n+2}} Q_{n+2} \cdots$

for some Q'_{n+1}, θ'_{n+1}, and θ'_{n+2} is such that:

- A_2 is the selected atom of Q_n
- $A_1\theta'_{n+1}$ is the selected atom of Q'_{n+1}
- $\theta'_{n+1}\theta'_{n+2} = \theta_{n+1}\theta_{n+2}$.

Proof: [Apt97, page 65]

Independence of Selection Rule

Theorem 3.33

Let ξ be a successful SLD derivation of $P \cup \{Q_0\}$.

Then for every selection rule $\mathcal{R}$ there exists a successful SLD derivation ξ' of $P \cup \{Q_0\}$ via $\mathcal{R}$ such that

- cas of Q_0 (w.r.t. ξ) = cas of Q_0 (w.r.t. ξ'),
- ξ and ξ' are of the same length.

This shows that choice 3 (choice of a selected atom) has no influence in case of successful queries.

Proof Sketch of Theorem 3.33.

Consider an SLD derivation $\xi = Q_0 \xrightarrow[c_1]{\theta_1} \cdots \xrightarrow[c_n]{\theta_n} Q_n = \Box$ that is not via $\mathcal{R}$.

Then there is a smallest $i \geq 1$ such that:

- ξ is via $\mathcal{R}$ up to Q_{i-1}.
- $\mathcal{R}$ selects A in Q_i.
- $A\theta_{i+1} \cdots \theta_{i+j}$ is the selected atom of Q_{i+j} in ξ for some $j \geq 1$ (ξ is successful).

$$\xi = Q_0 \quad \cdots \quad Q_i \quad \cdots \quad Q_{i+j-1} \xrightarrow[c_{i+j}]{\theta_{i+j}} Q_{i+j} \xrightarrow[c_{i+j+1}]{\theta_{i+j+1}} Q_{i+j+1} \quad \cdots \quad Q_n$$

Apply Switching Lemma once:

$$\xi = Q_0 \quad \cdots \quad Q_i \quad \cdots \quad Q_{i+j-1} \xrightarrow[c_{i+j+1}]{\theta'_{i+j}} Q'_{i+j} \xrightarrow[c_{i+j}]{\theta'_{i+j}} Q_{i+j+1} \quad \cdots \quad Q_n$$

Apply Switching Lemma further $j - 1$ times.

Choices

1	Choice of the renaming
2	Choice of the most general unifier
3	Choice of the selected atom in the query
4	**Choice of the program clause**

SLD Trees visualize Search Space

Definition

SLD Tree for $P \cup \{Q_0\}$ via selection rule $\mathcal{R}$ $:\Longleftrightarrow$

- the branches are SLD derivations of $P \cup \{Q_0\}$ via $\mathcal{R}$;
- every node Q with selected atom A has exactly one descendant for every clause c of P which is applicable to A. This descendant is a resolvent of Q and c w.r.t. A.

Definition

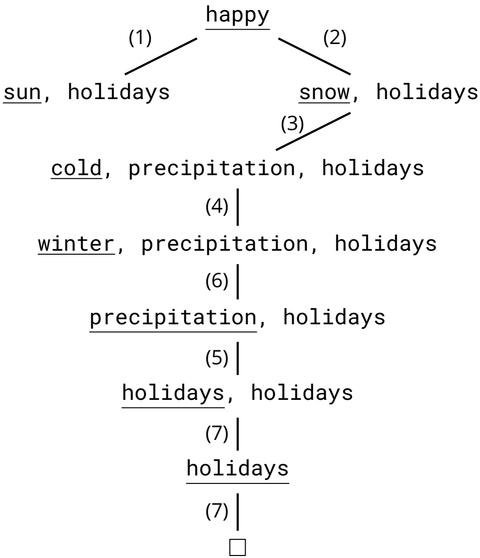
- SLD tree **successful** $:\Longleftrightarrow$ tree contains the node $\Box$.
- SLD tree **finitely failed** $:\Longleftrightarrow$ tree is finite and not successful.

SLD tree via “leftmost selection rule” corresponds to Prolog’s search space.

SLD Trees: Example

```
(1) happy :- sun, holidays.
(2) happy :- snow, holidays.
(3) snow :- cold, precipitation.
(4) cold :- winter.
(5) precipitation :- holidays.
(6) winter.
(7) holidays.
```

```
| ?- happy.
```



Variant Independence

Definition

A selection rule $\mathcal{R}$ is **variant independent**

$$:\Longleftrightarrow$$

in all initial fragments of SLD derivations that are similar (cf. slide 27), $\mathcal{R}$ chooses the atom in the same position in the last query.

Example

- The selection rule "select leftmost atom" is variant independent.
- The selection rule "select leftmost atom if query contains variable x, otherwise select rightmost atom" is variant dependent.

The Branch Theorem

Theorem 3.38

Consider an SLD tree $\mathcal{T}$ for $P \cup \{Q_0\}$ via a variant independent selection rule $\mathcal{R}$. Then every SLD derivation of $P \cup \{Q_0\}$ via $\mathcal{R}$ is similar to a branch in $\mathcal{T}$.

This shows that choice 4 (choice of a program clause) has no influence on the search space as a whole.

Proof Sketch of Theorem 3.38

Let $\xi = Q_0 \longrightarrow Q_1 \longrightarrow Q_2 \longrightarrow \ldots$ be an SLD derivation of $P \cup \{Q_0\}$ via $\mathcal{R}$.

By induction on $i \geq 0$ "find" branch $Q'_0, Q'_1, Q'_2, \ldots$ in $\mathcal{T}$ similar to ξ:

- $Q'_0 = Q_0$ (in particular they are variants).
- By definition of $\mathcal{T}$: The existence of Q'_i implies the existence of Q'_{i+1} (apply the same clause as to Q_i).
- Now $Q_0 \longrightarrow \ldots \longrightarrow Q_i$ and $Q'_0 \longrightarrow \ldots \longrightarrow Q'_i$ are similar.
- By variant independence of $\mathcal{R}$, in Q_i and Q'_i atoms in the same positions are selected.
- Thus $Q_0 \longrightarrow \ldots \longrightarrow Q_{i+1}$ and $Q'_0 \longrightarrow \ldots \longrightarrow Q'_{i+1}$ are also similar.

Conclusion

Summary

- A proof theory for (definite) logic programs is given by **SLD resolution**.
- A query is resolved with a (variant of a) program clause to another query.
- There are choices to be made (renaming of clause, mgu of query atom and clause, selected atom in query, program clause) with consequences.
- The search space can be visualized by (selection rule-induced) **SLD trees**.

Suggested action points:

- Clarify the relationship of SLD resolution and “ordinary” FOL resolution.
- Obtain SLD resolutions (with mgus) for the examples on Slide 15.
- Use Prolog’s `trace` predicate to check your results.