# Rational Inference and Defeasible Reasoning in Formal Concept Analysis

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# 1984 Congressional Voting Records

## Implications

- $R, w \to p$
- $R, a \to p$
- $D, h, p \to w$
- $D, R \to \bot$

Too strict for noisy or exception-prone data

$A \to B$ holds if every object with $A$ has $B$.

| #Congressmen | D | R | h | w | a | p |
|---|---|---|---|---|---|---|
| 79 | × | | × | | × | |
| 62 | × | | × | × | × | |
| 34 | × | | | × | × | |
| 3 | × | | | × | | × |
| 65 | | × | | | | × |
| 55 | | × | | × | | × |
| ... | ... | | | | | |

- $D$: Democrat
- $h$: handicapped infants
- $w$: water project cost sharing
- $R$: Republican
- $p$: physician fee freeze
- $a$: adoption of the budget resolution

# 1984 Congressional Voting Records

## Association rules

- $R, w \to p$
- $R, a \to p$
- $D, h, p \to w$
- $D, R \to \bot$

- $a \to D$ (91%)
- $D \to a$ (87%)
- $R \to p$ (97%)
- $p \to R$ (92%)
- …

| #Congressmen | D | R | h | w | a | p |
|---|---|---|---|---|---|---|
| 79 | × | | × | | × | |
| 62 | × | | × | × | × | |
| 34 | × | | | × | × | |
| 3 | × | | | × | | × |
| 65 | | × | | | | × |
| 55 | | × | | × | | × |
| … | … | | | | | |

$A \to B$ holds with confidence $\gamma$ if $\gamma \cdot 100\ \%$ of objects with $A$ have $B$.

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# 1984 Congressional Voting Records

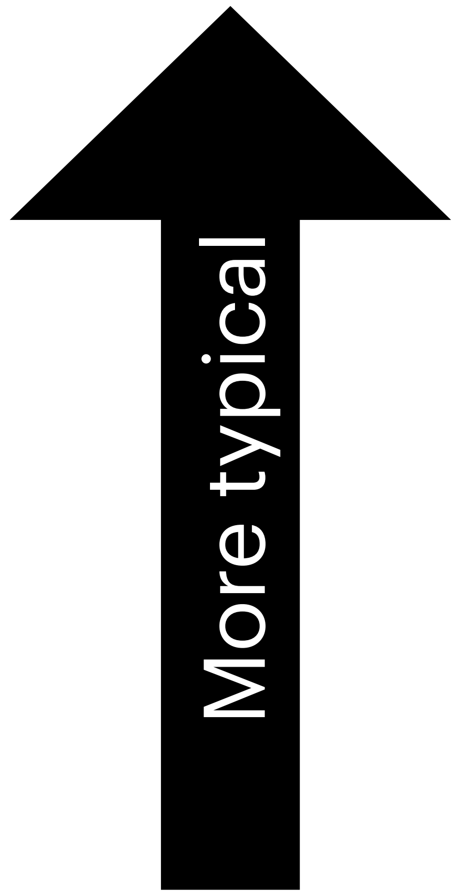
## Defeasible Conditionals

Treat some objects as less typical than others

- $D \mid\!\sim a$
- $p \mid\!\sim R$
- $w \mid\!\sim D$
- $w, p \not\mid\!\sim D$

Non-monotonic

$A \mid\!\sim B$ (defeasibly) holds if the most typical objects with $A$ have $B$.




| #Congressmen | *D* | *R* | *h* | *w* | *a* | *p* |
|---|---|---|---|---|---|---|
| 79 | × | | × | | × | |
| 65 | | × | | | | × |
| 62 | × | | × | × | × | |
| 55 | | × | | × | | × |
| 34 | × | | | × | × | |
| 3 | × | | | × | | × |

- $D$: Democrat
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# Why Not Go with Association Rules?

Fix a threshold $\gamma$ and say that $A \mid\!\sim B$ holds if $A \to B$ holds with confidence at least $\gamma$.

- For $\gamma = 0.51$, we have $w \mid\!\sim D$ but $w, p \mid\!\not\sim D$.

If typical objects with $A$ have $B$ and $C$, they should have $B \cup C$.

- However, $\mathtt{conf}(a \to h) > \mathtt{conf}(a \to w) > 0.51$, but $\mathtt{conf}(a \to h, w) < 0.51$.

| #Congressmen | D | R | h | w | a | p |
|---|---|---|---|---|---|---|
| 79 | × | | × | | × | |
| 62 | × | | × | × | × | |
| 34 | × | | | × | × | |
| 3 | × | | | × | | × |
| 65 | | × | | | | × |
| 55 | | × | | × | | × |

So, $a \mid\!\sim h$ and $a \mid\!\sim w$,

but $a \mid\!\not\sim h, w$

# KLM Postulates

## for Rational Consequence Relations

(REF) $$\frac{}{\phi \mid\sim \phi}$$

(LLE) $$\frac{\phi \equiv \psi, \quad \psi \mid\sim \gamma}{\phi \mid\sim \gamma}$$

(AND) $$\frac{\phi \mid\sim \psi, \quad \phi \mid\sim \gamma}{\phi \mid\sim \psi \wedge \gamma}$$

(CUT) $$\frac{\phi \wedge \psi \mid\sim \gamma, \quad \phi \mid\sim \psi}{\phi \mid\sim \gamma}$$

(RW) $$\frac{\psi \to \gamma, \quad \phi \mid\sim \psi}{\phi \mid\sim \gamma}$$

(OR) $$\frac{\phi \mid\sim \gamma, \quad \psi \mid\sim \gamma}{\phi \vee \psi \mid\sim \gamma}$$

(CM) $$\frac{\phi \mid\sim \psi, \quad \phi \mid\sim \gamma}{\phi \wedge \psi \mid\sim \gamma}$$

(RM) $$\frac{\phi \mid\sim \psi, \quad \phi \not\mid\sim \neg\gamma}{\phi \wedge \gamma \mid\sim \psi}$$

A consequence relation $\mid\sim$ is rational if it satisfies these postulates.

# Defeasible Conditionals in Formal Concept Analysis

# Formal Concept Analysis

**(Ganter, Wille 2024)**

- **Def:** A formal context is a triple $\mathbb{K} = (G, M, I)$, where $G$ is a set of objects, $M$ is a set of attributes, and $I \subseteq G \times M$.

- **Def:** For $A \subseteq G$ and $B \subseteq M$,

  $A' = \{m \in M \mid \forall g \in A\colon (g, m) \in I\}$ $\quad$ $B' = \{g \in G \mid \forall m \in B\colon (g, m) \in I\}$

- **Def:** For $X, Y \subseteq M$, an implication $X \to Y$ holds in $\mathbb{K}$ if $X' \subseteq Y'$. $\quad$ $\mathbb{K} \vDash X \to Y$

# Formal Concept Analysis

## Defeasible Conditionals

- **Def:** A ranked formal context $\mathbb{R} = (G, M, I, \mathsf{R})$ is a formal context $(G, M, I)$ supplied with a ranking function $\mathsf{R}\colon G \to \mathbb{N}$.
- **Def:** $\underline{A} := \{g \in A \mid \forall h \in A\colon \mathsf{R}(g) \leq \mathsf{R}(h)\}$ are the most typical objects in $A \subseteq G$.
- **Def:** A defeasible conditional $A \mid\!\sim B$ holds in $\mathbb{R}$ if $\underline{A'} \subseteq B'$. $\mathbb{R} \models A \mid\!\sim B$

- **Def:** An object $g \in G$ respects $A \mid\!\sim B$ if $g \notin A'$ or $g \in B'$.
- **Def:** An object $g \in G$ supports $A \mid\!\sim B$ if $g \in (A \cup B)'$.

# Formal Concept Analysis

## Compound attributes

The KLM postulates must be satisfied even with compound attributes in the context.

(OR) $$\frac{\phi \mid\!\sim \gamma, \quad \psi \mid\!\sim \gamma}{\phi \vee \psi \mid\!\sim \gamma}$$

(RM) $$\frac{\phi \mid\!\sim \psi, \quad \phi \not\mid\!\sim \neg\gamma}{\phi \wedge \gamma \mid\!\sim \psi}$$

| #Congressmen | *D* | *R* | *h* | *w* | *a* | *p* | *h* ∨ *w* | ¬*a* |
|---|---|---|---|---|---|---|---|---|
| 79 | × | | × | | × | | × | |
| 62 | × | | × | × | × | | × | |
| 34 | × | | | × | × | | × | |
| 3 | × | | | × | | × | × | × |
| 65 | | × | | | | × | | × |
| 55 | | × | | × | | × | × | × |

# Defeasible Conditionals

## in Ranked Contexts

- **Prop:** Defeasible conditionals of a ranked context satisfy the KLM postulates for rational consequence relations.
- **Thm:** A consequence relation $\mid\!\sim$ is rational if and only if it is induced by a ranked context.

# Non-monotonic Entailment in Formal Concept Analysis

Given a formal context and a set of defeasible conditionals, what else can be inferred?

# Determining the Typicality

## from Background Knowledge

- Start with background defeasible conditionals—your subjective view of the domain.
- Rank objects based on how well they agree with these conditionals.
- Do not treat an object as less typical than necessary.

- **Def:** For a set $\Delta$ of defeasible conditionals, a context $(G, M, I)$ is $\Delta$-compatible if there is a ranked context $(G, M, I, \mathsf{R})$ where all conditionals from $\Delta$ hold.

# Contextual Rational Closure

- **Def:** For ranked contexts $\mathbb{R}_1 = (G, M, I, \mathsf{R}_1)$ and $\mathbb{R}_2 = (G, M, I, \mathsf{R}_2)$,

$$\mathbb{R}_1 \preceq \mathbb{R}_2 \iff \forall g \in G\colon \mathsf{R}_1(g) \leq \mathsf{R}_2(g)$$

- **Prop:** For every $\Delta$ and $\Delta$-compatible $\mathbb{K} = (G, M, I)$, there is a unique $\preceq$-minimum ranked context $(G, M, I, \mathsf{R})$ satisfying $\Delta$. We denote it by $\mathbb{R}_{\text{min}}(\mathbb{K}, \Delta)$.

- **Def:** $\mathbb{K}, \Delta \models A \mid\!\sim B \iff \mathbb{R}_{\text{min}}(\mathbb{K}, \Delta) \models A \mid\!\sim B$

  $A \mid\!\sim B$ is in the rational closure of $\Delta$ w.r.t. $\mathbb{K}$

# Contextual Rational Closure

- Given
  - a set $\Delta$ of defeasible conditionals
  - and a $\Delta$-compatible context $\mathbb{K}$,
- rank $\mathbb{K}$'s objects as typical as possible to form a ranked context that satisfies $\Delta$,
- include in the rational closure of $\Delta$ w.r.t. $\mathbb{K}$ all conditionals that hold in this context.

# Determining the Typicality

## Object Rank

$\Delta: \quad D \mathrel{|\!\sim} a \qquad w \mathrel{|\!\sim} D \qquad w, p \mathrel{|\!\sim} R$

| #Congressmen | *D* | *R* | *h* | *w* | *a* | *p* | Rank |
|---|---|---|---|---|---|---|---|
| 79 | × | | × | | × | | |
| 62 | × | | × | × | × | | |
| 34 | × | | | × | × | | |
| 3 | × | | | × | | × | |
| 65 | | × | | | | × | |
| 55 | | × | | × | | × | |

# Determining the Typicality

## ObjectRank Algorithm

$\Delta: \qquad D \mathrel{|\!\sim} a \qquad w \mathrel{|\!\sim} D \qquad w, p \mathrel{|\!\sim} R$

- Find objects that respect $\Delta$.

| #Congressmen | *D* | *R* | *h* | *w* | *a* | *p* | Rank |
|---|---|---|---|---|---|---|---|
| 79 | × | | × | | × | | |
| 62 | × | | × | × | × | | |
| 34 | × | | | × | × | | |
| 3 | × | | | × | | × | |
| 65 | | × | | | | × | |
| 55 | | × | | × | | × | |

# Determining the Typicality

## ObjectRank Algorithm

$\Delta: \quad D \mid\!\sim a \quad w \mid\!\sim D \quad w, p \mid\!\sim R$

- Find objects that respect $\Delta$.
- Assign them rank 0.

| #Congressmen | *D* | *R* | *h* | *w* | *a* | *p* | Rank |
|---|---|---|---|---|---|---|---|
| 79 | × | | × | | × | | 0 |
| 62 | × | | × | × | × | | 0 |
| 34 | × | | | × | × | | 0 |
| 3 | × | | | × | | × | |
| 65 | | × | | | | × | 0 |
| 55 | | × | | × | | × | |

# Determining the Typicality

## ObjectRank Algorithm

$\Delta:$ $D \mathrel{|\!\sim} a$ $w \mathrel{|\!\sim} D$ $w, p \mathrel{|\!\sim} R$

- Find objects that respect $\Delta$.
- Assign them rank 0.
- Remove conditionals supported by 0-rank objects.

| #Congressmen | *D* | *R* | *h* | *w* | *a* | *p* | Rank |
|---|---|---|---|---|---|---|---|
| 79 | × | | × | | × | | 0 |
| 62 | × | | × | × | × | | 0 |
| 34 | × | | | × | × | | 0 |
| 3 | × | | | × | | × | |
| 65 | | × | | | | × | 0 |
| 55 | | × | | × | | × | |

# Determining the Typicality

## ObjectRank Algorithm

$\Delta :$ $D \mid\!\sim a$ $w \mid\!\sim D$ $w, p \mid\!\sim R$

- Find objects that respect $\Delta$.
- Assign them rank 0.
- Remove conditionals supported by 0-rank objects.
- Find unranked objects that respect the modified $\Delta$.

| #Congressmen | *D* | *R* | *h* | *w* | *a* | *p* | Rank |
|---|---|---|---|---|---|---|---|
| 79 | × | | × | | × | | 0 |
| 62 | × | | × | × | × | | 0 |
| 34 | × | | | × | × | | 0 |
| 3 | × | | | × | | × | |
| 65 | | × | | | | × | 0 |
| 55 | | × | | × | | × | |

# Determining the Typicality

## ObjectRank Algorithm

$\Delta : \quad D \mid\!\sim a \qquad w \mid\!\sim D \qquad w, p \mid\!\sim R$

- Find objects that respect Δ.
- Assign them rank 0.
- Remove conditionals supported by 0-rank objects.
- Find unranked objects that respect the modified Δ.
- Assign them rank 1.

| #Congressmen | *D* | *R* | *h* | *w* | *a* | *p* | Rank |
|---|---|---|---|---|---|---|---|
| 79 | × | | × | | × | | 0 |
| 62 | × | | × | × | × | | 0 |
| 34 | × | | | × | × | | 0 |
| 3 | × | | | × | | × | |
| 65 | | × | | | | × | 0 |
| 55 | | × | | × | | × | 1 |

# Determining the Typicality

## ObjectRank Algorithm

$\Delta$ :  $D \mid\!\sim a$  $w \mid\!\sim D$  $w, p \mid\!\sim R$

- Find objects that respect $\Delta$.
- Assign them rank 0.
- Remove conditionals supported by 0-rank objects.
- Find unranked objects that respect the modified $\Delta$.
- Assign them rank 1.
- And so on.

| #Congressmen | *D* | *R* | *h* | *w* | *a* | *p* | Rank |
|---|---|---|---|---|---|---|---|
| 79 | × | | × | | × | | 0 |
| 62 | × | | × | × | × | | 0 |
| 34 | × | | | × | × | | 0 |
| 3 | × | | | × | | × | 2 |
| 65 | | × | | | | × | 0 |
| 55 | | × | | × | | × | 1 |

# Determining the Typicality

## ObjectRank Algorithm

$\Delta$ :  $D \mid\!\sim a$  $w \mid\!\sim D$  $w, p \mid\!\sim R$

- Find objects that respect $\Delta$.
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|---|---|---|---|---|---|---|---|
| 79 | × | | × | | × | | 0 |
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| 34 | × | | | × | × | | 0 |
| 3 | × | | | × | | × | 2 |
| 65 | | × | | | | × | 0 |
| 55 | | × | | × | | × | 1 |

If $\Delta$ is not empty and there are unranked objects none of which respects $\Delta$, then $\mathbb{K}$ is not $\Delta$-compatible.

# Determining the Typicality

## ObjectRank Algorithm

$\Delta:\quad D \mid\sim a \quad w \mid\sim D \quad w, p \mid\sim R$

- Find objects that respect $\Delta$.
- Assign them rank 0.
- Remove conditionals supported by 0-rank objects.
- Find unranked objects that respect the modified $\Delta$.
- Assign them rank 1.
- And so on.

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|---|---|---|---|---|---|---|---|
| 79 | × | | × | | × | | 0 |
| 62 | × | | × | × | × | | 0 |
| 34 | × | | | × | × | | 0 |
| 3 | × | | | × | | × | 2 |
| 65 | | × | | | | × | 0 |
| 55 | | × | | × | | × | 1 |

Given $\Delta$ and a $\Delta$-compatible $\mathbb{K}$, ObjectRank computes $\mathbb{R}_{min}(\mathbb{K}, \Delta)$.

# Determining the Typicality

## ObjectRank Algorithm

$\Delta:$ $D \mid\!\sim a$ $w \mid\!\sim D$ $w, p \mid\!\sim R$

- Find objects that respect $\Delta$.
- Assign them rank 0.
- Remove conditionals supported by 0-rank objects.
- Find unranked objects that respect the modified $\Delta$.
- Assign them rank 1.
- And so on.

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| 65 | | × | | | | × | 0 |
| 55 | | × | | × | | × | 1 |

Runtime: $O(|G|^2|\Delta|)$

# Contextual Rational Closure

$\Delta :$ $D \mid\!\sim a$ $w \mid\!\sim D$ $w, p \mid\!\sim R$

$\Rightarrow$ $p \mid\!\sim R$

| #Congressmen | D | R | h | w | a | p | Rank |
|---|---|---|---|---|---|---|---|
| 79 | × | | × | | × | | 0 |
| 62 | × | | × | × | × | | 0 |
| 34 | × | | | × | × | | 0 |
| 3 | × | | | × | | × | 2 |
| 65 | | × | | | | × | 0 |
| 55 | | × | | × | | × | 1 |

# Contextual Rational Closure

$\Delta: \quad D \mid\!\sim a \quad w \mid\!\sim D \quad w, p \mid\!\sim R$

$\Rightarrow \quad p \mid\!\sim R \quad w, p \not\mid\!\sim D$

| #Congressmen | D | R | h | w | a | p | Rank |
|---|---|---|---|---|---|---|---|
| 79 | × | | × | | × | | 0 |
| 62 | × | | × | × | × | | 0 |
| 34 | × | | | × | × | | 0 |
| 3 | × | | | × | | × | 2 |
| 65 | | × | | | | × | 0 |
| 55 | | × | | × | | × | 1 |

# Contextual Rational Closure

## Non-monotonicity

$\Delta:$ $D \mid\sim a$ $w \mid\sim D$ $w, p \mid\sim R$

$\Rightarrow$ $p \mid\sim R$ $w, p \not\mid\sim D$

$\Delta:$ $w \mid\sim D$

| #Congressmen | D | R | h | w | a | p | Rank |
|---|---|---|---|---|---|---|---|
| 79 | × | | × | | × | | 0 |
| 62 | × | | × | × | × | | 0 |
| 34 | × | | | × | × | | 0 |
| 3 | × | | | × | | × | 0 |
| 65 | | × | | | | × | 0 |
| 55 | | × | | × | | × | 1 |

# Contextual Rational Closure

## Non-monotonicity

$\Delta:$ $D \mathrel{|\!\sim} a$ $w \mathrel{|\!\sim} D$ $w, p \mathrel{|\!\sim} R$

$\Rightarrow$ $p \mathrel{|\!\sim} R$ $w, p \mathrel{|\!\not\sim} D$

$\Delta:$ $w \mathrel{|\!\sim} D$

$\Rightarrow$ $p \mathrel{|\!\not\sim} R$

| #Congressmen | *D* | *R* | *h* | *w* | *a* | *p* | Rank |
|---|---|---|---|---|---|---|---|
| 79 | × | | × | | × | | 0 |
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| 34 | × | | | × | × | | 0 |
| 3 | × | | | × | | × | 0 |
| 65 | | × | | | | × | 0 |
| 55 | | × | | × | | × | 1 |

# Contextual Rational Closure

## Non-monotonicity

$\Delta:$ $D \mathrel{|\!\sim} a$ $w \mathrel{|\!\sim} D$ $w, p \mathrel{|\!\sim} R$

$\Rightarrow$ $p \mathrel{|\!\sim} R$ $w, p \mathrel{|\!\not\sim} D$

$\Delta:$ $w \mathrel{|\!\sim} D$

$\Rightarrow$ $p \mathrel{|\!\not\sim} R$ $w, p \mathrel{|\!\sim} D$

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|---|---|---|---|---|---|---|---|
| 79 | × | | × | | × | | 0 |
| 62 | × | | × | × | × | | 0 |
| 34 | × | | | × | × | | 0 |
| 3 | × | | | × | | × | 0 |
| 65 | | × | | | | × | 0 |
| 55 | | × | | × | | × | 1 |

# Rational Closure

## Context vs No Context

**Recall:** $\mathbb{K}, \Delta \models A \mid\!\sim B \iff \mathbb{R}_{\min}(\mathbb{K}, \Delta) \models A \mid\!\sim B$

$A \mid\!\sim B$ is in the rational closure of $\Delta$ w.r.t. $\mathbb{K}$

- The context $\mathbb{K}$ specifies background knowledge for the entailment relation by listing the allowed attribute combinations.
- For context-agnostic entailment, include all possible attribute combinations into $\mathbb{K}$.

# Conclusion

- We introduced defeasible conditionals into FCA via ranked contexts
  - $A \mid\!\sim B$ doesn't imply $A \cup C \mid\!\sim B$
- They satisfy the KLM postulates for rational consequence relations
- We defined contextual non-monotonic entailment for defeasible conditionals
  - $\Delta_1 \vDash A \mid\!\sim B$ doesn't mean that $\Delta_1 \cup \Delta_2 \vDash A \mid\!\sim B$

- Next:
  - Complexity of context-agnostic entailment
  - Typical concepts and the corresponding concept lattices
  - Other types of defeasible entailment