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What Can Approximation Fixpoint Theory Do For (Abstract) Argumentation?

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Overview

Motivation: Objective

Goal: Define semantics for (rule-based) KR formalisms in the presence of:

Recursion

- transitive closure
- indirect effects of actions

Negation

- shorter and more intuitive descriptions
- defaults and assumptions (e.g. closed world, non-effects of actions)

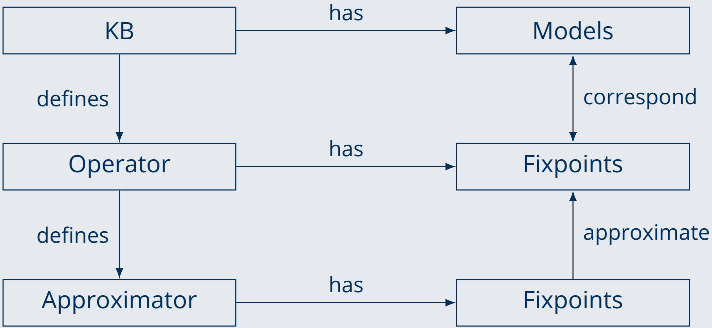
Recursion Through Negation

- mutually exclusive alternatives
- non-deterministic effects of actions

Motivation: Basic Idea

Approximation Fixpoint Theory

- Framework for studying semantics of (non-monotonic) KR formalisms
- Due to Denecker, Marek, and Truszczyński [2000, 2003, 2004]
- Based on lattice theory and fixpoint theory:



Motivation: History and Context

Approximation Fixpoint Theory

... emerged from similarities in the semantics of

- Default Logic [Reiter, 1980]
- Autoepistemic Logic [Moore, 1985]
- Logic Programs, in particular Stable Models [Gelfond and Lifschitz, 1988]

... and has since been applied to define/reconstruct semantics of ...

- Abstract Argumentation Frameworks
- Abstract Dialectical Frameworks
- Active Integrity Constraints
- Recursive SHACL

Agenda

Preliminaries
- Lattice Theory
- Abstract Argumentation Frameworks

Approximating Operators
- Approximator
- Defining Semantics

Abstract Dialectical Frameworks

Stable Operators
- Semantics via Fixpoints

Conclusion

Preliminaries

Partially Ordered Sets

Definition

A **partially ordered set** is a pair $(L, \leqslant)$ with

- L a set, and (carrier set)
- $\leqslant \subseteq L \times L$ a partial order. (reflexive, antisymmetric, transitive)

A partially ordered set $(L, \leqslant)$ has a

- **bottom element** $\bot \in L$ iff $\bot \leqslant x$ for all $x \in L$,
- **top element** $\top \in L$ iff $x \leqslant \top$ for all $x \in L$.

Examples

- $(\mathbb{N}, \leq)$: natural numbers with "usual" ordering, $\bot = 0$, no $\top$
- $(2^S, \subseteq)$: any powerset with subset relation, $\bot = \emptyset$, $\top = S$
- $(\mathbb{N}, |)$: natural numbers with divisibility relation, $\bot = 1$, $\top = 0$

Minimal, Maximal, Least, Greatest

Definition

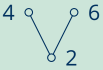
Let $(L, \leqslant)$ be a partially ordered set with $S \subseteq L$ and $x \in S$. We say that:

- x is a **minimal element** of S iff for each $y \in S$, $y \leqslant x$ implies $y = x$, dually,
- x is a **maximal element** of S iff for each $y \in S$, $x \leqslant y$ implies $y = x$;
- x is the **least element** of S iff for each $y \in S$, we have $x \leqslant y$, dually,
- x is the **greatest element** of S iff for each $y \in S$, we have $y \leqslant x$.

Example

In $(\mathbb{N}, |)$ (natural numbers with divisibility $a \mid b \iff (\exists k \in \mathbb{N}) a \cdot k = b$), ...

- the set $\{2, 3, 6\}$ has minimal elements 2 and 3, greatest element 6,
- the set $\{2, 4, 6\}$ has least element 2, and maximal elements 4 and 6.



Least Upper and Greatest Lower Bounds

Definition

Let $(L, \leqslant)$ be a partially ordered set with $S \subseteq L$ and $x \in L$.

- x is an **upper bound** of S iff for each $s \in S$, we have $s \leqslant x$, dually,
- x is a **lower bound** of S iff for each $s \in S$, we have $x \leqslant s$.

The set of all upper bounds of S is denoted by S^u, its lower bounds by S^ℓ.

- If S^u has a least element $z \in S$, z is the **least upper bound** of S, dually,
- if S^ℓ has a greatest element $z \in S$, z is the **greatest lower bound** of S.

We denote the **glb** of $\{x, y\}$ by $x \wedge y$, and the **lub** of $\{x, y\}$ by $x \vee y$.
We denote the glb of S by $\bigwedge S$, and the lub of S by $\bigvee S$.

Examples

- In $(2^S, \subseteq)$, $\wedge = \cap$ and $\vee = \cup$;
- in $(\mathbb{N}, |)$, $\wedge = \gcd$ and $\vee = \mathrm{lcm}$, e.g. $4 \vee 6 = 12$ and $23 \wedge 42 = 1$.

(Complete) Lattices

Definition

Let $(L, \leqslant)$ be a partially ordered set.

1. $(L, \leqslant)$ is a **lattice** if and only if for all $x, y \in L$, both $x \wedge y$ and $x \vee y$ exist;
2. $(L, \leqslant)$ is a **complete lattice** iff for all $S \subseteq L$, both $\bigwedge S$ and $\bigvee S$ exist.

In particular, a complete lattice has $\bigvee \emptyset = \bigwedge L = \bot$ and $\bigwedge \emptyset = \bigvee L = \top$.

Examples

- $(2^S, \subseteq)$ is a complete lattice for every set S.
- $(\mathbb{N}, \mid)$ is a complete lattice.
- $(\{M \subseteq \mathbb{N} \mid M \text{ is finite}\}, \subseteq)$ is a lattice.
- Every lattice $(L, \leqslant)$ with L finite is a complete lattice. (induction on $|S|$)

Further reading: **davey-priestley**

Operators and Their Properties

Definition

Let $(L, \leqslant)$ be a partially ordered set. An operator $O\colon L \to L$ is

- $\leqslant$-**monotone** iff for all $x, y \in L$, we find that $x \leqslant y$ implies $O(x) \leqslant O(y)$;
- $\leqslant$-**antimonotone** iff for all $x, y \in L$, we find that $x \leqslant y$ implies $O(y) \leqslant O(x)$.

Intuition: Operator application preserves/reverses ordering.

Example

Consider $(2^{\mathbb{N}}, \subseteq)$ with operator $O\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}, \quad M \mapsto \{\prod K \mid K \subseteq M,\ K \text{ finite}\}$.

- $O(\{2,3\}) = \{1,2,3,6\}$ and $O(\{2,3,5\}) = \{1,2,3,5,6,10,15,30\}$.
- O is $\subseteq$-monotone:
  - Let $M_1 \subseteq M_2 \subseteq \mathbb{N}$ and consider $k \in O(M_1)$.
  - Then there is a $K \subseteq M_1$ with $k = \prod K$.
  - By $K \subseteq M_1 \subseteq M_2$, we get $k \in O(M_2)$.

Fixpoints of Operators

Definition

Let $(L, \leqslant)$ be a partially ordered set and $O\colon L \to L$ be an operator.

- $x \in L$ is a **fixpoint** of O iff $O(x) = x$;
- $x \in L$ is a **prefixpoint** of O iff $O(x) \leqslant x$;
- $x \in L$ is a **postfixpoint** of O iff $x \leqslant O(x)$.

Theorem (Knaster/Tarski)

Let $(L, \leqslant)$ be a complete lattice and $O\colon L \to L$ be a monotone operator.
Then the set F of fixpoints of O has a least element and a greatest element.

Order-preserving operators on complete lattices have a fixpoint.

Example (Continued.)

Consider $(2^{\mathbb{N}}, \subseteq)$ with operator $O\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}, \quad M \mapsto \{\prod K \mid K \subseteq M,\ K \text{ finite}\}$.
O has least and greatest fixpoints: $O(\{1\}) = \{1\}$ and $O(\mathbb{N}) = \mathbb{N}$.

Fixpoints of Operators (2)

Theorem (Knaster/Tarski)

Let $(L, \leqslant)$ be a complete lattice and $O \colon L \to L$ be a monotone operator.
Then the set F of fixpoints of O has a least element and a greatest element.

Proof.

Define $A = \{x \in L \mid O(x) \leqslant x\}$ and $a = \bigwedge A$. $(A \neq \emptyset$ as $\top \in A$.)

- For every $x \in A$, we have $a \leqslant x$ and by monotonicity $O(a) \leqslant O(x) \leqslant x$.
- Thus $O(a)$ is a lower bound of A.
- Since a is the greatest lower bound of A, we get $O(a) \leqslant a$, that is, $a \in A$.
- Furthermore, monotonicity yields $O(O(a)) \leqslant O(a)$, whence $O(a) \in A$.
- Since a is a lower bound of A, we get $a \leqslant O(a)$, thus $O(a) = a$.
- Greatest fixpoint β is obtained dually: $B = \{x \in L \mid x \leqslant O(x)\}$, $\beta = \bigvee B$. $\square$

$(F, \leqslant)$ is a complete lattice: for $G \subseteq F$, take $([\bigvee G, \bigvee L], \leqslant)$ and $([\bigwedge L, \bigwedge G], \leqslant)$.

Fixpoints of Operators (3)

Nice to know there is one, but how do we get there?

Theorem

Let $(L, \leqslant)$ be a complete lattice and $O \colon L \to L$ be a $\leqslant$-monotone operator. For ordinals α, β, define

$$O^0(\bot) = \bot$$
$$O^{\alpha+1}(\bot) = O(O^\alpha(\bot)) \qquad \text{for successor ordinals}$$
$$O^\beta(\bot) = \bigvee \{O^\alpha(\bot) \mid \alpha < \beta\} \qquad \text{for limit ordinals}$$

Then for some ordinal α, the element $O^\alpha(\bot)$ is a fixpoint of O.

Example (Continued.)

Consider $(2^{\mathbb{N}}, \subseteq)$ with operator $O \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}, \quad M \mapsto \{\prod K \mid K \subseteq M,\ K \text{ finite}\}$.
We obtain the chain $O^0(\emptyset) = \emptyset \rightsquigarrow O^1(\emptyset) = \{1\} \rightsquigarrow O^2(\emptyset) = O(\{1\}) = \{1\}$.

Abstract Argumentation Frameworks

We assume some background reservoir of (abstract) arguments.

Definition (Dung, 1995)

An **argumentation framework** is a pair $F = (A, R)$ with $R \subseteq A \times A$.

A pair $(a, b) \in R$ expresses that a attacks b.

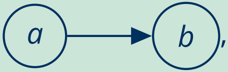
Definition (Dung, 1995)

For an AF $F = (A, R)$, its **characteristic operator** is given by

$$\Gamma_F : 2^A \to 2^A, \qquad S \mapsto \{a \in A \mid S \text{ defends } a\}$$

S **defends** a iff S attacks all attackers of a.

Example

In $F_1 =$ 

, we have $\Gamma_{F_1}(\emptyset) = \{a\}$ and $\Gamma_{F_1}(\{a\}) = \{a\}$.

Semantics via Operators

Observation

- For any AF F, the operator Γ_F is monotone in the complete lattice $(2^A, \subseteq)$.
- Therefore, Γ_F always has a least fixpoint.

Proposition

Let F be an argumentation framework.

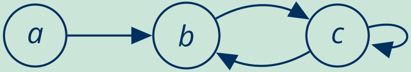
- The $\subseteq$-least fixpoint of Γ_F corresponds to the grounded extension of F.
- The conflict-free fixpoints of Γ_F correspond to complete extensions of F.

Open Questions

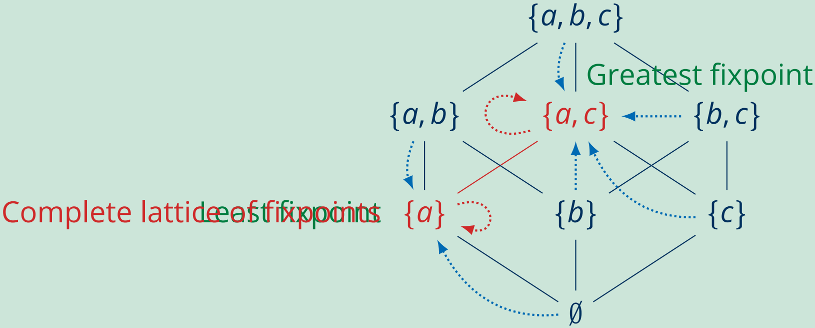
- Can other semantics also be recast in terms of operators?
- Can the extra condition of conflict-freeness be eliminated?

Characteristic Operator: Example

Example

Consider $A = \{a, b, c\}$ and the AF $F_2 =$  .

The operator Γ_{F_2} maps as follows:



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Pollock's Operator

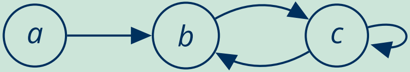
Definition (Pollock, 1987)

For an AF $F = (A, R)$, its **unattacked operator** is given by

$$U_F : 2^A \to 2^A, \quad S \mapsto A \setminus R(S) \;\text{ with }\; R(S) := \{a \in A \mid (b, a) \in R \text{ for some } b \in S\}\,.$$

Example

$F_2 =$



:

S	$U_{F_2}(S)$	S	$U_{F_2}(S)$
$\emptyset$	$\{a, b, c\}$	$\{a, b\}$	$\{a\}$
$\{a\}$	$\{a, c\}$	$\{a, c\}$	$\{a\}$
$\{b\}$	$\{a, b\}$	$\{b, c\}$	$\{a\}$
$\{c\}$	$\{a\}$	$\{a, b, c\}$	$\{a\}$

Proposition

For any AF $F = (A, R)$ and $S, T \subseteq A$, we have: $\quad S \subseteq T \implies U_F(T) \subseteq U_F(S)$.

Quiz: https://tud.link/jamqpw

Recall: $U_F(S) = A \setminus \{a \in A \mid (b, a) \in R \text{ for some } b \in S\}$.

Quiz

Consider the argumentation framework $F_3 = (A, R)$: ...

Pollock's Operator: Properties

Lemma 45 (Dung, 1995)

For any argumentation framework $F = (A, R)$ and $S \subseteq A$, $\Gamma_F(S) = U_F(U_F(S))$.

Proof.

$$
\begin{aligned}
a \notin \Gamma_F(S) \iff & \text{ there is a } b \in U_F(S) \text{ with } (b, a) \in R \\
\iff & \ a \in R(U_F(S)) \\
\iff & \ a \notin A \setminus R(U_F(S)) \\
\iff & \ a \notin U_F(U_F(S))
\end{aligned}
$$

□

Proposition

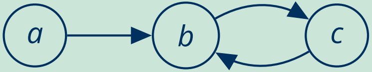
For any AF $F = (A, R)$ and $S \subseteq A$,

$$S \text{ is conflict-free} \iff S \subseteq U_F(S)$$

Pollock's Operator: Example

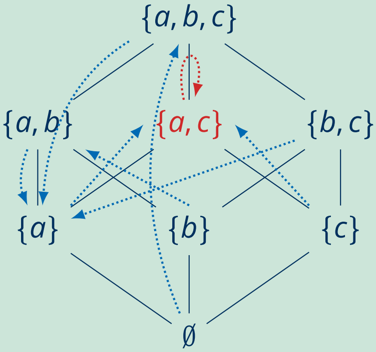
Example

Consider $F_4 = (A, R)$:



Operator U_F visualised by 

U_F has a fixpoint:
The stable extension of F_4.



- Does the correspondence fixpoints/stable extensions generalise?
- How to capture more semantics?

Characterising Semantics via Operators

Theorem

Let $F = (A, R)$ be an argumentation framework. A set $S \subseteq A$ is ...

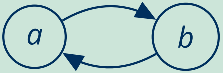
1. conflict-free iff $S \subseteq U_F(S)$;
2. admissible iff $S \subseteq U_F(S)$ and $S \subseteq \Gamma_F(S)$;
3. complete iff $S \subseteq U_F(S)$ and $S = \Gamma_F(S)$;
4. stable iff $S = U_F(S)$;
5. grounded iff it is the least fixpoint of Γ_F.

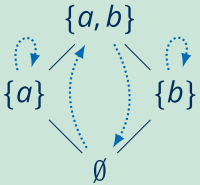
Proof.

4. S is stable iff S is conflict-free and S attacks all arguments in $A \setminus S$
iff S is conflict-free and $R(S) \supseteq A \setminus S$
iff $S \subseteq U_F(S)$ and $A \setminus R(S) \subseteq A \setminus (A \setminus S)$
iff $S \subseteq U_F(S)$ and $U_F(S) \subseteq S$ □

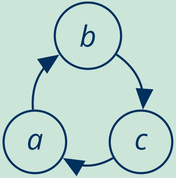
Why Is This Not Enough?

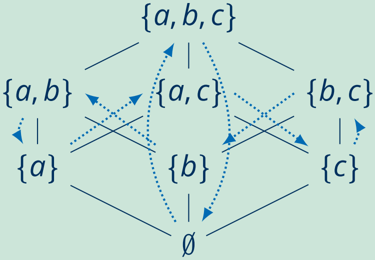
Example

- $F_5 =$ 

$\rightsquigarrow$ 

(no least fixpoint)

- $F_6 =$ 

$\rightsquigarrow$ 

(no fixpoint at all)

Stocktaking

- Monotone operators in complete lattices have (least and greatest) fixpoints.
- Operators can be associated with knowledge bases such that their fixpoints correspond to models.
- An AF F induces its characteristic operator Γ_F, whose least fixpoint is exactly the grounded extension of F.
- An AF F also induces its unattacked operator U_F, which characterises conflict-freeness and stable semantics.
- The unattacked operator U_F can emulate the characteristic operator Γ_F.
- Can semantics be formulated only in terms of U_F, and in a more uniform manner?

Approximating Operators

Approximating Operators

Main Idea

Use a more fine-grained structure to keep track of (partial) truth values.

Desiderata

- Preserve "interpretation revision" character of operators
- Preserve correspondence of fixpoints with models
- Obtain useful properties of operators

Approach

- Approximate sets of models by intervals.
- Use an information ordering on these approximations.
- Approximate operators by approximators – operators on intervals.
- Guarantee that fixpoints of approximators contain original fixpoints.

From Lattices to Bilattices

Definition

Let $(L, \leqslant)$ be a partially ordered set.
Its associated **information bilattice** is $(L^2, \leq_i)$ with $L^2 = L \times L$ and

$$(u, v) \leq_i (x, y) \quad \text{iff} \quad u \leqslant x \text{ and } y \leqslant v$$

- A pair (x, y) is **consistent** iff $x \leqslant y$; it approximates all $z \in L$ with $x \leqslant z \leqslant y$.
- For consistent pairs: Information ordering $\hat{=}$ interval inclusion:

$$(u, v) \leq_i (x, y) \text{ iff } [x, y] \subseteq [u, v]$$

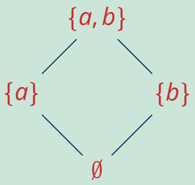
Proposition

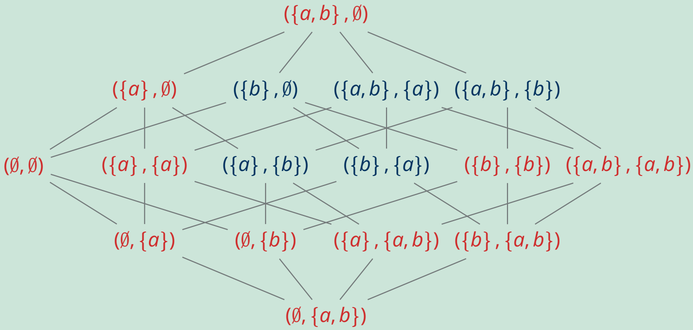
If $(L, \leqslant)$ is a complete lattice, then $(L^2, \leq_i)$ is a complete lattice. For $S \subseteq L^2$:

$$\bigwedge\nolimits_i S = \left(\bigwedge S', \bigvee S''\right) \quad \text{and} \quad \bigvee\nolimits_i S = \left(\bigvee S', \bigwedge S''\right) \qquad \begin{array}{l} S' = \{x \mid (x,y) \in S\} \\ S'' = \{y \mid (x,y) \in S\} \end{array}$$

From Lattice to Bilattice: Example

Example





Original lattice $\left(2^{\{a,b\}}, \subseteq\right)$

Bilattice $\left(2^{\{a,b\}} \times 2^{\{a,b\}}, \leq_i\right)$

Pairs in the bilattice correspond to four-valued interpretations $v\colon \{a,b\} \to \{\mathbf{t}, \mathbf{f}, \mathbf{u}, \mathbf{i}\}$.

We will mostly be concerned with the consistent pairs (x, y) with $x \leqslant y$.

Elements of the original lattice correspond to exact pairs (x, x).

Approximator

Recall approach: Approximate lattice operators on a richer structure.

Definition

Let $(L, \leqslant)$ be a complete lattice and $O\colon L \to L$ be an operator.
An operator $\mathcal{A}\colon L^2 \to L^2$ **approximates** O iff for all $x \in L$, we have

$$\mathcal{A}(x, x) = (O(x), O(x))$$

$\mathcal{A}$ is an **approximator** iff $\mathcal{A}$ approximates some O and $\mathcal{A}$ is $\leq_i$-monotone.

Approximator coincides with the operator on exact pairs.

$\mathcal{A}\colon L^2 \to L^2$ induces $\mathcal{A}', \mathcal{A}''\colon L^2 \to L$ with $\mathcal{A}(x, y) = (\mathcal{A}'(x, y), \mathcal{A}''(x, y))$.

Definition

An approximator is **symmetric** iff $\mathcal{A}'(x, y) = \mathcal{A}''(y, x)$.

If $\mathcal{A}$ is symmetric, then $\mathcal{A}(x, y) = (\mathcal{A}'(x, y), \mathcal{A}'(y, x))$, so $\mathcal{A}'$ fully specifies $\mathcal{A}$.

Approximator: Example

Example

An argumentation framework $F = (A, R)$ induces U_F with $U_F(S) = A \setminus R(S)$.
The **canonical approximator** of U_F is

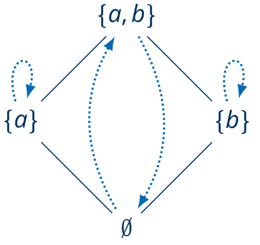
$$\mathcal{U}_F \colon 2^A \times 2^A \to 2^A \times 2^A, \qquad (X, Y) \mapsto (U_F(Y), U_F(X))$$

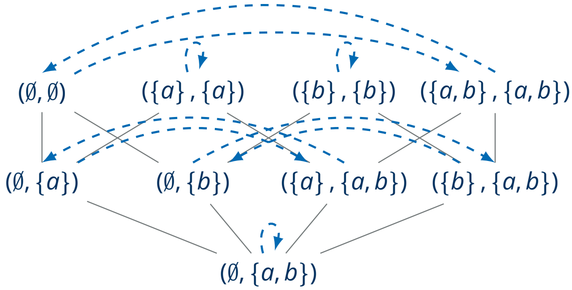
In other words, $\mathcal{U}_F$ is symmetric with $\mathcal{A}'(X, Y) = U_F(Y)$.

- $\mathcal{U}_F$ approximates U_F, as $\mathcal{U}_F(X, X) = (U_F(X), U_F(X))$.
- $\mathcal{U}_F$ is $\leq_i$-monotone:

$$\begin{aligned} (X_1, Y_1) \leq_i (X_2, Y_2) &\iff X_1 \subseteq X_2 \;\; \& \;\; Y_2 \subseteq Y_1 \\ &\implies U_F(X_2) \subseteq U_F(X_1) \;\; \& \;\; U_F(Y_1) \subseteq U_F(Y_2) \\ &\iff (U_F(Y_1), U_F(X_1)) \leq_i (U_F(Y_2), U_F(X_2)) \\ &\iff \mathcal{U}_F(X_1, Y_1) \leq_i \mathcal{U}_F(X_2, Y_2) \end{aligned}$$

Approximator $\mathcal{U}_F$: Example

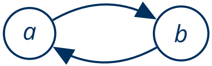




Original lattice $\left(2^{\{a,b\}},\subseteq\right)$

Bilattice $\left(2^{\{a,b\}}\times 2^{\{a,b\}},\leq_i\right)$

Argumentation Framework

$F_5 =$ 

Approximator $\mathcal{U}_F$ for U_F: - - ▶

Operator U_F: ·······▶

Quiz: Approximator $\mathcal{U}_F$

https://tud.link/8jn6f9

Recall: $\mathcal{U}_F(X, Y) = (U_F(Y), U_F(X))$, with $U_F(S) = A \setminus R(S)$.

Quiz

Consider the following argumentation framework: …

Approximator: Observations (1)

Lemma

Let $(L, \leqslant)$ be a complete lattice and $\mathcal{A}$ an approximator on $(L^2, \leq_i)$.

1. If C is a non-empty chain of consistent pairs, then $\bigvee_i C$ is consistent.
2. If (x, y) is consistent, then $\mathcal{A}(x, y)$ is consistent.

Approximators map consistent pairs to consistent pairs.

Proof.

1. Let $a, b \in C$. Since C is a chain, $a \leq_i b$ (then $a' \leqslant b' \leqslant b''$) or $b \leq_i a$ (then $a' \leqslant a'' \leqslant b''$). In any case, $a' \leqslant b''$. So every $c'' \in C''$ is an upper bound of C', and $\bigvee C' \leqslant c''$. Hence $\bigvee C'$ is a lower bound of C'' and $\bigvee C' \leqslant \bigwedge C''$.
2. If $x \leqslant y$, then for z with $x \leqslant z \leqslant y$ we have $(x, y) \leq_i (z, z)$. $\mathcal{A}$ is $\leq_i$-monotone, thus $\mathcal{A}(x, y) \leq_i \mathcal{A}(z, z)$. $\mathcal{A}$ approximates some O, thus $\mathcal{A}(z, z) = (O(z), O(z))$. In combination $\mathcal{A}'(x, y) \leqslant O(z) \leqslant \mathcal{A}''(x, y)$. □

Approximator: Observations (2)

Theorem

Let $(L, \leqslant)$ be a complete lattice with $O : L \to L$, and $\mathcal{A}$ an approximator for O.

1. $\mathcal{A}$ has a $\leq_i$-least fixpoint (x^*, y^*) with $x^* \leqslant y^*$.
2. Every fixpoint z of O satisfies $x^* \leqslant z \leqslant y^*$.

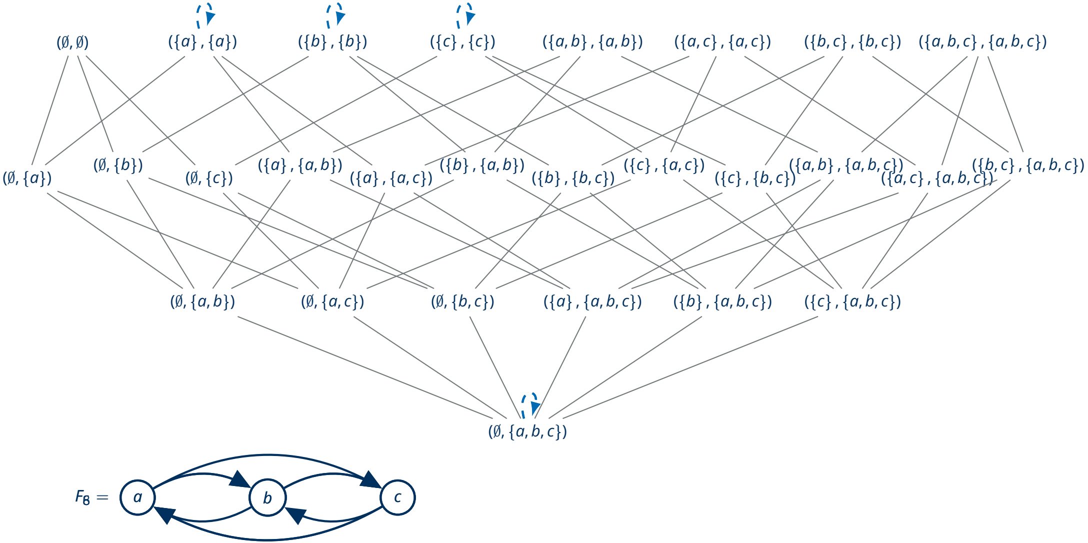
The least fixpoint of $\mathcal{A}$ is consistent and approximates all fixpoints of O.

Proof.

1. By Knaster/Tarski, $\mathcal{A}$ has a $\leq_i$-least fixpoint (x^*, y^*). It is also consistent: Define $Q = \left\{(x,y) \in L^2 \mid x \leqslant y \ \ \& \ \ (x,y) \leq_i \mathcal{A}(x,y) \ \ \& \ \ (x,y) \leq_i (x^*,y^*)\right\}$. Q is non-empty as $(\bot, \top) \in Q$. Each non-empty chain in Q has an upper bound in Q, therefore by Zorn's Lemma, Q has a maximal element, p. Since p is maximal, $p \leq_i \mathcal{A}(p)$ directly yields $\mathcal{A}(p) = p = (x^*, y^*)$.
2. If $O(z) = z$ then $\mathcal{A}(z,z) = (O(z), O(z)) = (z,z)$ and $(x^*, y^*) \leq_i (z,z)$. □

Let $\emptyset \neq C \subseteq Q$ be a chain. Define $d = \bigvee_i C$. (1) By the previous lemma, d is consistent. (2) For every $c \in C$ we have $c \leq_i d$ and thus $c \leq_i \mathcal{A}(c) \leq_i \mathcal{A}(d)$; thus $\mathcal{A}(d)$ is an upper bound of C, whence $d \leq_i \mathcal{A}(d)$. (3) We know that $C \subseteq Q$ whence (x^*, y^*) is an upper bound of C, thus $d \leq_i (x^*, y^*)$.

Approximator $\mathcal{U}_F$: Examples



Recovering Semantics

Approximator fixpoints give rise to several semantics.

Theorem

Let $F = (A, R)$ be an argumentation framework and $X \subseteq Y \subseteq A$.

- X is stable for F iff $\mathcal{U}_F(X, X) = (X, X)$.
- (X, Y) is complete for F iff $\mathcal{U}_F(X, Y) = (X, Y)$.
- (X, Y) is grounded for F iff $(X, Y) = \mathrm{lfp}(\mathcal{U}_F)$.
- (X, Y) is admissible for F iff $(X, Y) \leq_i \mathcal{U}_F(X, Y)$.

Further semantics (e.g. preferred, ideal) via maximisation/intersection/...

So what does it buy us?

For a new formalism, we only have to define an approximator!

Abstract Dialectical Frameworks

Abstract Dialectical Frameworks: Syntax

Main Idea: Allow for more flexible specification of argument relationships.

Definition (Brewka and Woltran, 2010)

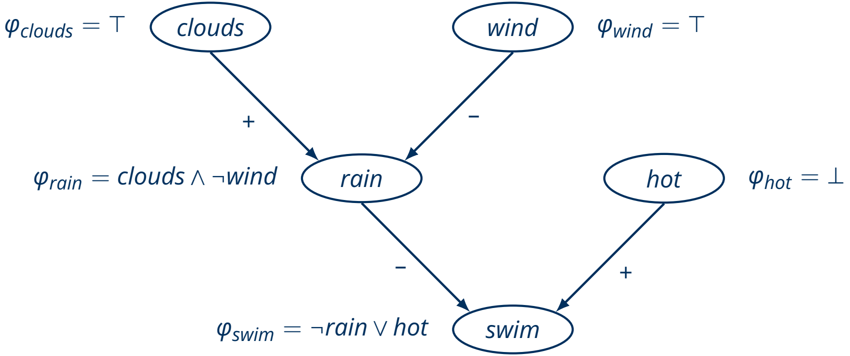
An **abstract dialectical framework** (ADF) is a triple $D = (S, L, C)$ with

- a finite set S of statements (arguments),
- a set $L \subseteq S \times S$ of links, $(par(s) = \{r \in S \mid (r, s) \in L\})$
- a family $C = \{C_s\}_{s \in S}$ of acceptance conditions $C_s \colon 2^{par(s)} \to \{\mathbf{t}, \mathbf{f}\}$.

A set $M \subseteq S$ is a **model** for D iff
for all $s \in S$, we have $s \in M$ iff $C_s(M \cap par(s)) = \mathbf{t}$.

- For $M \subseteq par(s)$, $C_s(M) = \mathbf{t}$ expresses that s can be accepted if all statements in M are accepted (and all statements in $par(s) \setminus M$ are not accepted).
- An acceptance condition C_s is typically represented by a propositional formula φ_s over $par(s)$, with all $M \subseteq par(s)$ satisfying $C_s(M) = \mathbf{t}$ iff $M \models \varphi_s$.

Abstract Dialectical Frameworks: Example



Single model: $M = \{clouds, wind, swim\}$

Bipolar: All links are attacking (–) or supporting (+).

Link (r, s) is **attacking** iff for all $M \subseteq par(s)$, if $C_s(M) = \mathbf{f}$ then $C_s(M \cup \{r\}) = \mathbf{f}$;
link (r, s) is **supporting** iff for all $M \subseteq par(s)$, if $C_s(M) = \mathbf{t}$ then $C_s(M \cup \{r\}) = \mathbf{t}$.

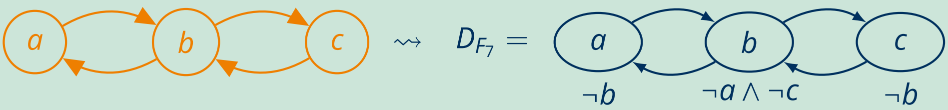
From AFs to ADFs: Translation

Definition

Let $F = (A, R)$ be an argumentation framework. Define its corresponding ADF $D_F = (S, L, C)$ by setting $S = A$, $L = R$, and for every $s \in S$:

$$C_s \colon 2^{par(s)} \to \{\mathbf{t}, \mathbf{f}\}, \qquad M \mapsto \begin{cases} \mathbf{t} & \text{if } M = \emptyset, \\ \mathbf{f} & \text{otherwise.} \end{cases}$$

Example

$F_7 =$ 

Proposition

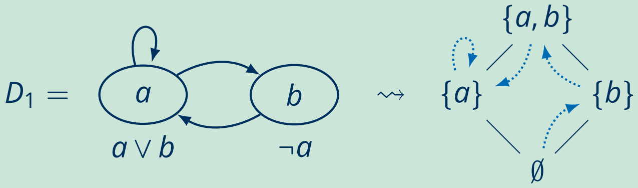
For any $F = (A, R)$: $M \subseteq A$ is stable for F iff M is a model of D_F.

ADFs: Operator

Definition

Let $D = (S, L, C)$ be an abstract dialectical framework. A consequence operator is given by $G_D\colon 2^S \to 2^S$ with $M \mapsto \{s \in S \mid C_s(M \cap par(s)) = \mathbf{t}\}$.

Example



Proposition

Let $D = (S, L, C)$ be an abstract dialectical framework. For any $M \subseteq S$:

$$G_D(M) = M \text{ if and only if } M \text{ is a model for } D.$$

Quiz: `https://tud.link/djuy7e`

Recall: $G_D(M) = \{s \in S \mid C_s(M \cap par(s)) = \mathbf{t}\}$

Quiz

Consider the following ADF: . . .

ADFs: Approximator

Main Benefit of Approximation Fixpoint Theory

To obtain semantics for ADFs, we only need to define an approximator.

Definition

Let $D = (S, L, C)$ be an ADF. Define approximator $\mathcal{G}_D : (2^S \times 2^S) \to (2^S \times 2^S)$ via

$$(X, Y) \mapsto \left(\bigcap_{X \subseteq Z \subseteq Y} G_D(Z), \bigcup_{X \subseteq Z \subseteq Y} G_D(Z) \right)$$

- $\mathcal{G}_D$ approximates G_D, as $\mathcal{G}_D(X, X) = (G_D(X), G_D(X))$.
- $\mathcal{G}_D$ is $\leq_i$-monotone: $(X_1, Y_1) \leq_i (X_2, Y_2)$ implies $X_1 \subseteq X_2 \subseteq Z \subseteq Y_2 \subseteq Y_1$.
- This construction is known as ultimate approximation (Denecker, Marek, and Truszczyński, 2004).

From AFs to ADFs: Defining Semantics

Definition

Let $D = (S, L, C)$ be an ADF. A pair (X, Y) is ...

- **admissible** iff $(X, Y) \leq_i \mathcal{G}_D(X, Y)$;
- **complete** iff $\mathcal{G}_D(X, Y) = (X, Y)$;
- **preferred** iff (X, Y) is $\leq_i$-maximal w.r.t. $\mathcal{G}_D(X, Y) = (X, Y)$;
- **grounded** iff $(X, Y) = \mathrm{lfp}(\mathcal{G}_D)$.

Theorem

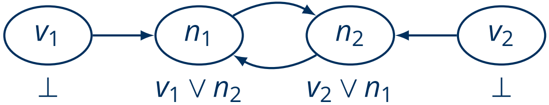
Let $F = (A, R)$ be an AF and D_F its corresponding ADF, and $X \subseteq Y \subseteq A$.

- (X, Y) is admissible for F iff (X, Y) is admissible for D_F;
- (X, Y) is complete for F iff (X, Y) is complete for D_F;
- (X, Y) is grounded for F iff (X, Y) is grounded for D_F;
- (X, X) is stable for F iff X is a model of D_F.

Towards Stable Model Semantics

Consider this simplified model of a fuel system for an aircraft:
Node n_1 is pressurised by valve v_1 or node n_2; symmetrically for node n_2.

We can model the behaviour of this system as an ADF as follows:

$D_2 =$ 

What are the models of D_2?

There are two models.

Is this desired?

Stable Operators

Stable Operator: Intuition

The Gelfond-Lifschitz Reduct of a logic program P . . .

- . . . starts out with a two-valued interpretation $M \subseteq S$;
- . . . removes all rules requiring some $a \in M$ to be false;
- . . . assumes all $a \notin M$ to be false in the remaining rules.

- To obtain ADF reduct D^M, assume all and only atoms $a \notin M$ to be false.
- Using D^M, try to constructively prove all and only atoms $a \in M$ to be true.
- Try to ensure that G_{D^M} is a $\subseteq$-monotone operator on $(2^S, \subseteq)$.

Expressing the Reduct via an Operator

- For pair (X, Y), an $a \in S$ is true iff $a \in X$; atom a is false iff $a \notin Y$.
- Use $\mathcal{G}_D{}'$ to reconstruct what is true, fixing the upper bound to M:

$$\mathcal{G}_D{}'(\cdot, M) \colon 2^A \to 2^A, \quad X \mapsto \mathcal{G}_D{}'(X, M)$$

Stable Operator: Preparation

Proposition

Let $(L, \leqslant)$ be a complete lattice and $\mathcal{A}$ be an approximator on $(L^2, \leq_i)$.
For every pair $(x, y) \in L^2$, the following operators are $\leqslant$-monotone:

$$\mathcal{A}'(\cdot, y) \colon L \to L, \quad z \mapsto \mathcal{A}'(z, y) \qquad \text{and} \qquad \mathcal{A}''(x, \cdot) \colon L \to L, \quad z \mapsto \mathcal{A}''(x, z)$$

Proof.

1. Let $x_1 \leqslant x_2$ and $y \in L$.
   Then $(x_1, y) \leq_i (x_2, y)$ and $\mathcal{A}(x_1, y) \leq_i \mathcal{A}(x_2, y)$, thus $\mathcal{A}'(x_1, y) \leqslant \mathcal{A}'(x_2, y)$.
2. Let $x \in L$ and $y_1 \leqslant y_2$.
   Then $(x, y_2) \leq_i (x, y_1)$ and $\mathcal{A}(x, y_2) \leq_i \mathcal{A}(x, y_1)$, thus $\mathcal{A}''(x, y_1) \leqslant \mathcal{A}''(x, y_2)$. □

- $\mathcal{A}'(\cdot, y)$ has a $\leqslant$-least fixpoint, denoted $\mathrm{lfp}(\mathcal{A}'(\cdot, y))$;
- $\mathcal{A}''(x, \cdot)$ has a $\leqslant$-least fixpoint, denoted $\mathrm{lfp}(\mathcal{A}''(x, \cdot))$.

Stable Operator: Definition

Definition

Let $(L, \leqslant)$ be a complete lattice and $\mathcal{A}$ be an approximator on $(L^2, \leq_i)$.
The **stable approximator** for $\mathcal{A}$ is given by $\mathcal{SA}\colon L^2 \to L^2$ with

$$\begin{aligned} \mathcal{SA}'\colon L^2 &\to L, & (x,y) &\mapsto \mathrm{lfp}(\mathcal{A}'(\cdot,y)) \\ \mathcal{SA}''\colon L^2 &\to L, & (x,y) &\mapsto \mathrm{lfp}(\mathcal{A}''(x,\cdot)) \end{aligned}$$

- $\mathcal{SA}'$: improve lower bound for all fixpoints of O at or below upper bound;
- $\mathcal{SA}''$: obtain tightmost new upper bound (eliminate non-minimal fixpoints).

Proposition

Let (x,y) be a postfixpoint of approximator $\mathcal{A}$. Then

$$a \in [\bot, y] \text{ implies } \mathcal{A}'(a,y) \in [\bot, y] \quad \text{and} \quad b \in [x, \top] \text{ implies } \mathcal{A}''(x,b) \in [x, \top].$$

In particular, $\mathrm{lfp}(\mathcal{A}'(\cdot,y)) \leqslant y$ and $x \leqslant \mathrm{lfp}(\mathcal{A}''(x,\cdot))$.

Stable Operator: Observations

Theorem

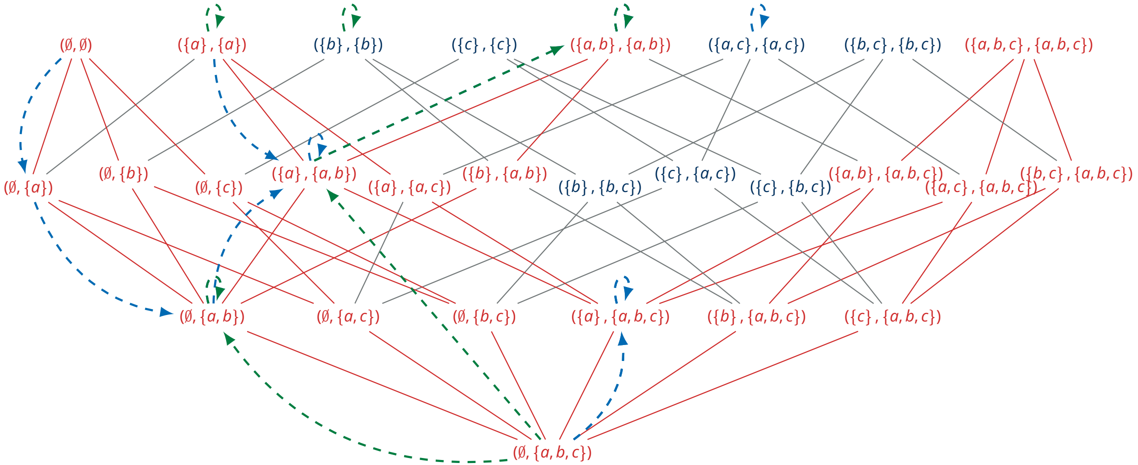
Let $(L, \leqslant)$ be a complete lattice and $\mathcal{A}$ be an approximator on $(L^2, \leq_i)$.

1. $\mathcal{SA}$ is $\leq_i$-monotone.
2. If (x, y) is a consistent postfixpoint of $\mathcal{A}$, then $\mathcal{SA}(x, y)$ is consistent.

Proof.

1. Let $(u, v) \leq_i (x, y)$. Now $y \leqslant v$ implies $\mathcal{A}'(z, v) \leqslant \mathcal{A}'(z, y)$ for all $z \in L$ since $\mathcal{A}$ is $\leq_i$-monotone. In particular, for $z^* = \text{lfp}(\mathcal{A}'(\cdot, y))$, $\mathcal{A}'(z^*, v) \leqslant \mathcal{A}'(z^*, y) = z^*$ whence z^* is a prefixpoint of $\mathcal{A}'(\cdot, v)$. Thus $\text{lfp}(\mathcal{A}'(\cdot, v)) \leqslant z^* = \text{lfp}(\mathcal{A}'(\cdot, y))$. In combination, $\mathcal{SA}'(u, v) = \text{lfp}(\mathcal{A}'(\cdot, v)) \leqslant \text{lfp}(\mathcal{A}'(\cdot, y)) = \mathcal{SA}'(x, y)$. $\mathcal{SA}''$: dual.
2. Let $x \leqslant y$ with $(x, y) \leq_i \mathcal{A}(x, y)$. For every $z \in L$ with $x \leqslant z \leqslant y$, we have $\mathcal{SA}'(x, y) \leqslant \mathcal{SA}'(z, z) = \text{lfp}(\mathcal{A}'(\cdot, z)) \leqslant z \leqslant \text{lfp}(\mathcal{A}''(z, \cdot)) = \mathcal{SA}''(z, z) \leqslant \mathcal{SA}''(x, y)$. □

Stable Operator $\mathcal{SG}_D$: Example



$D_3: \quad \varphi_a = \neg b, \quad \varphi_b = a \wedge \neg c, \quad \varphi_c = c$ [illegible]

Stable Semantics: Definition via Operators

Definition

Let $(L, \leqslant)$ be a complete lattice, $O\colon L \to L$ be an operator.
Let $\mathcal{A}\colon L^2 \to L^2$ be an approximator of O in $(L^2, \leq_i)$. A pair $(x, y) \in L^2$ is

- a **two-valued stable model of** $\mathcal{A}$ iff $x = y$ and $\mathcal{S}\mathcal{A}(x, y) = (x, y)$;
- a **three-valued stable model of** $\mathcal{A}$ iff $x \leqslant y$ and $\mathcal{S}\mathcal{A}(x, y) = (x, y)$;
- the **well-founded model of** $\mathcal{A}$ iff it is the least fixpoint of $\mathcal{S}\mathcal{A}$.

The names are inspired by notions from logic programming.

Theorem

1. $\text{lfp}(\mathcal{A}) \leq_i \text{lfp}(\mathcal{S}\mathcal{A})$;
2. $\mathcal{S}\mathcal{A}(x, y) = (x, y)$ implies $\mathcal{A}(x, y) = (x, y)$;
3. if $\mathcal{S}\mathcal{A}(x, x) = (x, x)$ then x is a $\leqslant$-minimal fixpoint of O;

Reprise: How to Find an Approximator?

Definition

Let $O: L \to L$ be an operator in a complete lattice $(L, \leqslant)$.
Define the **ultimate approximator of** O as follows:

$$\mathcal{X}_O : L^2 \to L^2, \qquad (x, y) \mapsto \left(\bigwedge \{ O(z) \mid x \leqslant z \leqslant y \}, \bigvee \{ O(z) \mid x \leqslant z \leqslant y \} \right)$$

Intuition: Consider glb and lub of applying O pointwise to given interval.

Theorem

For every approximator $\mathcal{A}$ of O and consistent pair $(x, y) \in L^2$, we find

$$\mathcal{A}(x, y) \leq_i \mathcal{X}_O(x, y)$$

Ultimate approximator is most precise approximator possible.

Used e.g. for standard semantics of ADFs (Brewka et al., 2013).

Conclusion

Conclusion

Summary

- Operators in complete lattices can be used to define semantics of KR formalisms.
- Approximation fixpoint theory provides a general account of operator-based semantics.
- Stable approximator reconstructs well-founded and stable model semantics of logic programming.
- To define semantics for new formalisms, only an approximator needs to be defined, AFT does the rest.
- With ultimate approximation, only a consequence operator needs to be defined.

Outlook

What else can Approximation Fixpoint Theory do for Argumentation?

Open Topics

AFT could be used to analyse/define/compare semantics of . . .

- . . . argumentation frameworks with set attacks;
- . . . argumentation frameworks with supports/necessities;
- . . . gradual and probabilistic argumentation;
- . . . assumption-based argumentation;
- . . . the formalism you are interested in?

References

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