

Sebastian Rudolph

(based on slides by Bernardo Cuenca Grau, Ian Horrocks, Przemysław Wałęga)

Faculty of Computer Science, Institute of Artificial Intelligence, Computational Logic Group

Horn Logics and Datalog

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Recap

- Looked at using Propositional Logic (PL) for representing knowledge
- effectively implementable (SAT solvers), but lacks expressiveness

Want a KR language that can ...

1. Represent **sets of objects**
2. Represent **relationships between objects**
3. Write statements that are true for **some** or **all** objects satisfying certain conditions
4. Express everything we can express in propositional logic (*and, or, implies, not, ...*)

↪ First-order logic (FOL)

However: FOL satisfiability is undecidable

↪ cannot hope for implementations

Propositional Horn Fragment

PL Horn Fragment: only allows the following kinds of formulas ($n > 0$):

$$\begin{array}{ll} P_1 \wedge \dots \wedge P_n \rightarrow Q & \text{rules} \\ P & \text{facts} \end{array}$$

With P_i, Q being atoms, and where Q can be $\perp$.

Horn Clauses: Clauses with at most one positive literal.

$$\neg P_1 \vee \dots \vee \neg P_n \vee Q$$

(Fact) Entailment. Input: set $\mathcal{H}$ of Horn formulas and atom P
Answer: **true** if every model of $\mathcal{H}$ is also a model of P
and **false** otherwise.

PL Horn entailment is solvable in **polynomial time**.

Lifting PL Horn to FOL Horn

First-Order Horn Clauses: Clauses with at most one positive literal

But now, atoms can contain variables, constants, and function symbols.

Implicit assumption: all variables $\forall$ -quantified, but quantifiers not written

Some examples of First-Order Horn clauses:

$$\begin{aligned} & \neg \text{JuvArthritis}(x) \vee \text{Arthritis}(x) \\ & \neg \text{Arthritis}(x) \vee \neg \text{JuvDisease}(x) \vee \text{JuvArthritis}(x) \\ & \neg \text{Child}(x) \vee \neg \text{Adult}(x) \\ & \neg \text{Affects}(x, y) \vee \text{Person}(y) \\ & \neg \text{JuvDisease}(x) \vee \text{Affects}(x, f(x)) \\ & \text{JuvDisease}(\text{JRA}) \end{aligned}$$

Horn Logics

Horn Formulas: FOL sentences whose (Skolem) CNF yield Horn clauses.

Horn Logics: Syntactic FOL fragments allowing only Horn Formulas.

Some examples of Horn formulas:

$$\forall x.(\text{Arthritis}(x) \wedge \text{JuvDisease}(x) \rightarrow \text{JuvArthritis}(x))$$

$$\forall x.(\text{Child}(x) \wedge \text{Adult}(x) \rightarrow \perp)$$

$$\forall x.(\forall y.(\text{Affects}(x,y) \rightarrow \text{Person}(y)))$$

$$\forall x.(\text{JuvDisease}(x) \rightarrow \exists y.(\text{Affects}(x,y) \wedge \text{Child}(y)))$$

$$\forall x.(\forall y.(\forall z.(\text{fatherOf}(x,y) \wedge \text{brotherOf}(x,z) \rightarrow \text{uncleOf}(z,y))))$$
$$\text{JuvDisease(JRA)}$$

Expressivity

We **cannot** express “disjunctive information”:

- Covering statements:

$$\forall x.(\textit{Person}(x) \rightarrow \textit{Adult}(x) \vee \textit{Child}(x) \vee \textit{Teenager}(x))$$

- Negation on the left of implication

$$\forall x.(\textit{Person}(x) \wedge \neg \textit{Woman}(x) \rightarrow \textit{Man}(x))$$

As well as many others ...

Note, however, that some formulas apparently “disjunctive” are Horn:

$$\forall x.(\textit{Adult}(x) \vee \textit{Child}(x) \vee \textit{Teenager}(x) \rightarrow \textit{Person}(x))$$

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Note, however, that some formulas apparently “disjunctive” are Horn:

$$\forall x.(\textit{Adult}(x) \vee \textit{Child}(x) \vee \textit{Teenager}(x) \rightarrow \textit{Person}(x))$$

...because they can be rewritten into formulas that are obviously Horn:

$$\forall x.(\textit{Adult}(x) \rightarrow \textit{Person}(x))$$

$$\forall x.(\textit{Child}(x) \rightarrow \textit{Person}(x))$$

$$\forall x.(\textit{Teenager}(x) \rightarrow \textit{Person}(x))$$

Existential Rules

$\forall \vec{x}. \forall \vec{z}. (\varphi(\vec{x}, \vec{z}) \rightarrow \exists \vec{y}. \psi(\vec{x}, \vec{y}))$ *Existential Rule*

$\forall \vec{x}. \forall \vec{z}. (\varphi(\vec{x}, \vec{z}) \rightarrow \perp)$ $\perp$ -Rule

$P(\vec{a})$ *Fact*

$\varphi(\vec{x}, \vec{z})$: conjunction of function-free atoms with vars $\vec{x} \cup \vec{z}$.

$\psi(\vec{x}, \vec{y})$: conjunction of function-free atoms with vars $\vec{x} \cup \vec{y}$.

$\forall x. (\text{Arthritis}(x) \wedge \text{JuvDisease}(x) \rightarrow \text{JuvArthritis}(x))$ *Rule*

$\forall x. (\text{Child}(x) \wedge \text{Adult}(x) \rightarrow \perp)$ $\perp$ -Rule

$\forall x. (\text{JuvDisease}(x) \rightarrow \exists y. (\text{Affects}(x, y) \wedge \text{Child}(y)))$ *Rule*

$\text{JuvDisease}(\text{JRA})$ *Fact*

Examples of Horn formulas outside this logic:

$\forall x. (\text{Adult}(x) \vee \text{Child}(x) \vee \text{Teenager}(x) \rightarrow \text{Person}(x))$

Reasoning with Existential Rules

Fact Entailment:

Input: pair $\langle \mathcal{R}, \mathcal{F} \rangle$ of rules and facts and a fact P .

Answer is **true** iff $\langle \mathcal{R}, \mathcal{F} \cup \{\neg P\} \rangle$ is unsatisfiable.

Resolution (inferencing calculus for FOL) can be optimised for Horn clauses.

General strategy: allow only certain kinds of resolution inferences:

- Need to show **completeness**
Unsatisfiability must imply that the empty clause is derivable.
- **No** need to show soundness
Still just resolution, which is sound.

Recall FOL Resolution Rule

$$\frac{\alpha \vee \varphi \quad \neg\beta \vee \psi}{(\varphi \vee \psi)MGU(\alpha, \beta)} \quad \begin{array}{l} \alpha, \beta \text{ are atoms} \\ MGU(\alpha, \beta) \text{ is Most General Unifier of } \alpha \text{ and } \beta \end{array}$$

Examples:

$$\frac{(\neg \text{ArthritisPat}(x) \vee \text{Affects}(f(x), x)) \quad \text{ArthritisPat}(g(a))}{\text{Affects}(f(g(a)), g(a))} \quad \{x \mapsto g(a)\}$$

$$\frac{\text{Affects}(x, \text{John}) \quad \neg \text{Affects}(\text{JRA}, y)}{\square} \quad \{x \mapsto \text{JRA}, y \mapsto \text{John}\}$$

$$\frac{\text{JuvDisease}(h(g(f(x), a))) \quad \neg \text{JuvDisease}(h(g(y, y)))}{\text{Rule not applicable}}$$

Recall FOL Factoring Rule

$$\frac{\gamma \vee \delta \vee \psi}{(\gamma \vee \psi)MGU(\gamma, \delta)} \quad \gamma, \delta \text{ literals, same sign}$$

Examples:

$$\frac{\text{ArthritisPat}(x) \vee \text{Affects}(f(x), x) \vee \text{ArthritisPat}(g(a))}{\text{Affects}(f(g(a)), g(a)) \vee \text{ArthritisPat}(g(a))} \quad \{x \mapsto g(a)\}$$

$$\frac{\text{Affects}(x, \text{John}) \vee \text{Affects}(\text{JRA}, y)}{\text{Affects}(\text{JRA}, \text{John})} \quad \{x \mapsto \text{JRA}, y \mapsto \text{John}\}$$

$$\frac{\neg \text{JuvDisease}(h(g(f(x), a))) \vee \neg \text{JuvDisease}(h(g(y, z)))}{\neg \text{JuvDisease}(h(g(f(x), a)))} \quad \{y \mapsto f(x), z \mapsto a\}$$

Recall FOL Resolution Procedure

```
1: procedure Sat( $\mathcal{S}$ )
2:   repeat
3:     for all clauses  $C_1, C_2$  in  $\mathcal{S}$  do
4:        $\mathcal{S} := \mathcal{S} \cup \text{resolve}(C_1, C_2)$ 
5:     end for
6:   until No new clause can be added to  $\mathcal{S}$  or  $\square \in \mathcal{S}$ 
7:   If  $\square \in \mathcal{S}$  return false
8:   return true
9: end procedure
```

Function $\text{resolve}(C_1, C_2)$ applies FO resolution in all possible ways, and then applies factoring in all possible ways.

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Function $\text{resolve}(C_1, C_2)$ applies FO resolution in all possible ways, and then applies factoring in all possible ways.

Wait ... in all possible ways?

Resolution with Free Selection

Resolution with free selection: a complete strategy

- Calculus parameterised by a Selection Function S
- S assigns to each Horn clause C a non-empty subset of its literals:
 - $S(C)$ contains the single positive literal, OR
 - $S(C)$ contains a subset of negative literals
- Restrict resolution such that we only resolve on selected literals

We are free to design the selection function ourselves:

If we satisfy the basic constraints, completeness is guaranteed.

Resolution with Free Selection

A reasonable selection function:

- Select the set of all negative literals in each clause
- If there is no negative literal, select the (unique) positive literal

As usual, we prepare for resolution (Skolemisation, CNF, clause form)

$$\begin{aligned} A(x) \rightarrow \exists y. R(x, y) \wedge B(y) &\rightsquigarrow \neg A(x) \vee R(x, f(x)) \\ &\quad \neg A(x) \vee B(f(x)) \end{aligned}$$

$$B(x) \rightarrow C(x) \rightsquigarrow \neg B(x) \vee C(x)$$

$$R(x, y) \wedge C(y) \rightarrow D(x) \rightsquigarrow \neg R(x, y) \vee \neg C(y) \vee D(x)$$

$$A(a) \rightsquigarrow A(a)$$

We now want to see whether $D(a)$ follows ...

Resolution with Free Selection

$$\neg A(x) \vee R(x, f(x)) \quad (1)$$

$$\neg A(x) \vee B(f(x)) \quad (2)$$

$$\neg B(x) \vee C(x) \quad (3)$$

$$\neg R(x, y) \vee \neg C(y) \vee D(x) \quad (4)$$

$$A(a) \quad (5)$$

$$\neg D(a) \quad (6)$$

With this selection, we don't need to resolve (1) and (4)

Observation: This strategy amounts to **Unit Resolution**

One of the premises of resolution must be a unit clause!

Resolution with Free Selection

$$\neg A(x) \vee R(x, f(x)) \quad (1)$$

$$\neg A(x) \vee B(f(x)) \quad (2)$$

$$\neg B(x) \vee C(x) \quad (3)$$

$$\neg R(x, y) \vee \neg C(y) \vee D(x) \quad (4)$$

$$A(a) \quad (5)$$

$$\neg D(a) \quad (6)$$

$$R(a, f(a)) \quad (1) + (5) \quad (7)$$

$$B(f(a)) \quad (2) + (5) \quad (8)$$

$$C(f(a)) \quad (8) + (3) \quad (9)$$

$$\neg C(f(a)) \vee D(a) \quad (7) + (4) \quad (10)$$

$$D(a) \quad (9) + (10) \quad (11)$$

$$\square \quad (11) + (6) \quad (12)$$

Resolution with Free Selection

We still have termination problems ...

$$A(x) \rightarrow \exists y. R(x, y) \wedge A(y) \rightsquigarrow \neg A(x) \vee R(x, f(x))$$

$$\neg A(x) \vee A(f(x))$$

$$A(a) \rightsquigarrow A(a)$$

$$\neg A(x) \vee R(x, f(x))$$

$$\neg A(x) \vee A(f(x))$$

$$A(a)$$

$$R(a, f(a))$$

$$A(f(a))$$

$$R(a, f(f(a)))$$

$$A(f(f(a)))$$

...

Resolution with Free Selection

We still have termination problems ...

$$A(x) \rightarrow \exists y. R(x, y) \wedge A(y) \rightsquigarrow \neg A(x) \vee R(x, f(x))$$

$$\neg A(x) \vee A(f(x))$$

$$A(a) \rightsquigarrow A(a)$$

$$\neg A(x) \vee R(x, f(x))$$

$$\neg A(x) \vee A(f(x))$$

$$A(a)$$

$$R(a, f(a))$$

$$A(f(a))$$

$$R(a, f(f(a)))$$

$$A(f(f(a)))$$

...

Theorem

Unsatisfiability and fact entailment over existential rules are **undecidable** (semi-decidable).

That is, as difficult as checking unsatisfiability in FOL.

Datalog

To achieve decidability we need to sacrifice (even more) expressivity.

Datalog: The quintessential rule-based KR language

$$\forall \vec{x}. \forall \vec{z}. (\varphi(\vec{x}, \vec{z}) \rightarrow \psi(\vec{x})) \quad \textit{Rule}$$

$$\forall \vec{x}. \forall \vec{z}. (\varphi(\vec{x}, \vec{z}) \rightarrow \perp) \quad \perp\textit{-Rule}$$

$$P(\vec{a}) \quad \textit{Fact}$$

$\varphi(\vec{x}, \vec{z})$ and $\psi(\vec{x})$: conjunctions of function-free atoms

We can still express

$$\forall x. (\forall y. (\forall z. (\textit{fatherOf}(x, y) \wedge \textit{brotherOf}(x, z) \rightarrow \textit{uncleOf}(z, y))))$$

$$\forall x. (\forall y. (\textit{Affects}(x, y) \rightarrow \textit{Person}(y)))$$

But, we can no longer express

$$\forall x. (\textit{JuvDisease}(x) \rightarrow \exists y. (\textit{Affects}(x, y) \wedge \textit{Child}(y))))$$

Decidability of Entailment

Theorem

Fact entailment in Datalog is **decidable**.

Decidability follows directly from **Herbrand's theorem**

- Our problem reduces to unsatisfiability of $\mathcal{S} = \mathcal{R} \cup \mathcal{F} \cup \{\neg P\}$
- $\mathcal{R} \cup \mathcal{F} \cup \{\neg P\}$ is a set of clauses **without function symbols** so Herbrand universe **finite**
- Gilmore's FOL unsatisfiability algorithm terminates.

Decidability of Entailment

Our **algorithm** is an adaptation of Gilmore's when Herbrand universe is finite

```
1: procedure Datalog-Gil( $\langle \mathcal{R}, \mathcal{F} \rangle, P$ )  
2:   Compute Herbrand Universe  $U$   
3:    $\mathcal{R}' := \text{ground}(\mathcal{R}, U)$   
4:   return Horn-Prop( $\langle \mathcal{R}', \mathcal{F} \rangle, P$ )  
5: end procedure
```

Subroutine Horn-Prop solves entailment problem for Horn PL

Complexity Considerations

$$\forall x.(\forall y.(\forall z.(fatherOf(x,y) \wedge brotherOf(x,z) \rightarrow uncleOf(z,y))))$$

fatherOf(John, Mary)
brotherOf(John, Peter)

Herbrand universe: constants in $\langle \mathcal{R}, \mathcal{F} \rangle$

$$U = \{John, Mary, Peter\}$$

Grounding leads to exponential size set of propositional clauses

fatherOf(John, John) ∧ brotherOf(John, John) → uncleOf(John, John)
fatherOf(John, Mary) ∧ brotherOf(John, Mary) → uncleOf(Mary, Mary)
fatherOf(John, Peter) ∧ brotherOf(John, Peter) → uncleOf(Peter, Peter)
and so on

Size of the grounding grows as $\mathcal{O}(c^v)$, where

- c is the max. number of constants in facts.
- v is the max. number of variables in rules.

Complexity Considerations

Propositional entailment in Horn PL can be decided in **polynomial time**.
Overall process takes **exponential time** (because of grounding).

Theorem

Fact entailment in Datalog is **decidable in ExpTime**.

In fact, the problem is also **ExpTime-hard** (beyond this course).

⇒ Naive grounding algorithm is worst-case optimal.

Practical Considerations

From a **practical point of view**, we can do much better:

- Avoid computing the grounding upfront
- Instantiate variables to constants “on the fly”

We develop two **resolution-based** strategies:

1. **Forward chaining:**

Start from facts and instantiate rules to derive new facts whenever possible until goal is derived

2. **Backward chaining:**

Start from goal and proceed “backwards” to derive the empty clause

Both strategies can be seen as **Resolution with Free Selection**.

Forward Chaining (Example)

Start from facts and instantiate rules to derive new facts whenever possible until goal (or $\square$) is derived.

Example: Check if *Child(John)* follow from the following:

$$\forall x.(\text{JuvArthritis}(x) \rightarrow \text{JuvDisease}(x)) \quad (13)$$

$$\forall x.(\forall y.(\text{JuvDisease}(x) \wedge \text{Affects}(x, y) \rightarrow \text{Child}(y))) \quad (14)$$

$$\text{JuvArthritis}(\text{JRA}) \quad (15)$$

$$\text{Affects}(\text{JRA}, \text{John}) \quad (16)$$

Match existing facts to rule bodies to derive new facts.

From Fact (15) and Rule (13) we obtain the following by unit resolution

$$\text{JuvDisease}(\text{JRA}) \quad (17)$$

From Facts (17) and (16) and Rule (14), derive goal and stop.

$$\text{Child}(\text{John})$$

Forward Chaining and Resolution

$\mathcal{S}_{fw}$: select all negative literals in clauses, and the (unique) positive literal if the clause does not have negative literals.

$$\neg JuvArthritis(x) \vee JuvDisease(x)$$
$$JuvArthritis(JRA)$$

We obtain the following by resolution:

$$JuvDisease(JRA)$$

Forward Chaining and Resolution

$\mathcal{S}_{fw}$: select all negative literals in clauses, and the (unique) positive literal if the clause does not have negative literals.

Deriving a new fact by matching other facts to a rule may require several resolution steps (**Hyperresolution**).

$$\begin{aligned} &\neg JuvDisease(x) \vee \neg Affects(x, y) \vee Child(y) \\ &\quad Affects(JRA, John) \\ &\quad JuvDisease(JRA) \end{aligned}$$

We obtain the following by resolution:

$$\begin{aligned} &\neg JuvDisease(JRA) \vee Child(John) \\ &\quad Child(John) \end{aligned}$$

In forward chaining, we do both steps in one.

Forward Chaining

```
1: procedure Forward( $\langle \mathcal{R}, \mathcal{F} \rangle, P$ )
2:    $\mathcal{F}' := \mathcal{F}$ 
3:   repeat
4:     for each rule  $R = \neg B_1 \vee \neg B_2 \vee \dots, \vee \neg B_n \vee H \in \mathcal{R}$  do
5:       if  $\{D_1, \dots, D_n\} \subseteq \mathcal{F}'$  such that  $B_i$  unifies with  $D_i$  then
6:          $\theta := \text{Unify}(\{B_1 \doteq D_1, \dots, B_n \doteq D_n\})$ 
7:          $\mathcal{F}' := \mathcal{F}' \cup \{H\theta\}$ 
8:       end if
9:     end for
10:    until No new atom can be added to  $\mathcal{F}'$  or  $P \in \mathcal{F}'$  or  $\square \in \mathcal{F}'$ 
11:    if  $P \in \mathcal{F}'$  or  $\square \in \mathcal{F}'$  then
12:      return true
13:    else
14:      return false
15:    end if
16: end procedure
```

Backward Chaining (Example)

Check whether following rules and facts imply *Child(John)*:

$$\forall x.(\text{JuvArthritis}(x) \rightarrow \text{JuvDisease}(x)) \quad (18)$$

$$\forall x.(\forall y.(\text{JuvDisease}(x) \wedge \text{Affects}(x, y) \rightarrow \text{Child}(y))) \quad (19)$$

$$\text{JuvArthritis}(\text{JRA}) \quad (20)$$

$$\text{Affects}(\text{JRA}, \text{John}) \quad (21)$$

Match “goal” *Child(John)* to rule heads and facts to derive new goals.

To prove *Child(John)*, by Rule (19) it is sufficient to show

$$\text{JuvDisease}(x) \quad \text{and} \quad \text{Affects}(x, \text{John})$$

Then, by Fact (21), it would be sufficient to show *JuvDisease(JRA)*.

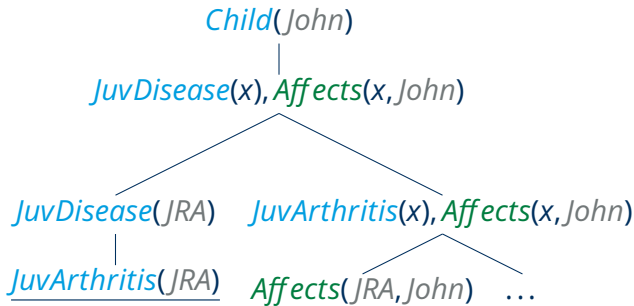
Another possibility is to use Rule (18) and get the following sub-goals

$$\text{JuvArthritis}(x) \quad \text{and} \quad \text{Affects}(x, \text{John})$$

And so on ...

Backward Chaining (Example)

We can represent this kind of backwards reasoning in an AND-OR tree:


$$\forall x. (JuvArthritis(x) \rightarrow JuvDisease(x))$$
$$\forall x. (\forall y. (JuvDisease(x) \wedge Affects(x, y) \rightarrow Child(y)))$$

$JuvArthritis(JRA)$
 $Affects(JRA, John)$

Backward Chaining and Resolution

$\mathcal{S}_{bw}$: select the unique positive literal in clauses, and
all negative literals if the clause does not have positive literals.

Matching the goal to a rule head or a fact corresponds to one resolution step.

$$\frac{\neg \text{JuvDisease}(x) \vee \neg \text{Affects}(x, y) \vee \text{Child}(y) \quad \neg \text{Child}(\text{John})}{\neg \text{JuvDisease}(x) \vee \neg \text{Affects}(x, \text{John})}$$

Termination Issues

Resolution with free selection may not terminate with $\mathcal{S}_{bw}$.

Example: Show that John is a Scientist.

$$\neg \text{worksWith}(x, y) \vee \neg \text{Scientist}(y) \vee \text{Scientist}(x) \quad (22)$$

$$\text{worksWith}(\text{John}, \text{Mary}) \quad (23)$$

$$\neg \text{Scientist}(\text{John}) \quad (24)$$

We start resolving on selected atoms:

$$\neg \text{worksWith}(\text{John}, y) \vee \neg \text{Scientist}(y) \quad (22) + (24) \quad (25)$$

$$\neg \text{worksWith}(\text{John}, y_1) \vee \neg \text{worksWith}(y_1, y_2) \vee \neg \text{Scientist}(y_2) \quad (22) + (25) \quad (26)$$

...

Keep on generating clauses with chains of *worksWith* atoms of **increasing length** (variable proliferation).

Thus, the backward chaining tree can have infinite branches.

Other Considerations

Implementing Forward and Backward chaining efficiently is **non-trivial**:

- Forward chaining: set of deduced facts might get huge
- Backward chaining: recursion may be too deep or search tree too wide.

There are many ways to optimise these algorithms

Semi-naive evaluation, Magic sets, ...

But, this is beyond the scope of this course.

There are many optimised systems that implement forward/backward chaining.

The KR languages we have described are related to:

- **Databases**: **Datalog** query language, and **deductive databases**
- **Logic programming**: **Prolog**