Complexity and Expressive Power of Description Logics with Numerical Constraints

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January 15, 2026

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ScaDS.AI DRESDEN LEIPZIG

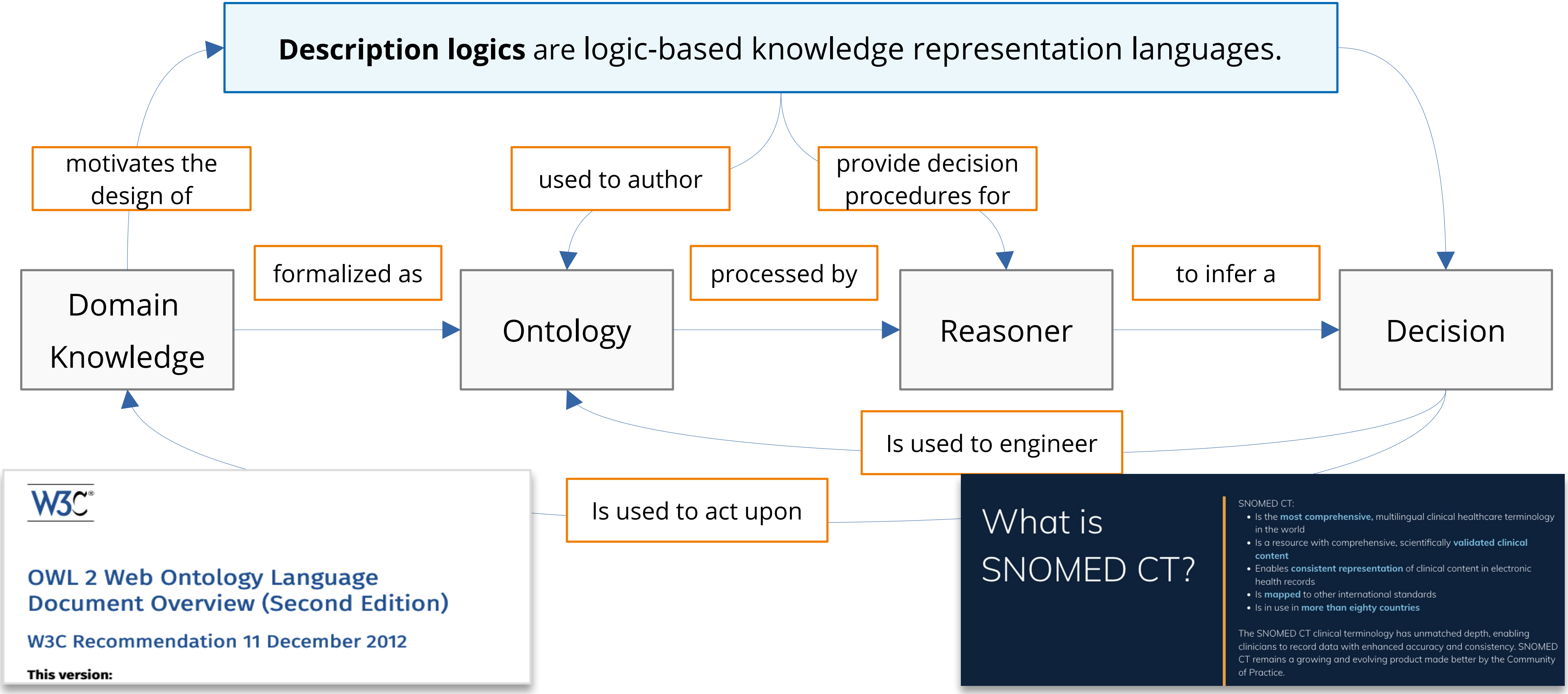
SACHSEN Diese Maßnahme wird gefördert durch die Bundesregierung aufgrund eines Beschlusses des Deutschen Bundestages. Diese Maßnahme wird mitfinanziert durch Steuermittel auf der Grundlage des von den Abgeordneten des Sächsischen Landtags beschlossenen Haushaltes.

Bundesministerium für Forschung, Technologie und Raumfahrt

QuantLA

DFG

Description Logics



Representing Knowledge with Description Logics

Syntax

Natural language statement

"…has a friend that is a musician."

First-order logic (FO) formula

∃y. (friend(x,y) ∧ Musician(y))

≡ *equivalent* (same models)

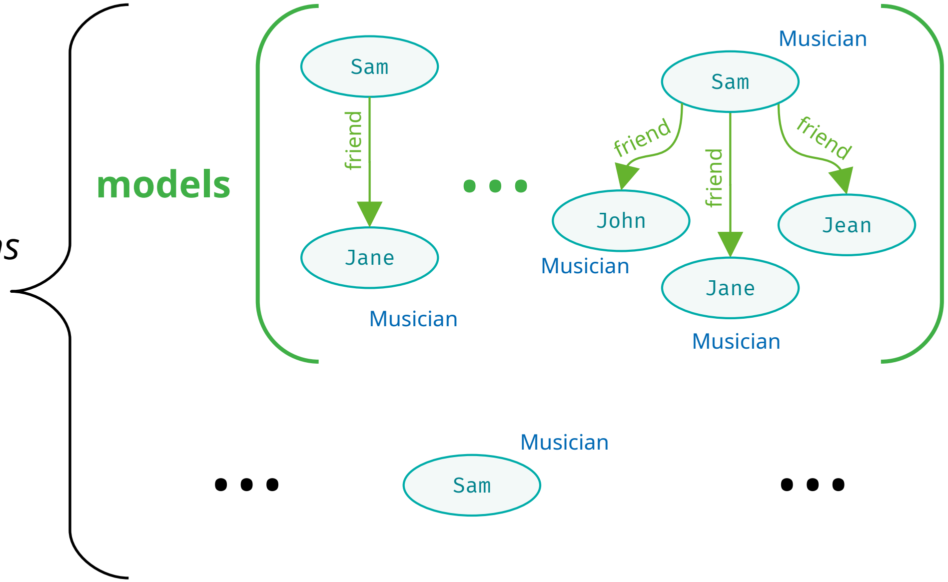
FO-definable (equiv. to FO formula)

Description Logics concept

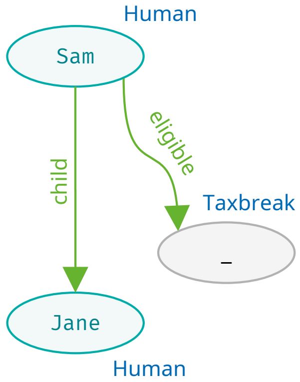
∃friend.Musician

Semantics

interpretations



Reasoning tasks in Description Logics



$\models$

Ontology

$C ::= A \mid \neg C \mid C \sqcap C \mid \exists r.C$ ← ALC

$\text{Human} \sqcap \exists \text{child}.\text{Human} \sqsubseteq \exists \text{eligible}.\text{TaxBreak}$ ← concept inclusions

Human(Sam), child(Sam, Jane) ← concept/role assertions

"If … is human and has a human child, then … is eligible for a tax break."

"Sam is human."

"Jane is child of Sam. "

Ontology consistency: does the input ontology have a model?

Ontology consistency is ExpTime-complete for ALC ontologies.

Adding Numerical Constraints to Description Logics

Cardinality Constraints

"...has at least three friends that are musicians."

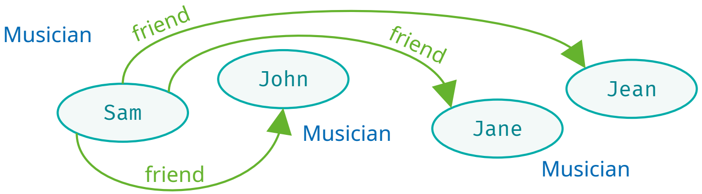
❌ $(\geqslant 3\ \text{friend}.\text{Musician})$

ALC + qualified number restrictions ← ALCQ

"the majority of ...'s friends are musicians."

✅ $\text{succ}(|\text{friend} \cap \text{Musician}| = |\text{friend} \cap \neg\text{Musician}|)$

ALC + (role) successor cardinality constraints ← ALCSCC



Concrete Domain Reasoning

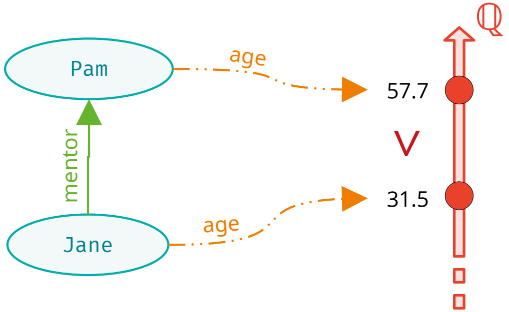
"...has a mentor whose age is greater than theirs."

$\exists y.\ (\text{mentor}(x,y) \wedge \text{age}(x) < \text{age}(y))$

FO with concrete domain restrictions w.r.t. $\mathfrak{D}$ ← FO($\mathfrak{D}$)

$\exists \text{age}, \text{mentor age}. <$

ALC + concrete domain restrictions w.r.t. $\mathfrak{D}$ ← ALC($\mathfrak{D}$)



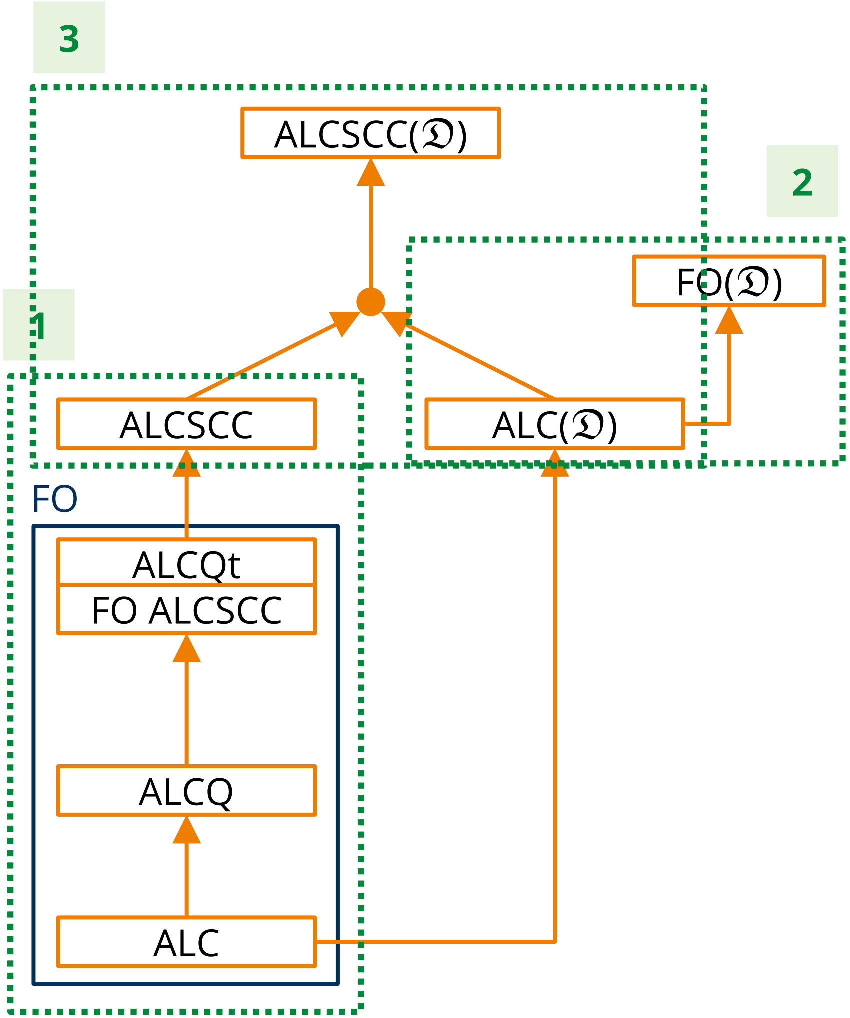
Research Questions

Expressive power

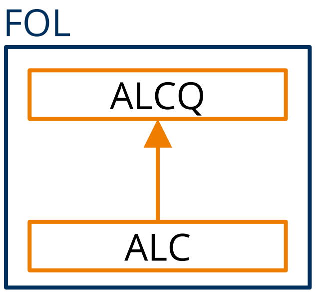
- What are the structural properties shared by the models of concepts with numerical constraints?
- Can we use them to **characterize the expressive power of description logics with numerical constraints**?

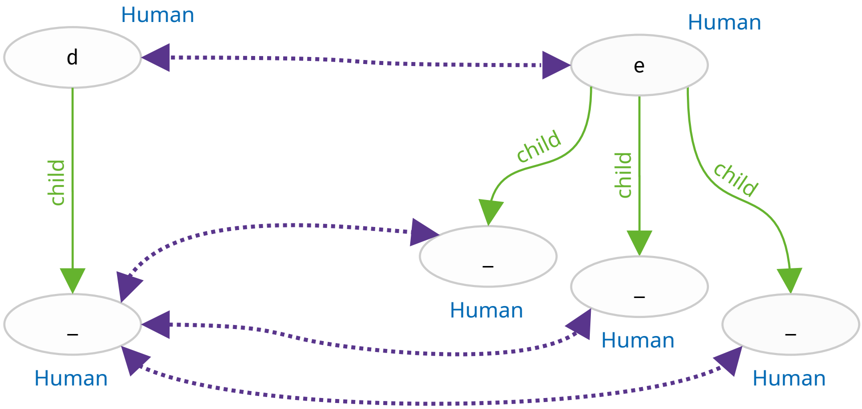
Decidability and complexity

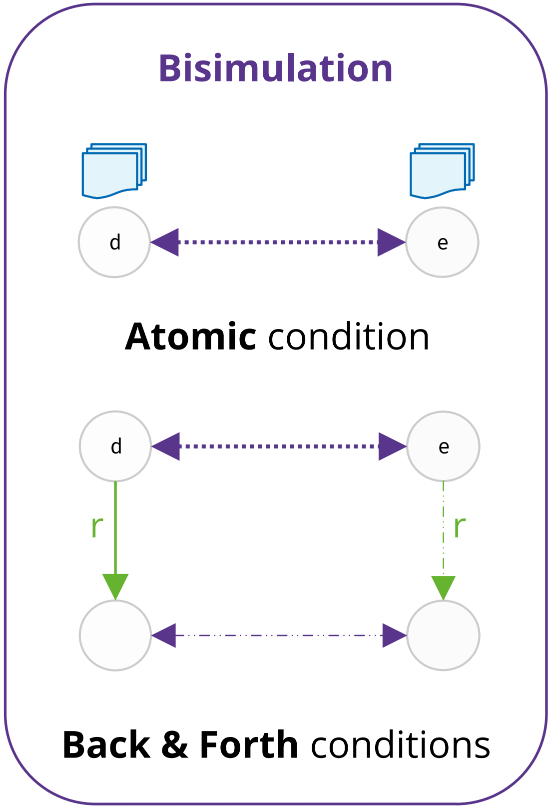
- What conditions on a concrete domain ensure decidability of reasoning? Can we find (tight) complexity bounds?
- Can we **combine cardinality constraints and concrete domain reasoning** and retain decidability and complexity bounds?



Bisimulations, characterization and non-expressivity







Van Benthem/Rosen theorem:
”φ(x) is invariant under bisimulation ↔️ φ(x) is equivalent to ALC concept.”

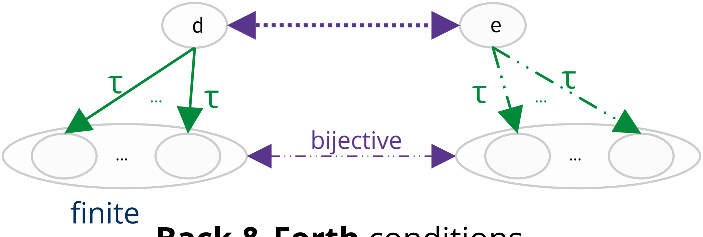
(⩽ 1 child. Human) is *not* invariant under bisimulation

- it cannot be expressed in ALC!

Expressive power: major results (FroCoS '19, '25)

Presburger Bisimulation

Atomic as before



Back & Forth conditions

For all FO(𝔇) formulae φ(x), the following are equivalent*:

1) φ(x) is invariant under bisimulation.
2) φ(x) is equivalent to some ALC() concept.

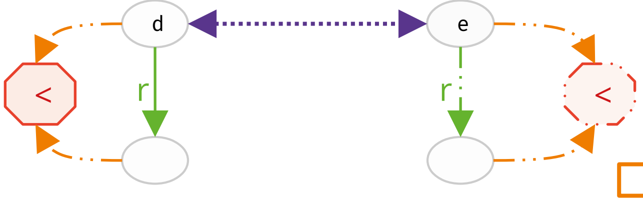
For all FO formulae φ(x), the following are equivalent:

1) φ(x) is equivalent to some ALCSCC concept.
2) φ(x) is invariant under Presburger bisimulation.
3) φ(x) is equivalent to some ALCQt concept.

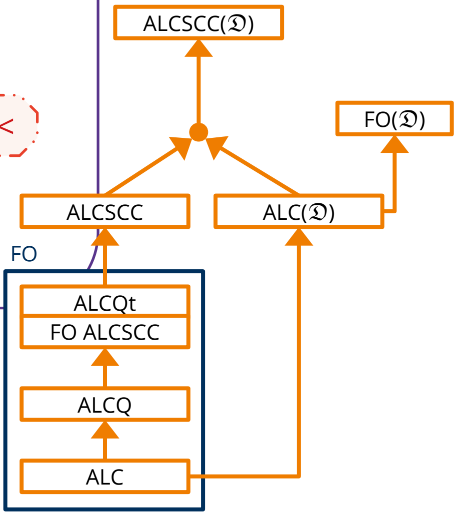
𝔇 Bisimulation

Atomic as before

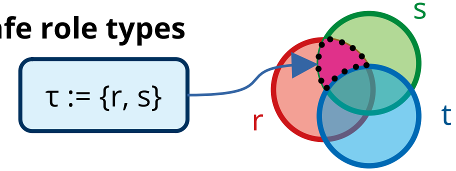
Back & Forth as before



Feature conditions



Safe role types



*the predicates of 𝔇 must be finitely many, and closed under negation.

Cardinality Constraints meet Concrete Domains

ALCSCC + ALC($\mathfrak{D}$) + *feature roles* in successor restrictions

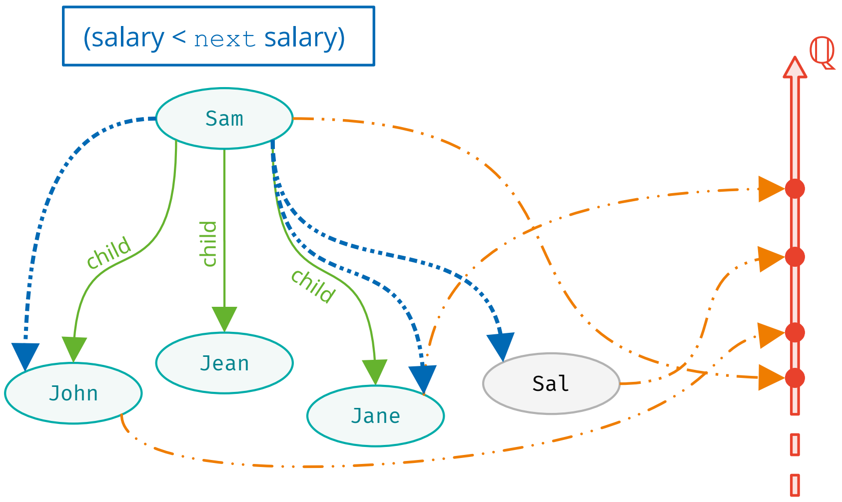
ALCSCC($\mathfrak{D}$)

ALC($\mathfrak{D}$)

ALCSCC

Feature roles

$$\mathrm{succ}(|\mathrm{child} \cap (\mathrm{salary} < \mathtt{next}\ \mathrm{salary})| > |\mathrm{child} \cap (\mathrm{salary} < \mathtt{next}\ \mathrm{salary})^c|)$$



"the salary of… is smaller than that of the majority of their children."

It cannot be expressed without feature roles!

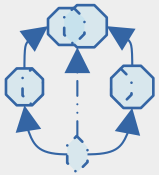
$$(\exists \mathrm{salary}, \mathrm{child}\ \mathrm{salary}.<) \sqsubseteq (\mathrm{succ}(|\mathrm{child}| \leq 0))$$

Inconsistent, due to the interaction of cardinality constraints and concrete domain!

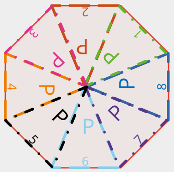
Decidability/Complexity: major results (CADE '25)

ExpTime-ω-admissible concrete domains

- Finitely many predicates
- Constraint satisfaction is decidable in exponential time
- Can express equality
- Homomorphism ω-compact



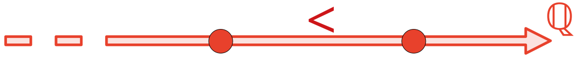
Amalgamation



Jointly Exhaustive, Pairwise Disjoint

CADE '25

If $\mathfrak{D}$ is ExpTime-ω-admissible then consistency of ALCSCC($\mathfrak{D}$) ontologies is ExpTime-complete.



Rationals with comparison

Allen's interval algebra

Undecidable extensions (CADE '25):

- ALCSCC($\mathfrak{D}$) with mixed numerical constraints*
  - *when concrete domains have domain set $\mathbb{N}$, $\mathbb{Q}$ or $\mathbb{Z}$ and can express equality
  - "The h-index of … is equal to the number of authored works that have a citation count that is at least h-index."
  - succ(h-index = |author ∩ (h_index ⩽ `next` citecount)|)
- ALCSCC^{++}($\mathfrak{D}$) (successor constraints evaluated globally) if $\mathfrak{D}$ is infinite and can express equality
- SSCC = ALCSCC + transitivity axioms trans(r) for roles

Citations and Reception

Thanks for listening! :-)

Research cited in recent works on neurosymbolic AI...

- Description Logics with neural-like constructors: *Succinctness and complexity of ALC with counting perceptrons (2023)*
- Expressive Power of Graph Neural Networks (GNNs): *Are Targeted Messages More Effective? (2024), Verifying quantized graph neural networks is PSPACE-complete (2025)*

...and in other works at the intersection of logic, numbers and counting:

- Temporal/modal logics: *On the effects of adding assignments in linear-time temporal logics modulo theories (2025), Finite Traces and Definite Descriptions: a Knowledge Representation Journey (2022)*
- Logics with Counting: *Maximum Entropy Reasoning via Model Counting in (Description) Logics that Count (2025), Interleaving logic and counting (2023), Expressive description logics with rich yet affordable numeric constraints (2025)*
- Logics with Concrete Domain Reasoning: *Reasoning in OWL 2 EL with Hierarchical Concrete Domains (2025)*

FroCoS '19: Best Student Paper

SAC '24: Best Paper

FroCoS '25: Best Student Paper