Decidable (Ac)counting with Parikh and Muller: Adding Presburger Arithmetic to Monadic Second-Order Logic over Tree-Interpretable Structures

Luisa Herrmann, Vincent Peth, and Sebastian Rudolph

Naples, CSL 2024, Feb 23

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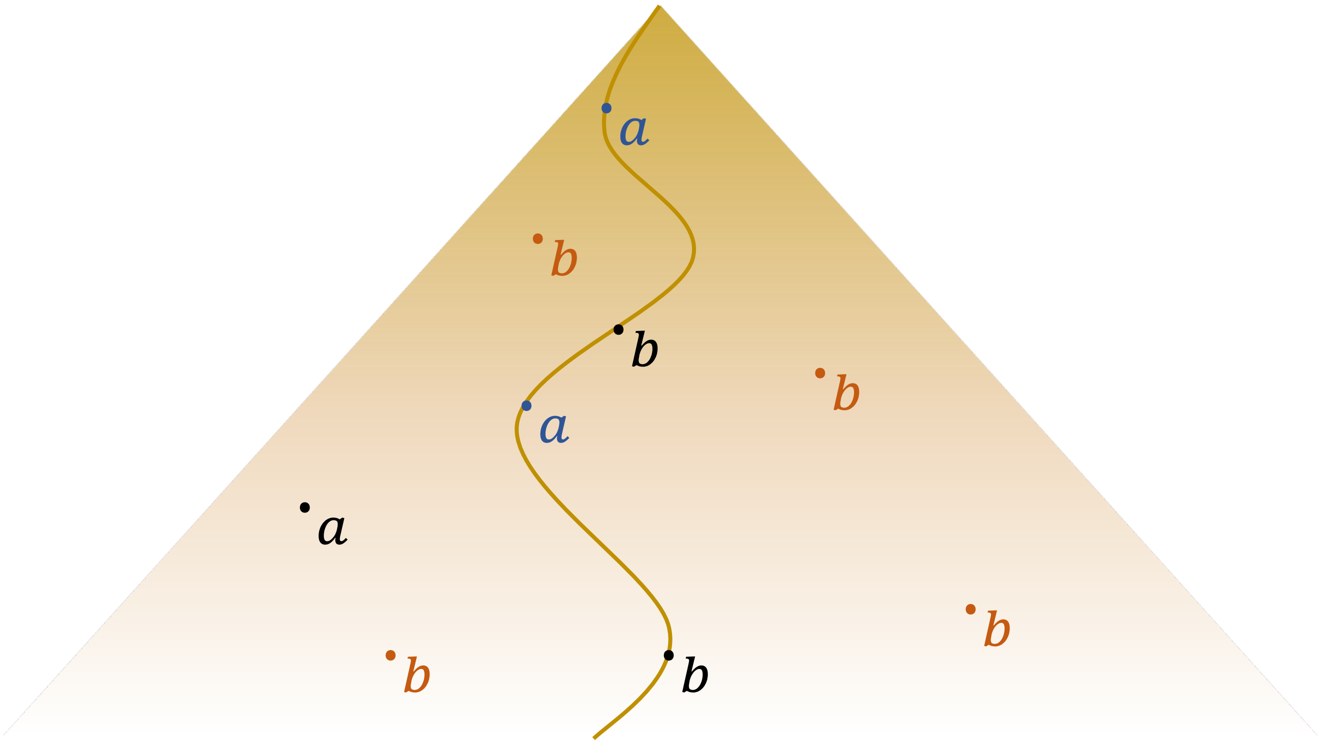

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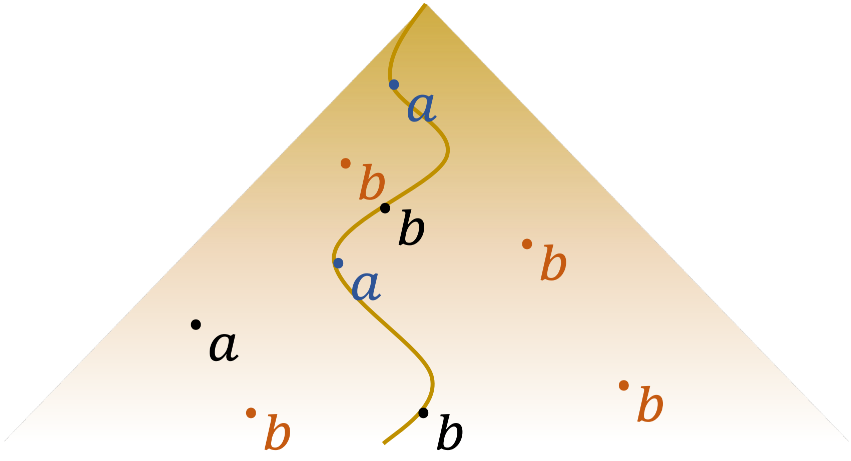

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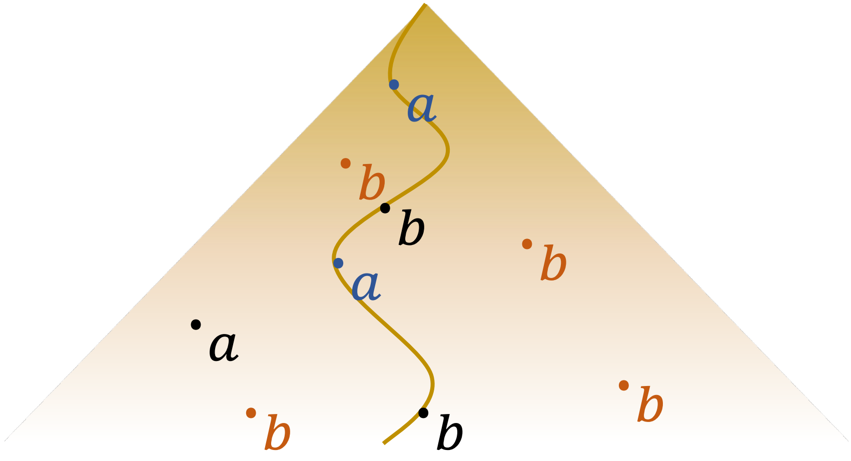

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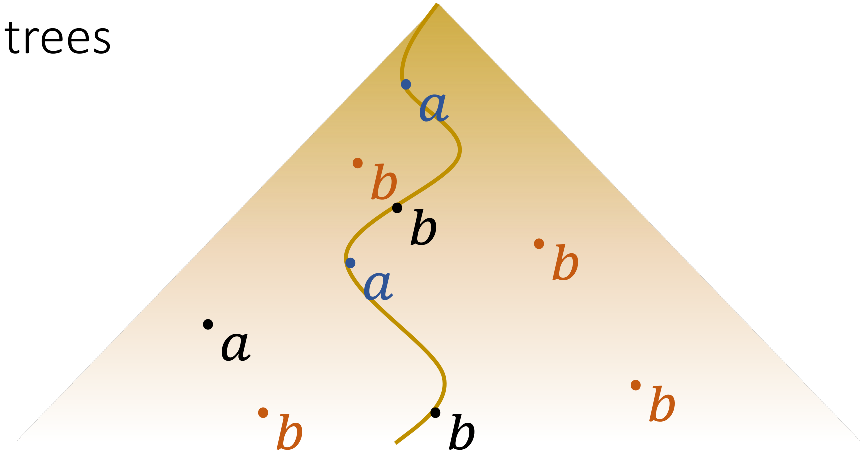

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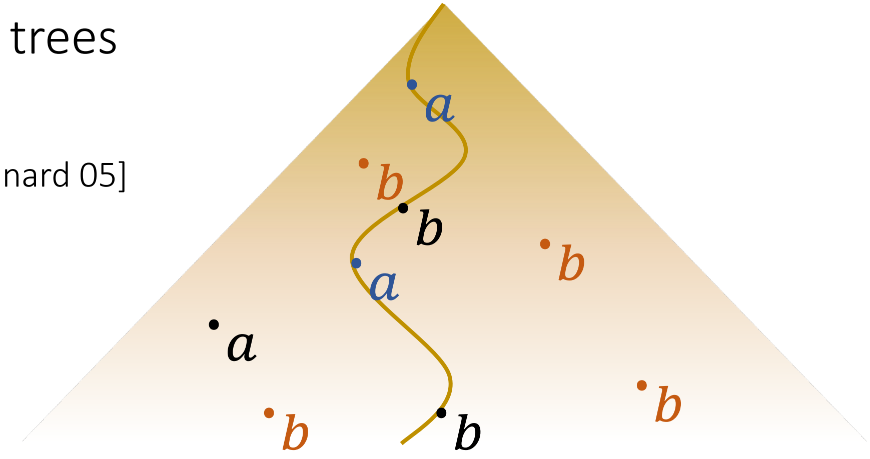

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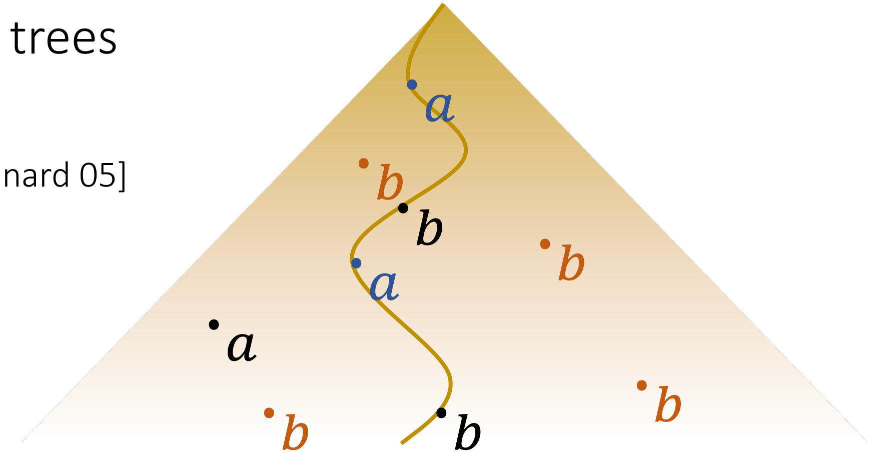
↪ we combine these approaches



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- the fragment ωMSO⋈BAPA
- ... and its normal form

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Decidability

From CMSO to ωMSO·BAPA

signature $\mathbb{S} = \mathbb{S}_C \cup \mathbb{S}_P$

countable $\mathbb{S}$-structure $\mathfrak{A} = (A, \cdot^{\mathfrak{A}})$

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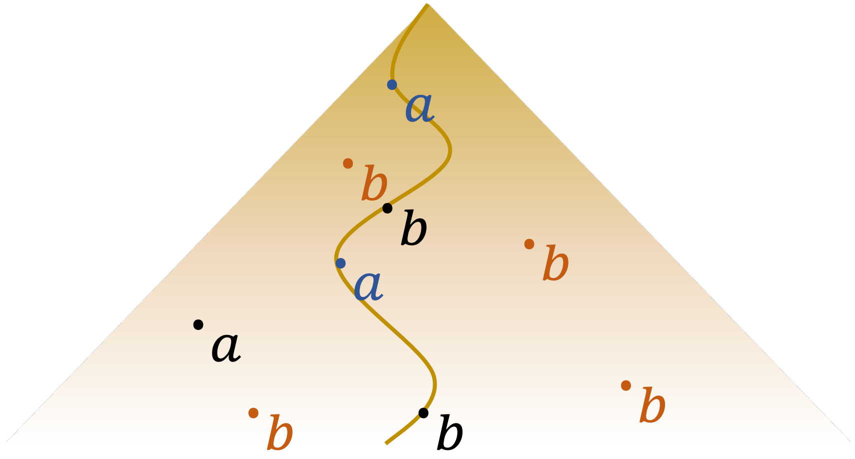

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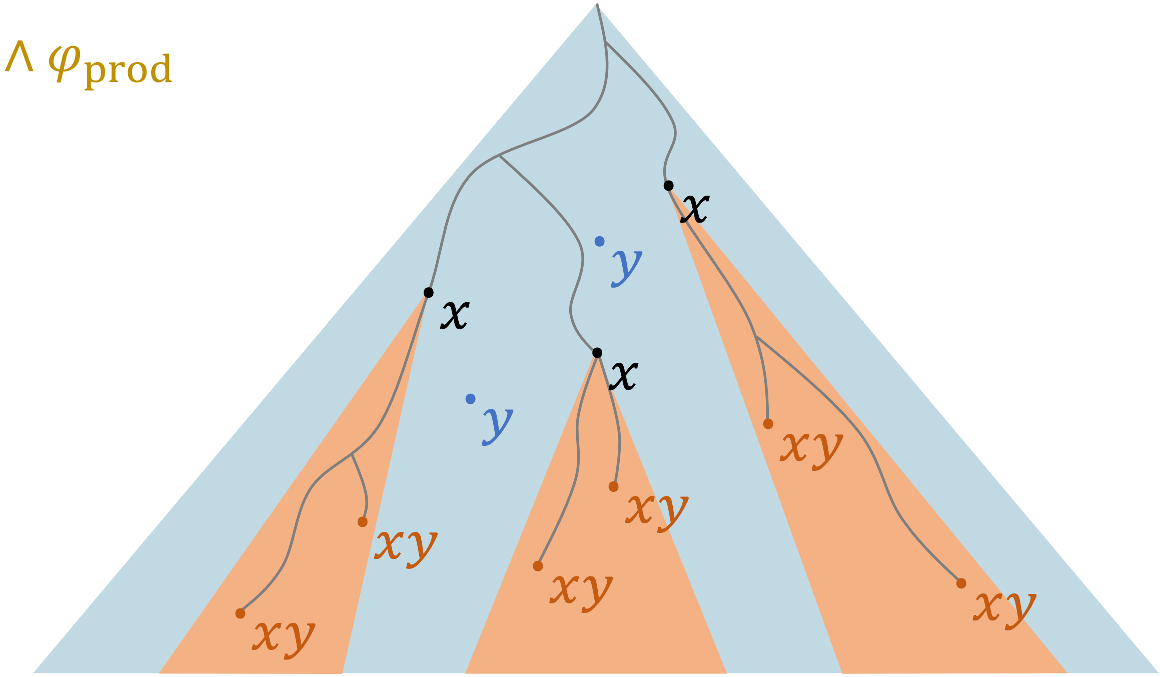

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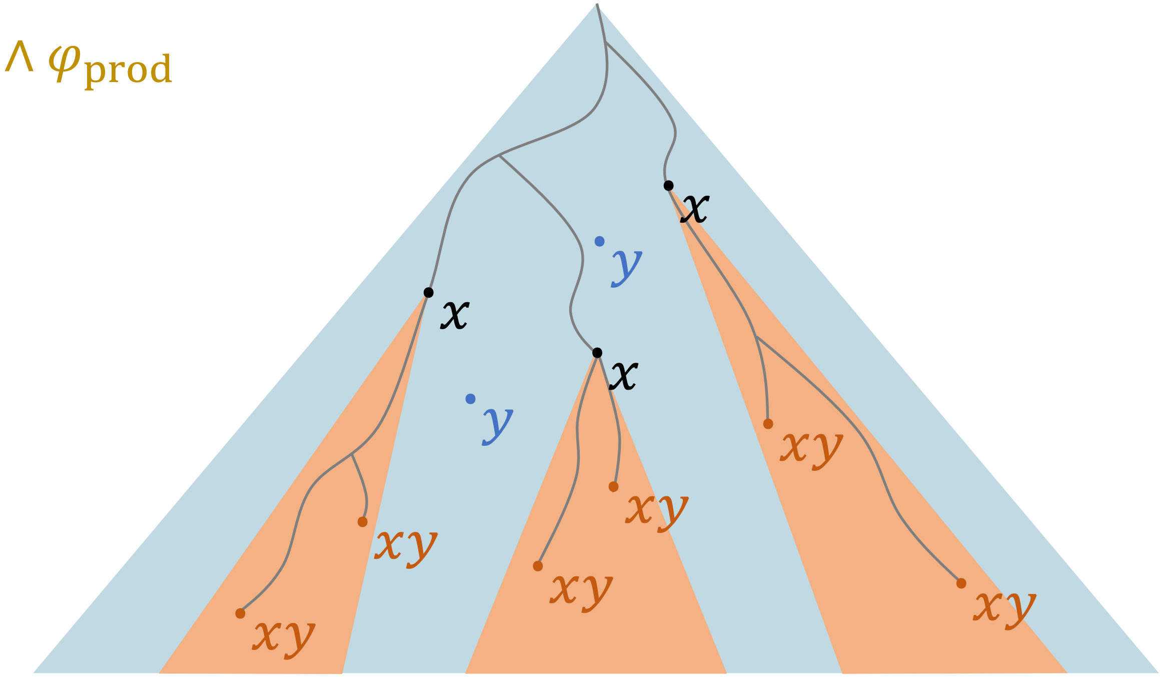

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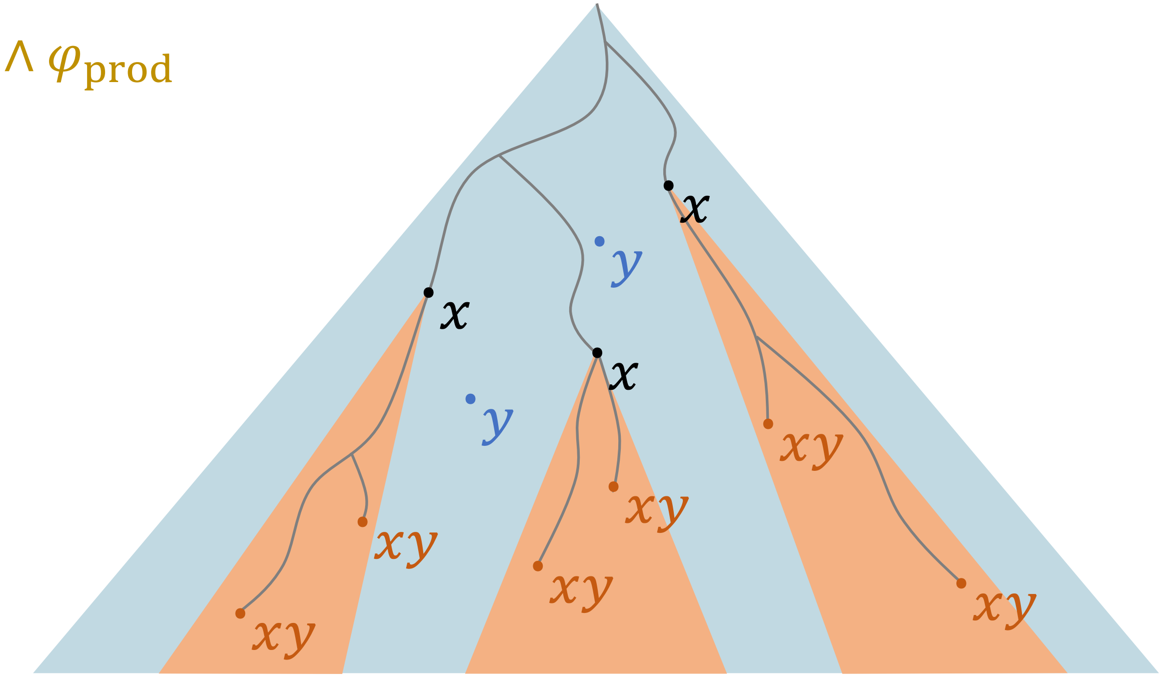

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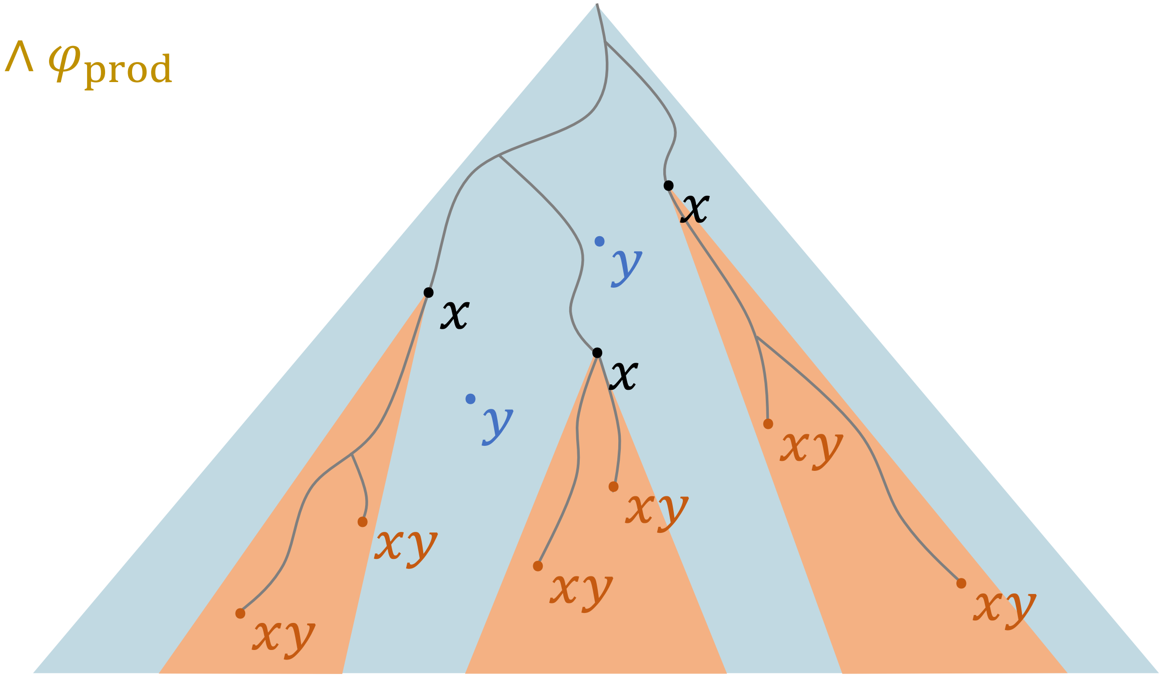

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Proposition. Satisfiability of the class of ωMSO·BAPA sentences of the shape $\varphi_{\mathcal{D}}$ is undecidable.

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$$\exists X \exists V \forall Y \exists Y'.\ \mathrm{Path}(X) \wedge \varphi_{\mathrm{MSO}}(V) \wedge$$
$$\#(X \cap Y \cap V) \leq \#(Y' \cap V) \wedge (\forall z. Y'(z) \Rightarrow \mathrm{P_{red}}(z))$$

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bad! ☹

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$\chi_{i,j}$ Parikh constraints

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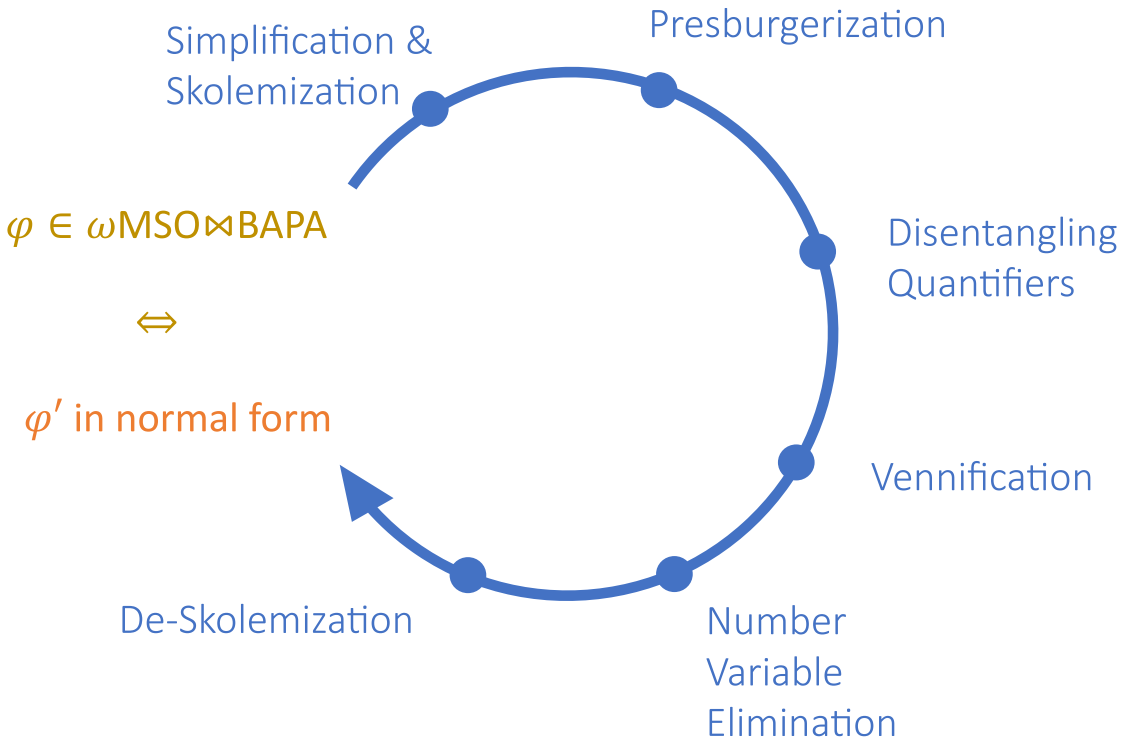

$\varphi \in \omega$MSO⋈BAPA

⇔

φ' in normal form

Simplification & Skolemization

Presburgerization

Disentangling Quantifiers

Vennification

Number Variable Elimination

De-Skolemization

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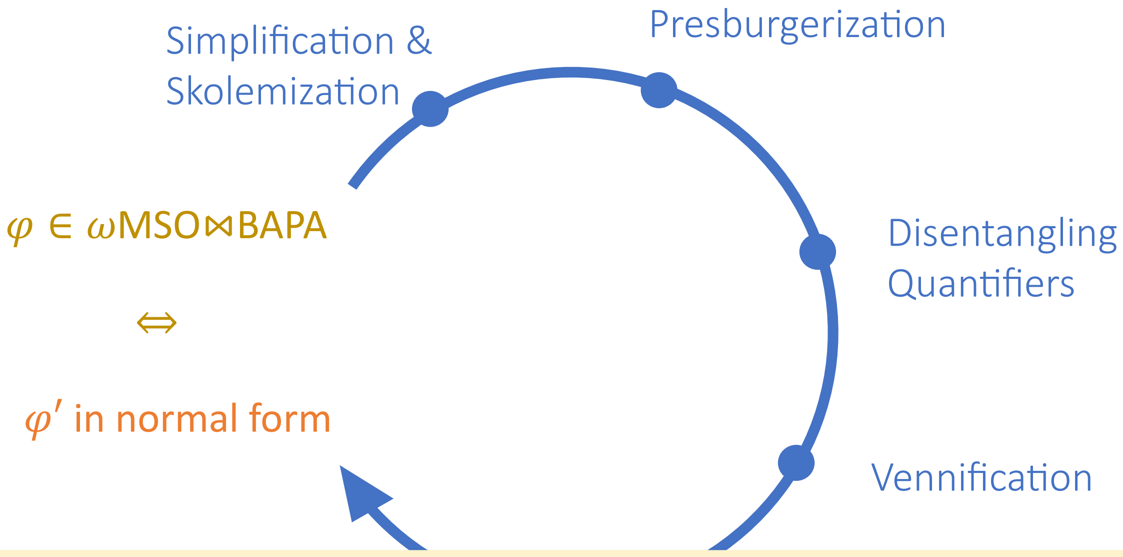
Simplification & Skolemization
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Proposition. For each ωMSO⋈BAPA formula φ we can compute an equivalent formula φ' of the form

$$\varphi' = \exists X_1 \dots X_n. \bigvee_i (\varphi_i \wedge \bigwedge_j \chi_{i,j})$$

where φ_i are CMSO formulae and $\chi_{i,j}$ are Parikh constraints.

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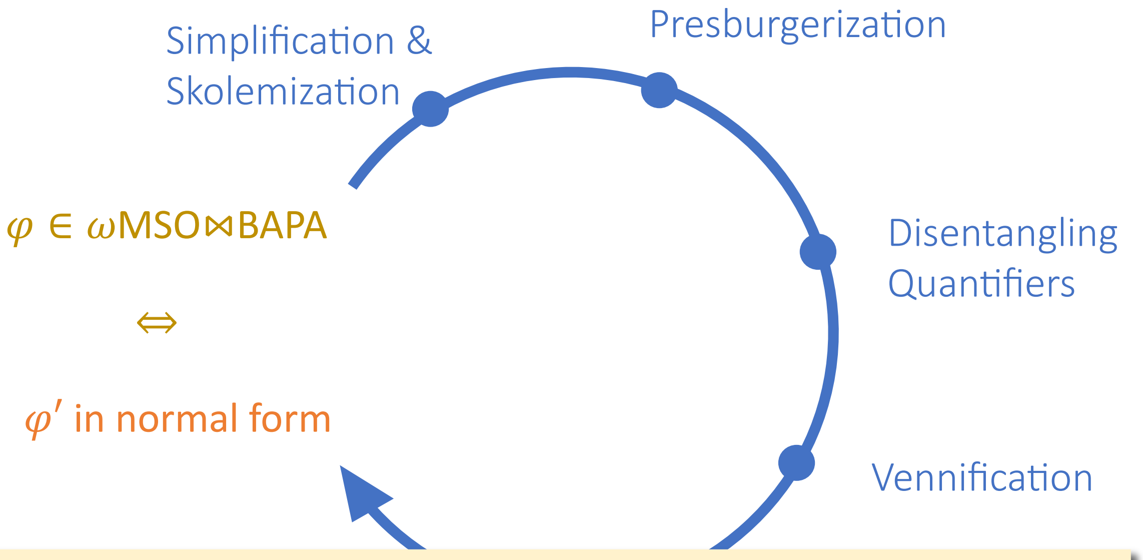

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where φ_i are MSO formulae and $\chi_{i,j}$ are Parikh constraints that is equivalent to φ over labeled infinite binary trees.

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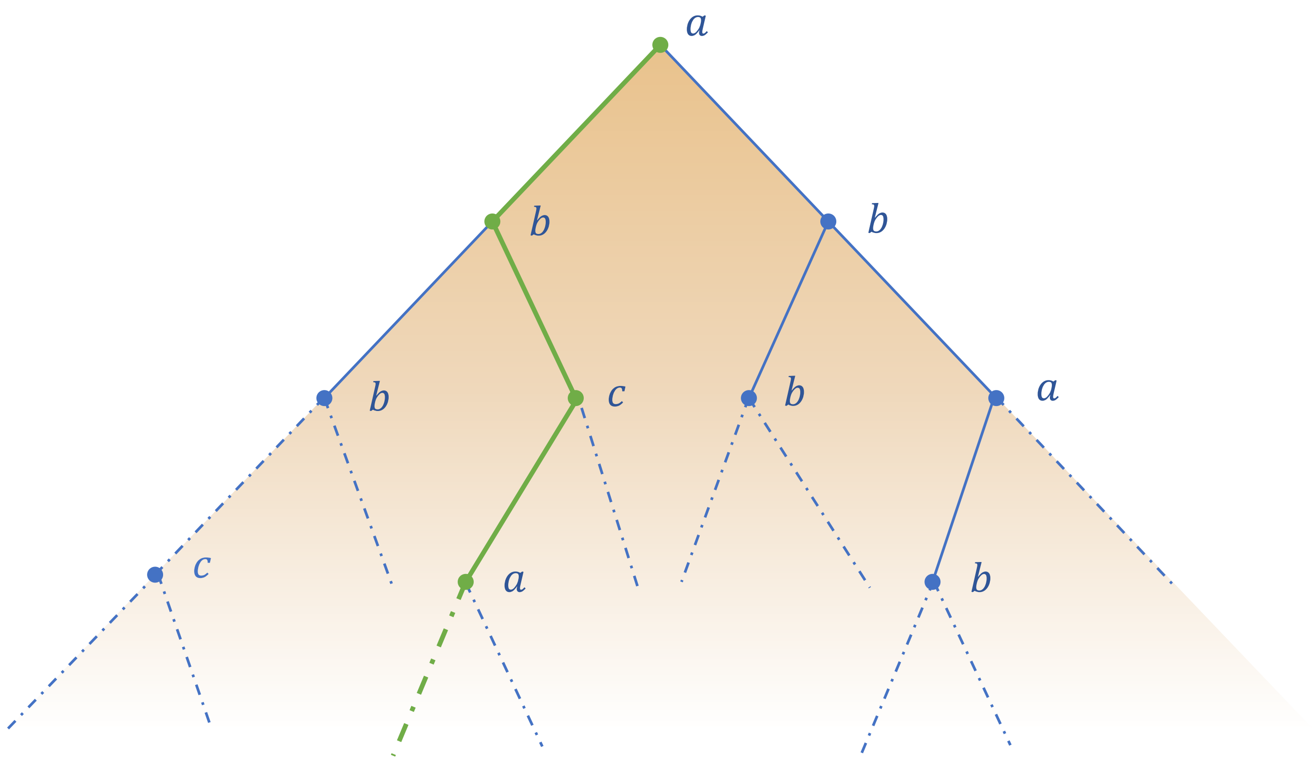
↓

Tree Automata!

Counting with Tree Automata



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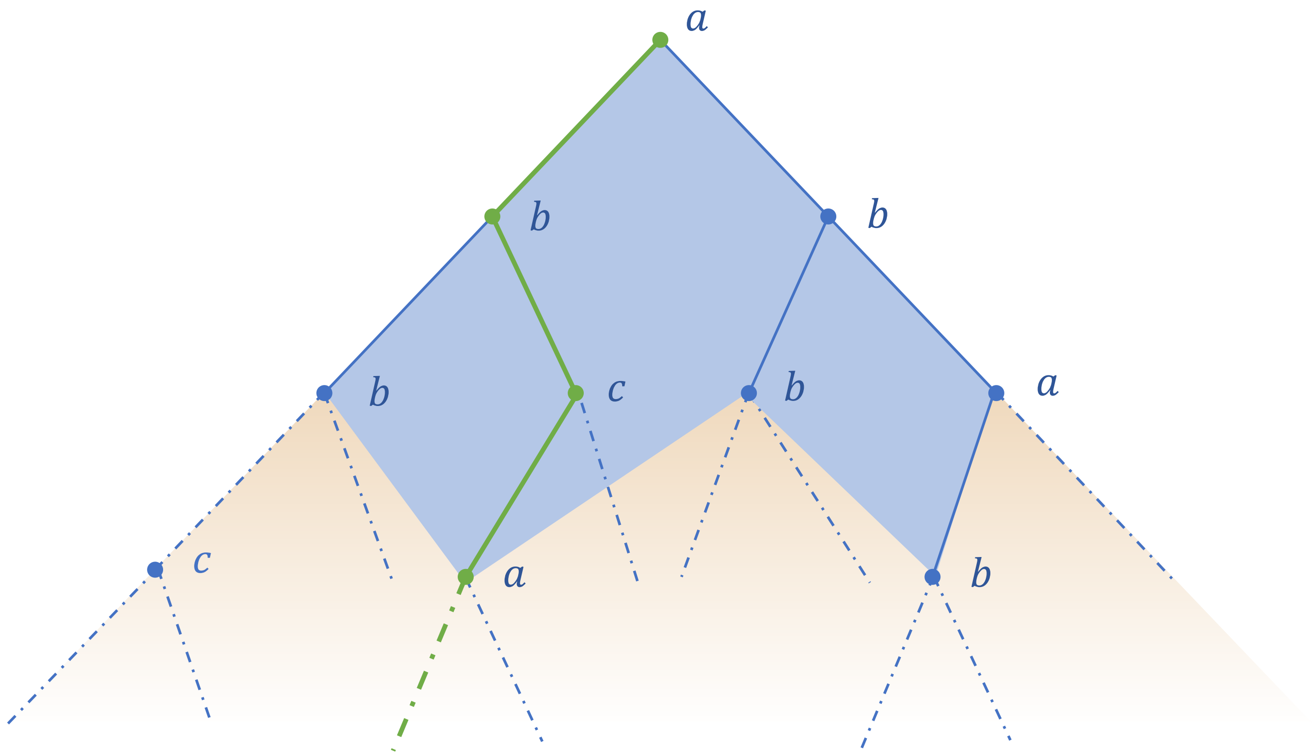
Counting with Tree Automata



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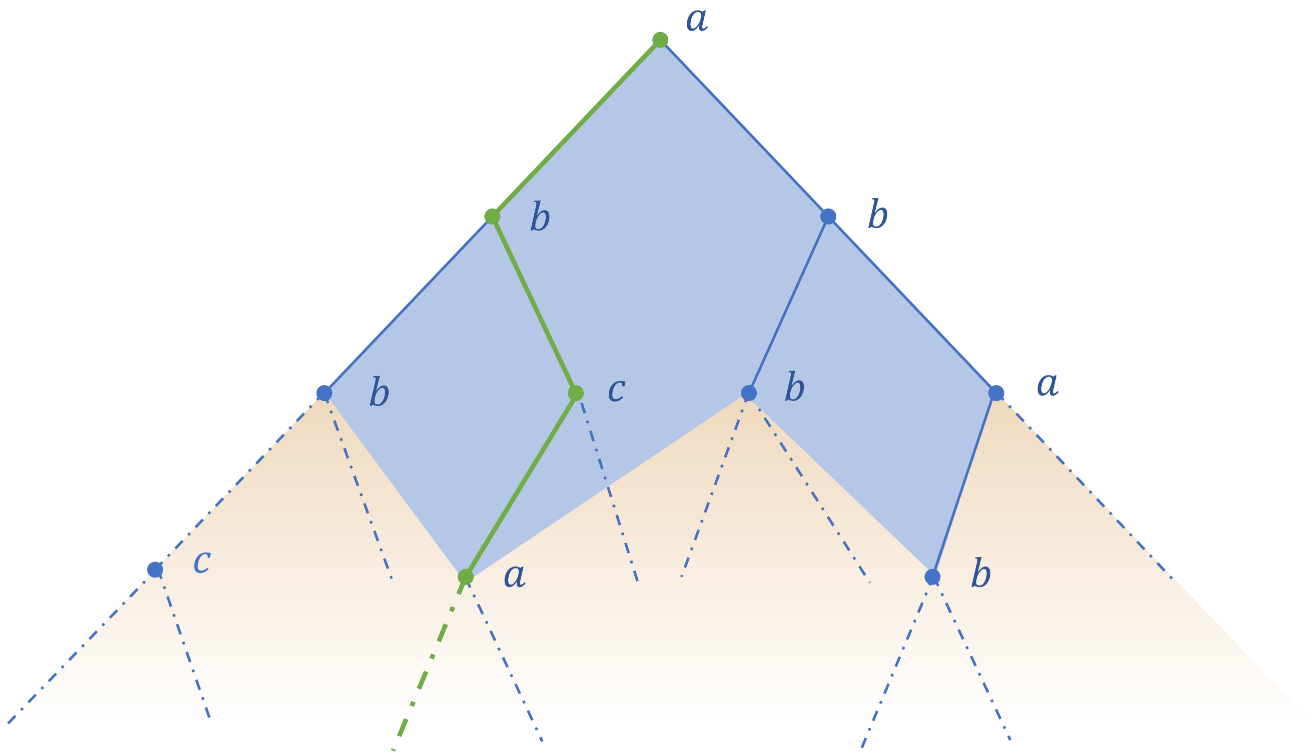
Counting with Tree Automata

Idea: generalize Parikh Automata [Klaedtke, Rueß 03]



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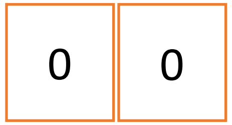

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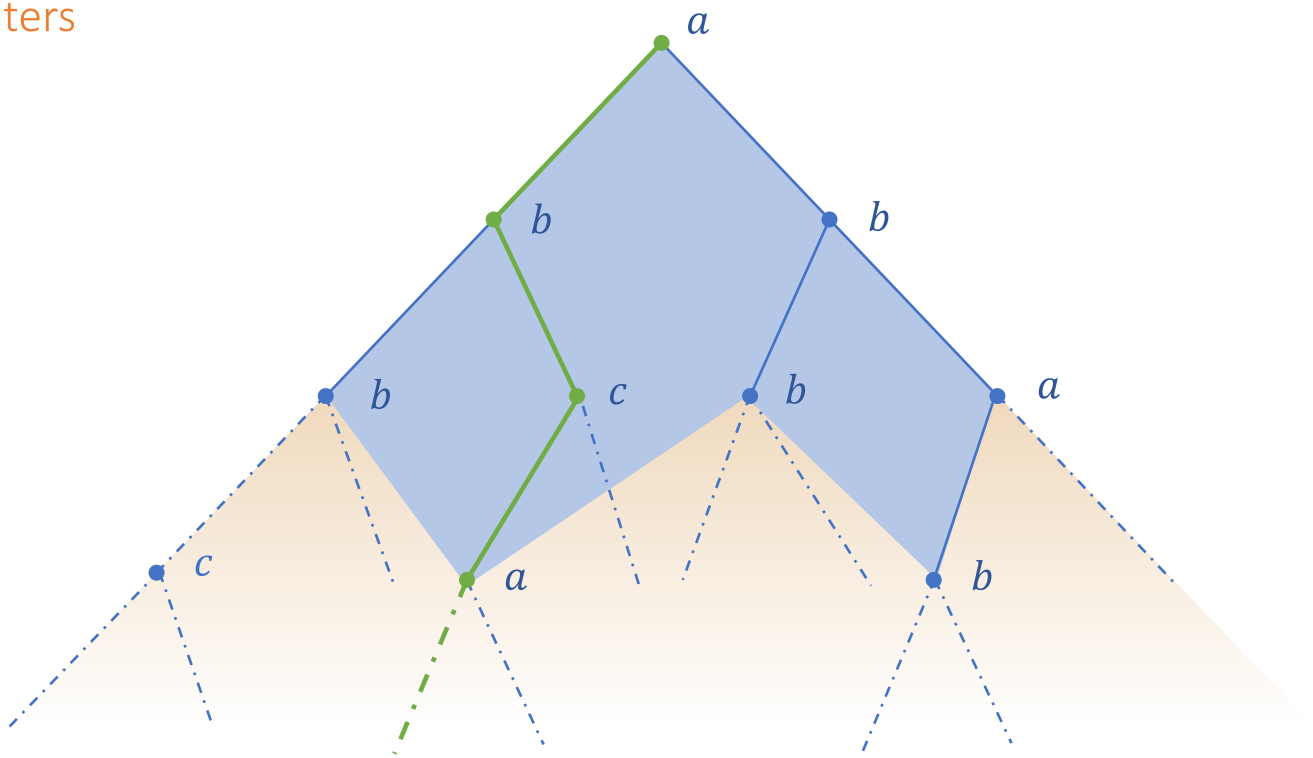
- use finite number of global counters



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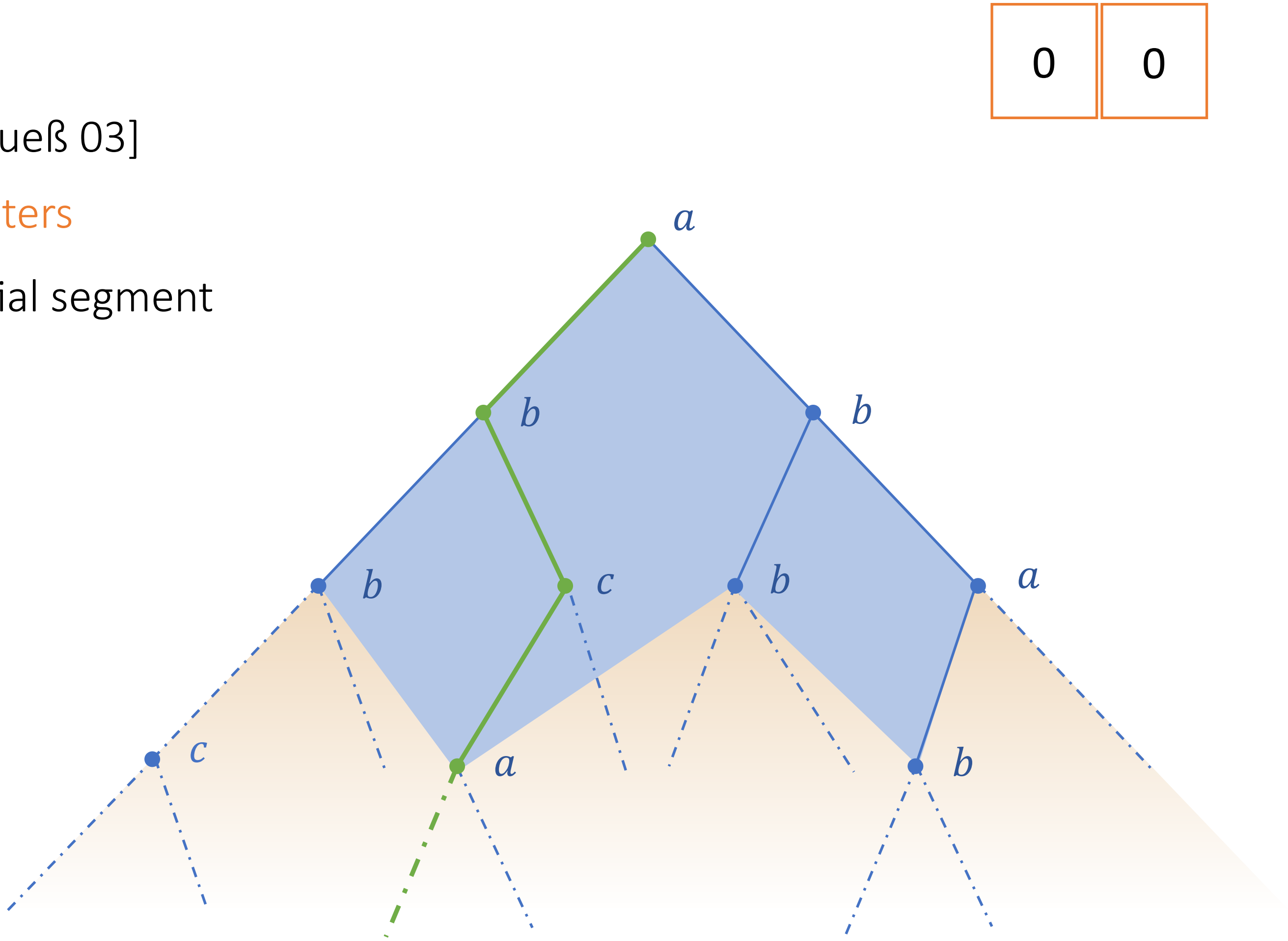
- increment counters on finite initial segment



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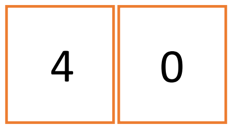

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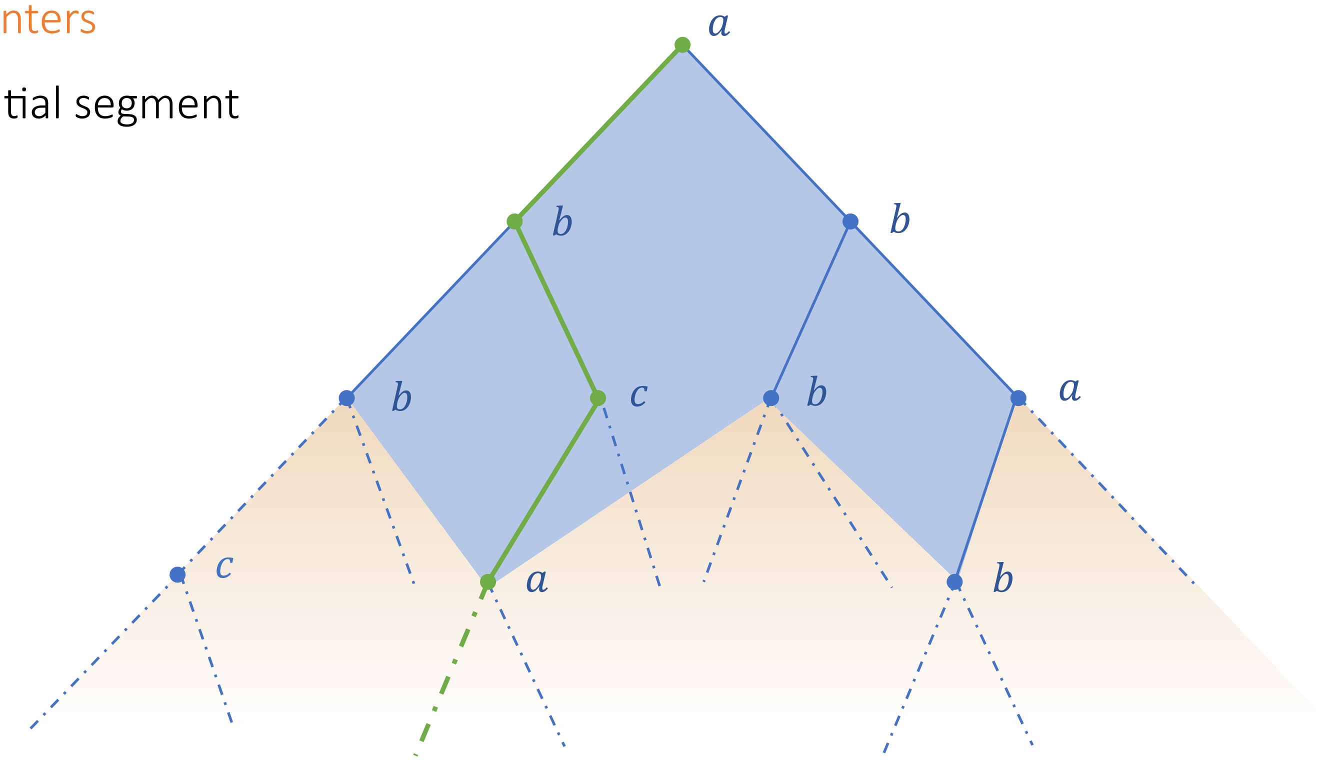

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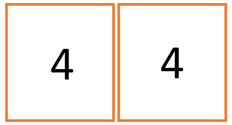

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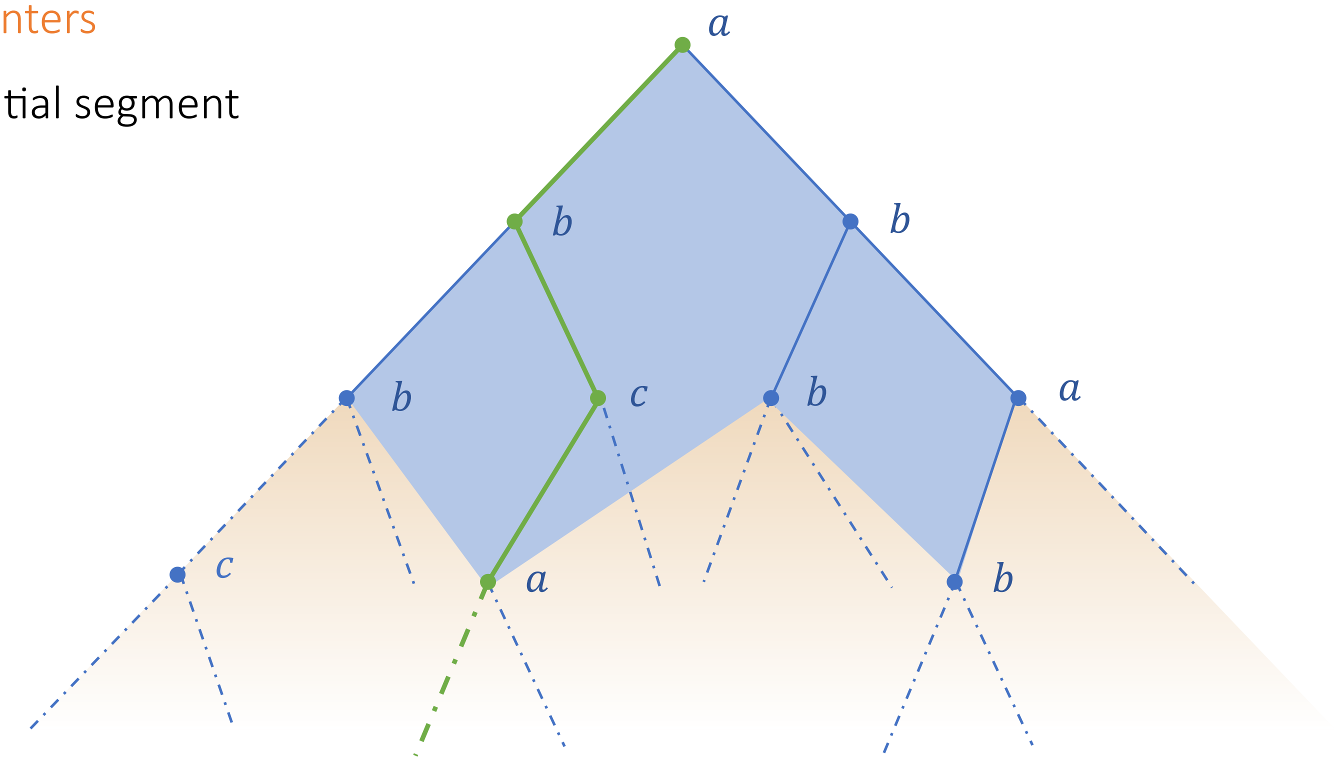

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Counting with Tree Automata

Idea: generalize Parikh Automata [Klaedtke, Rueß 03]

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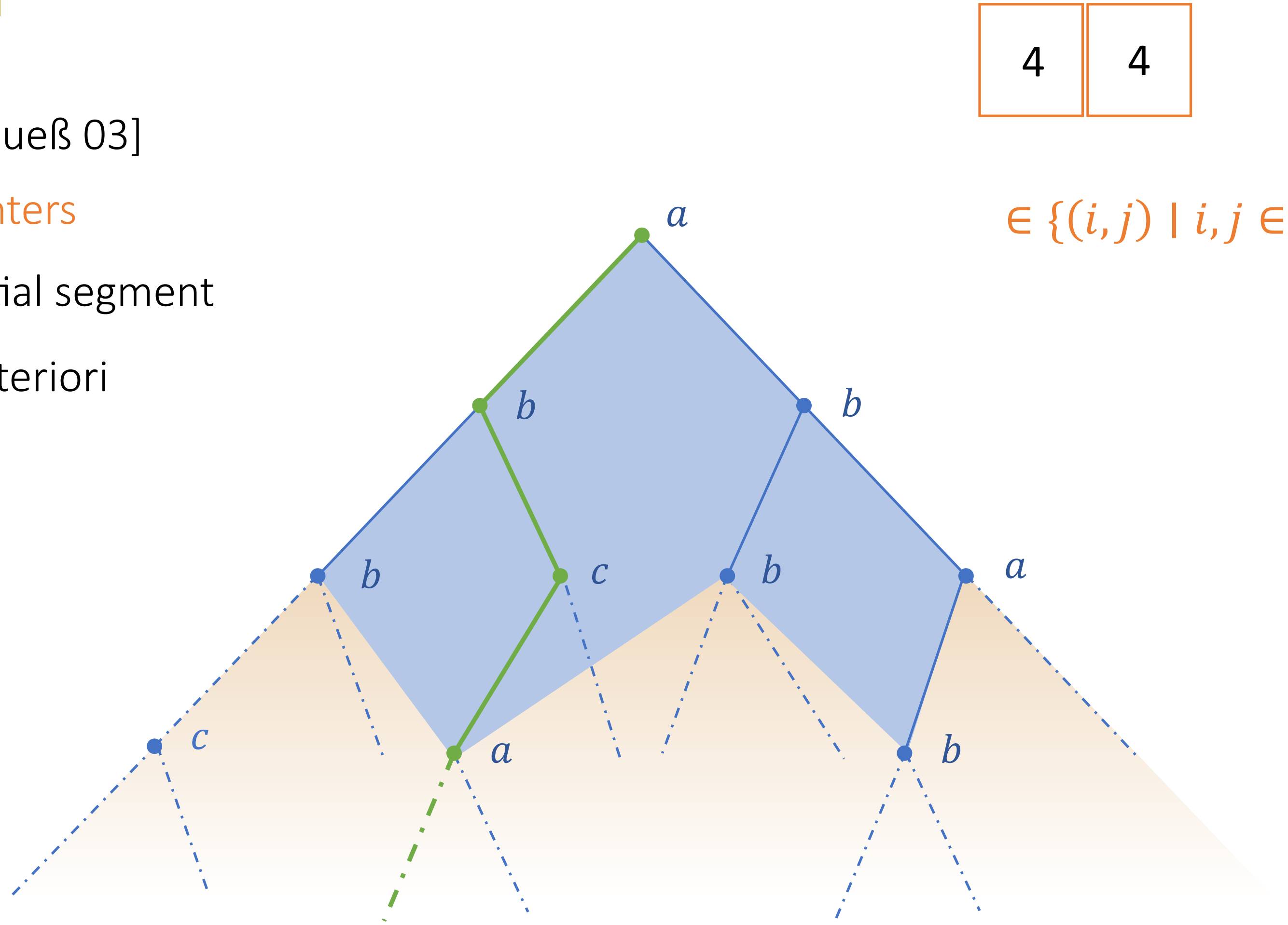
- increment counters on finite initial segment
- test counter configuration a posteriori



$\in \{(i,j) \mid i,j \in \mathbb{N}, i \leq j\}$?

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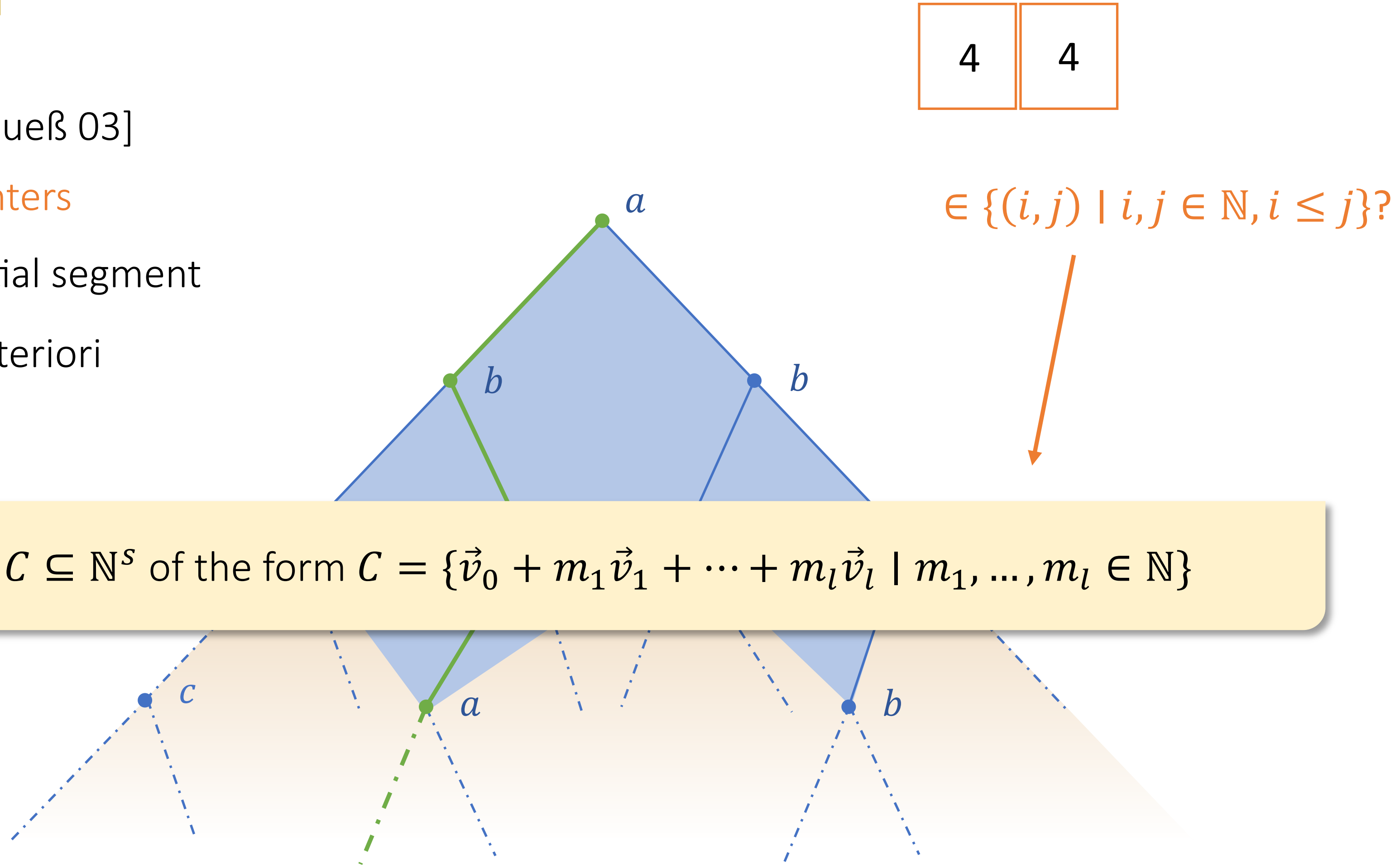

semilinear set: finite union of sets $C \subseteq \mathbb{N}^s$ of the form $C = \{\vec{v}_0 + m_1\vec{v}_1 + \cdots + m_l\vec{v}_l \mid m_1, \ldots, m_l \in \mathbb{N}\}$

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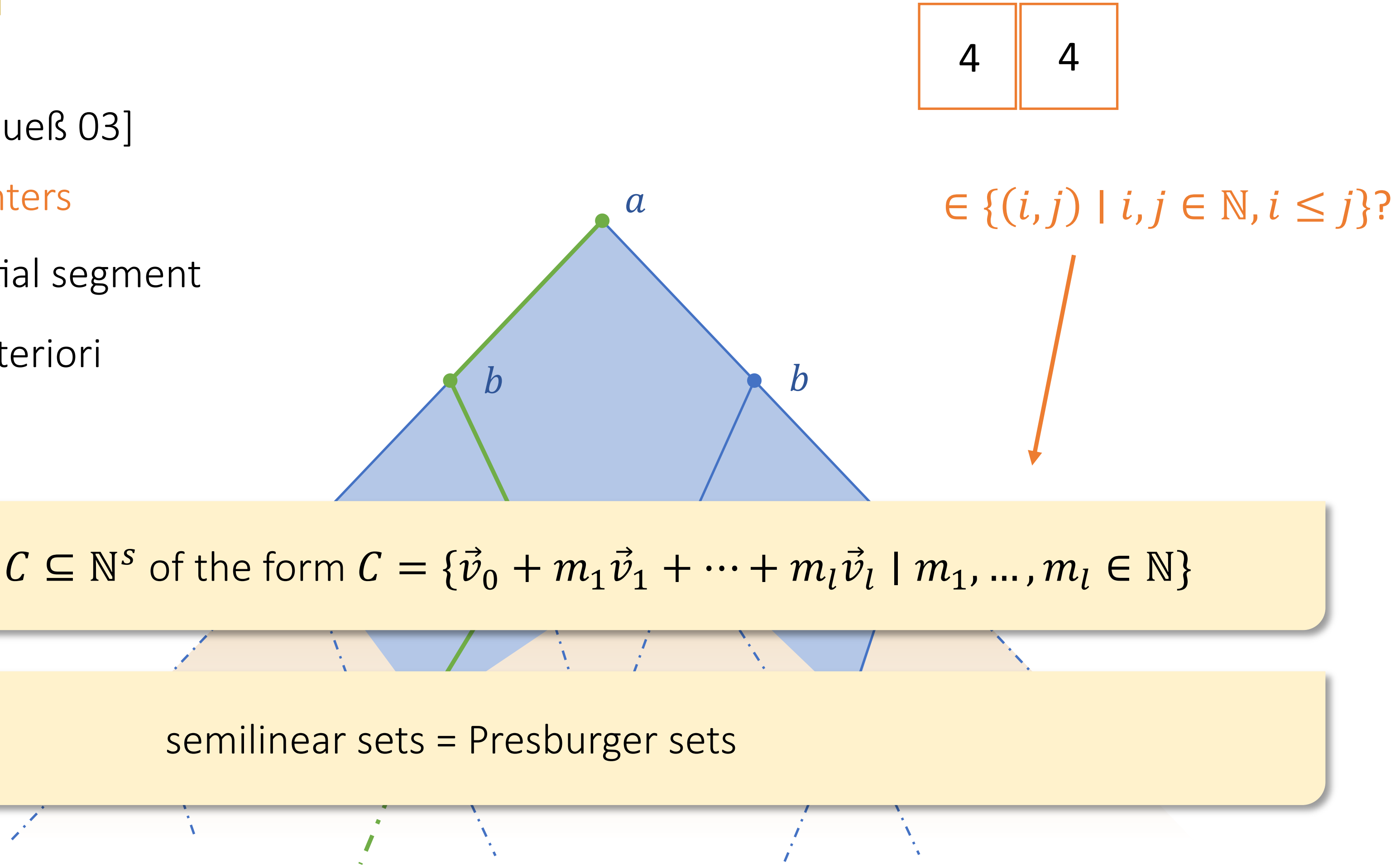

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Theorem [Ginsburg, Spanier 64]. semilinear sets = Presburger sets

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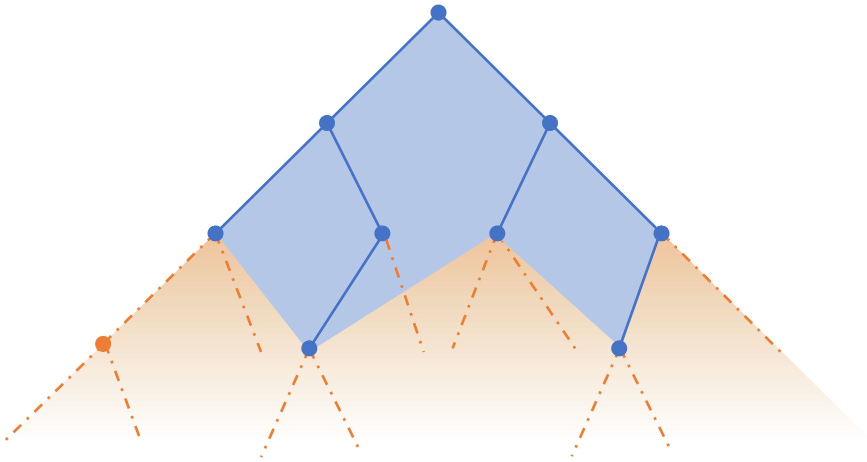
Parikh-Muller Tree Automata

dimension $s \in \mathbb{N}$

counter increments $D \subseteq_{\text{fin}} \mathbb{N}^s$

PMTA $\mathcal{A} = (Q, \Xi, q_I, \Delta, \mathcal{F}, C)$



tree $\zeta \in \mathrm{T}_{\Xi}^{\omega}$

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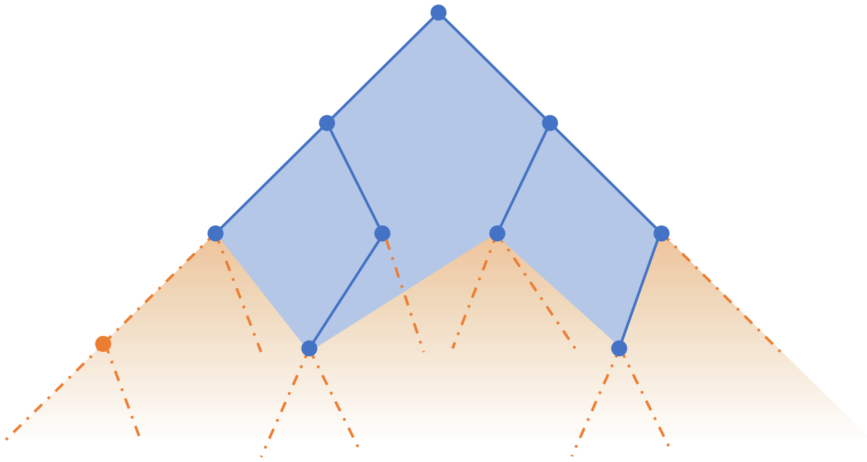

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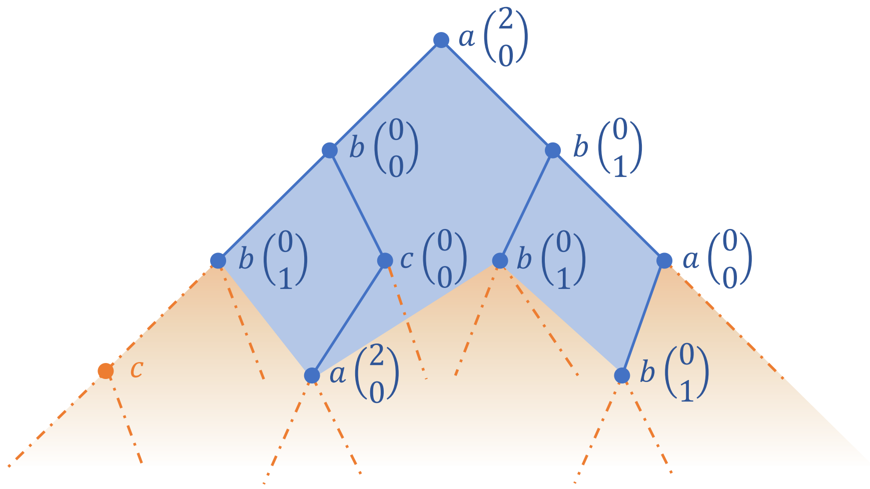
- $\Xi = (\Sigma \times D) \cup \Sigma$ alphabet



tree $\zeta \in \mathrm{T}^\omega_\Xi$

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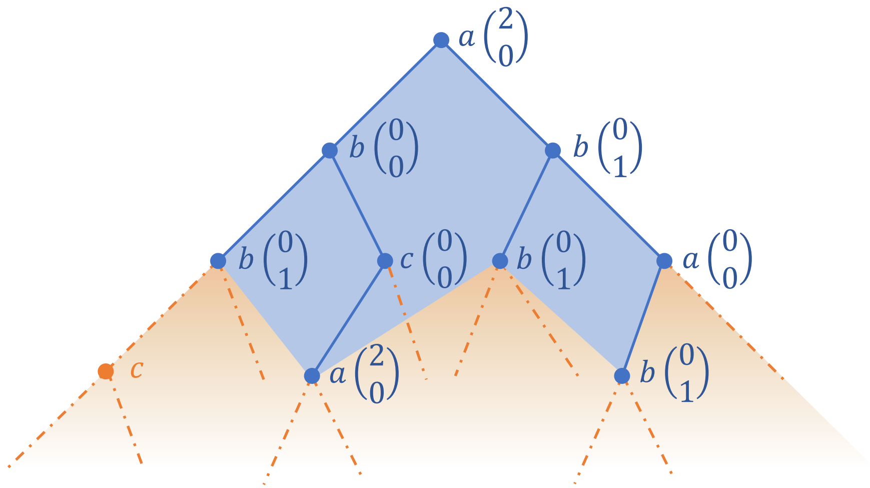
- $\Delta = \Delta_P \cup \Delta_\omega$ transitions

Δ_P: $p \to a \binom{2}{0} \langle p_1, p_2 \rangle$ reading the initial “counting“ segment

$p \to b \binom{0}{1} \langle q_1, p_1 \rangle$

Δ_ω: $q \to c \ \langle q_1, q_2 \rangle$ reading the remaining tree



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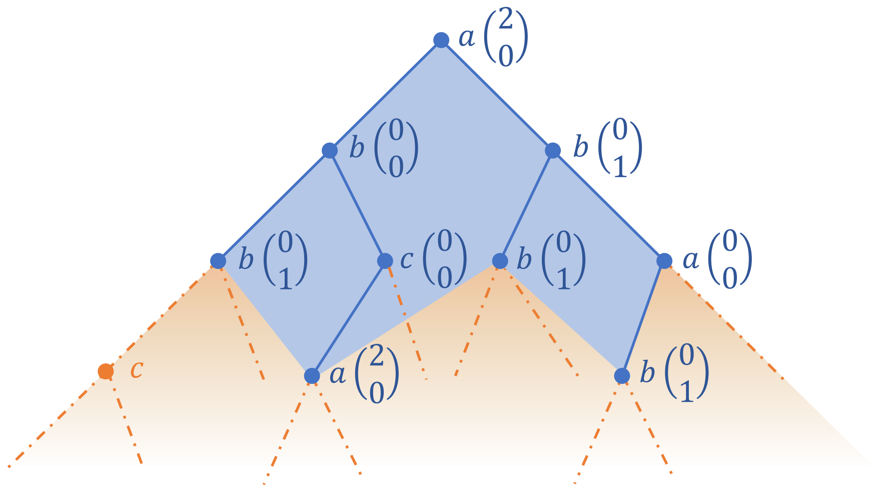
Δ_ω: $q \to c\ \langle q_1, q_2 \rangle$ reading the remaining tree

- $\mathcal{F} \subseteq 2^{Q_\omega}$ final state sets
- $C \subseteq \mathbb{N}^s$ semilinear set



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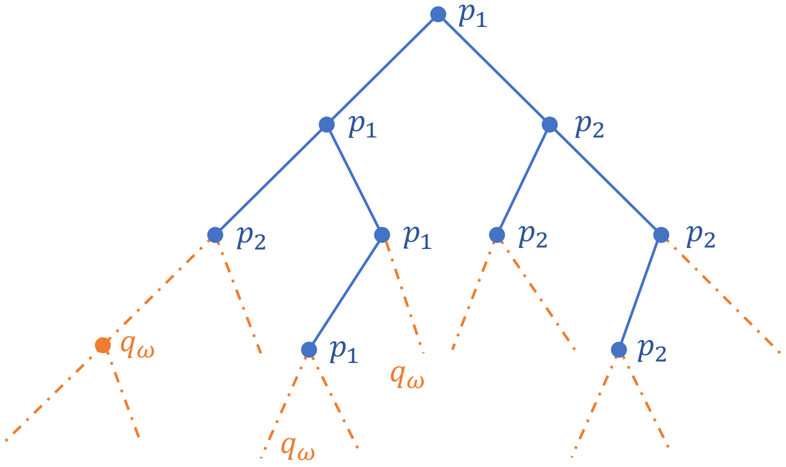
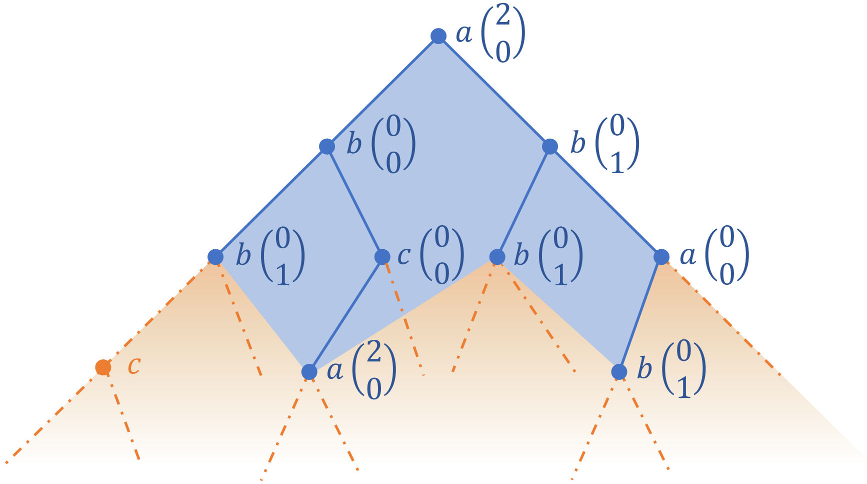


run κ of ζ



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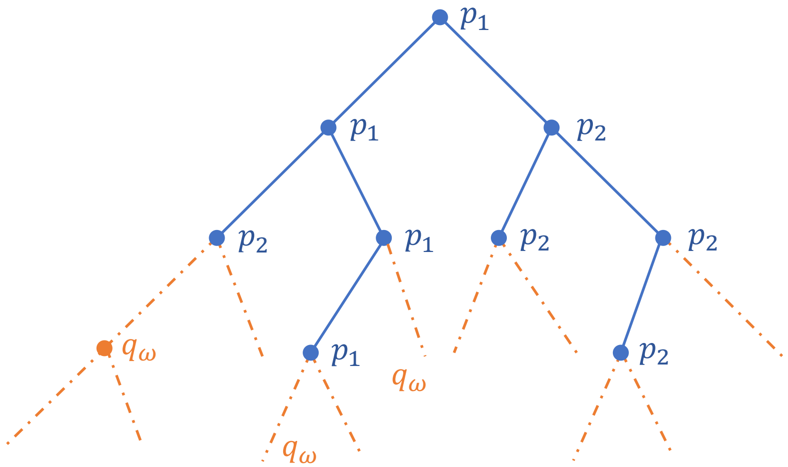

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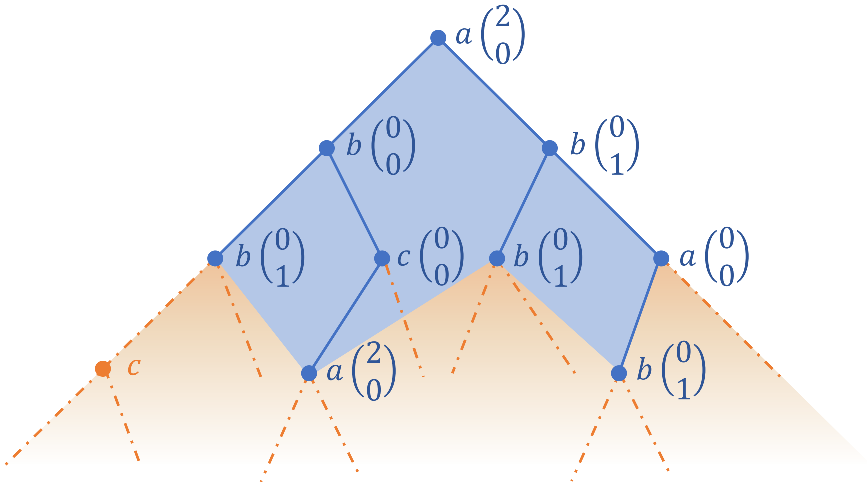

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run κ of ζ is successful if

- Muller acceptance condition holds

Parikh-Muller Tree Automata

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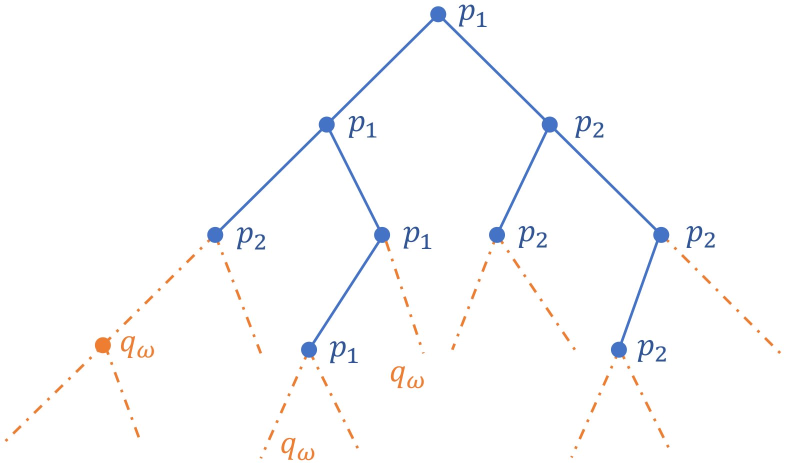

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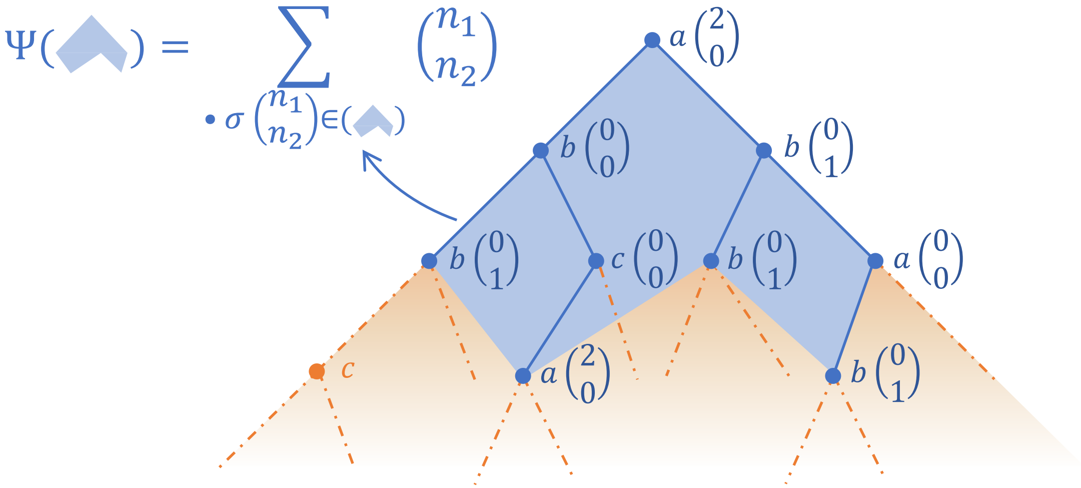
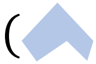


tree $\zeta \in \mathrm{T}_\Xi^\omega$

run κ of ζ is successful if

- Muller acceptance condition holds
- Parikh condition: $\Psi($$) \in C$ holds

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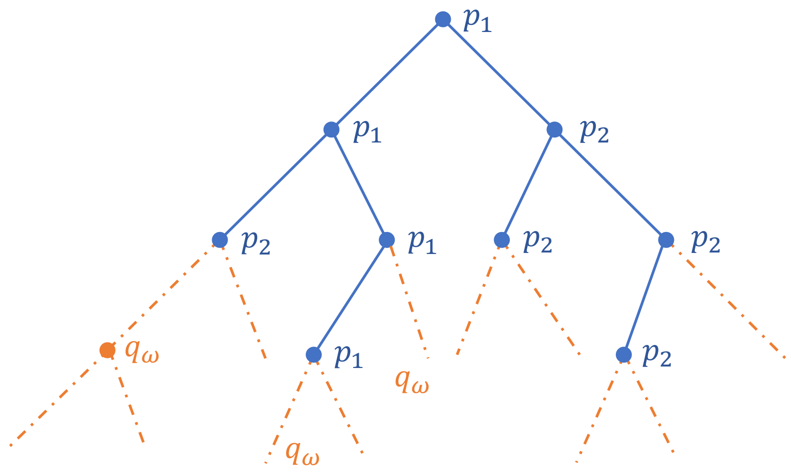
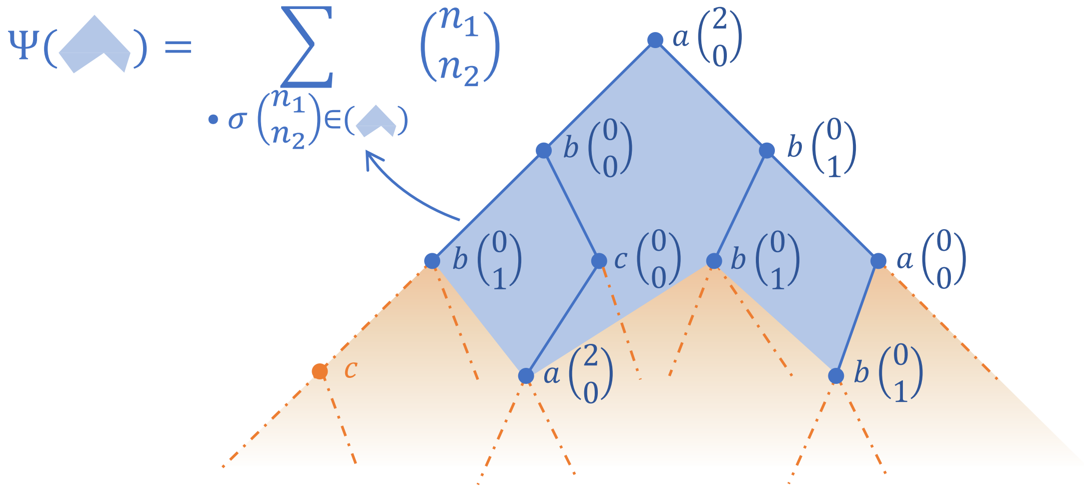


run κ of ζ



tree $\zeta \in T_\Xi^\omega$

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- Parikh condition: $\Psi($▲$) \in C$ holds

$$\mathcal{L}(\mathcal{A}) = \{\xi \in T_\Sigma \mid \exists \zeta \in T_\Xi^\omega \text{ with } (\zeta)_\Sigma = \xi \text{ and } \exists \text{ successful run } \kappa \text{ on } \zeta\}$$

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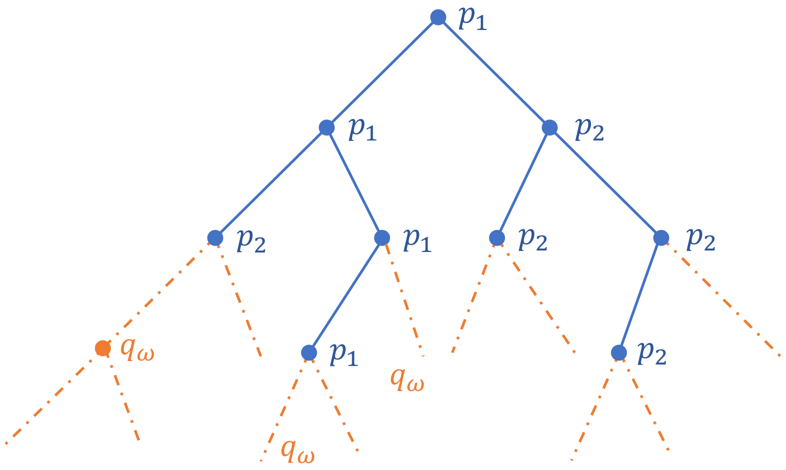
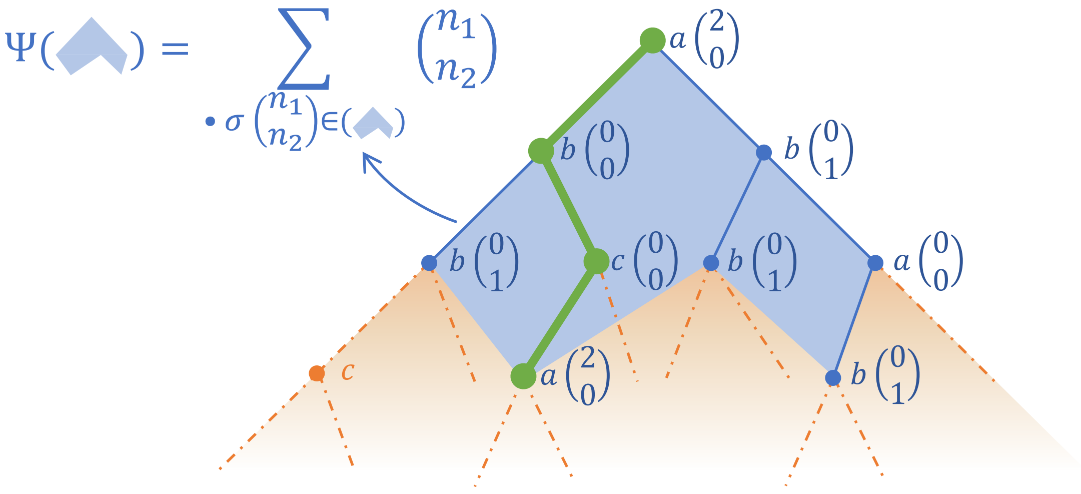


run κ of ζ



tree $\zeta \in T_\Xi^\omega$

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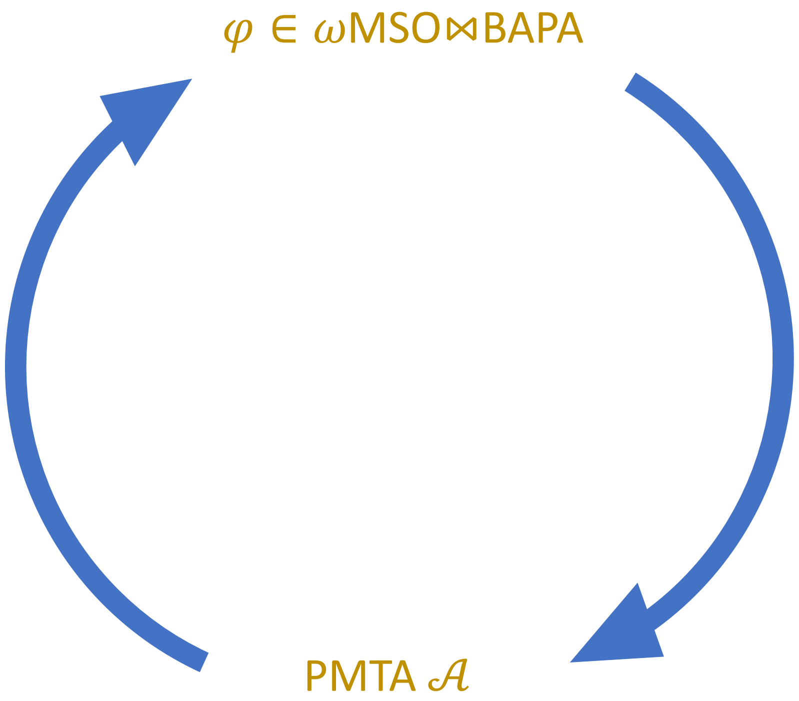
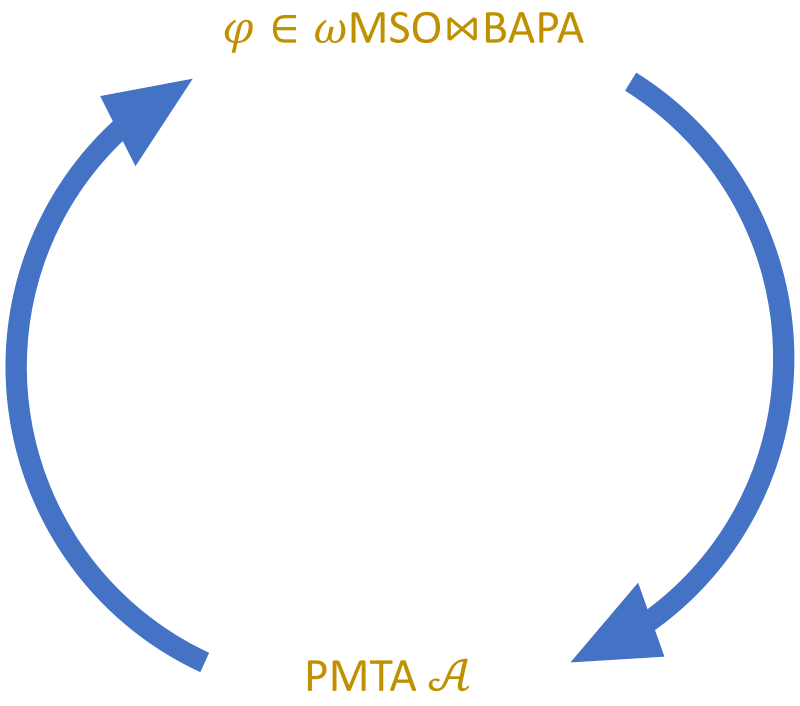
Correspondence of ωMSO⋈BAPA and PMTA



$\varphi \in \omega$MSO⋈BAPA

PMTA $\mathcal{A}$

Correspondence of $\omega\mathrm{MSO}\bowtie\mathrm{BAPA}$ and PMTA



$\varphi \in \omega\mathrm{MSO}\bowtie\mathrm{BAPA}$

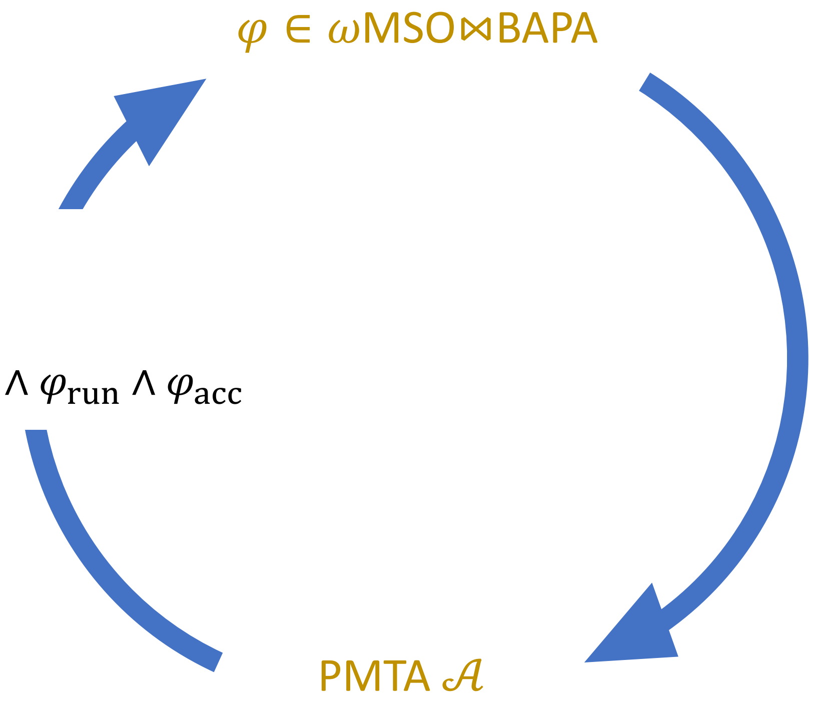
PMTA $\mathcal{A}$

Induction on structure of

$$\varphi = \exists X_1 \dots X_n . \bigvee_i (\varphi_i \wedge \bigwedge_j \chi_{i,j})$$

- PMTA recognize MSO sentences [Rabin]
- PMTA recognize Parikh constraints
- PMTA are closed under ∪, ∩, relabeling

Correspondence of ωMSO⋈BAPA and PMTA



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PMTA $\mathcal{A}$

Construction of formula φ

$$\exists X_1 \ldots X_{|Q|} \forall P.\, \varphi_{\text{part}} \wedge \varphi_{\text{run}} \wedge \varphi_{\text{acc}}$$

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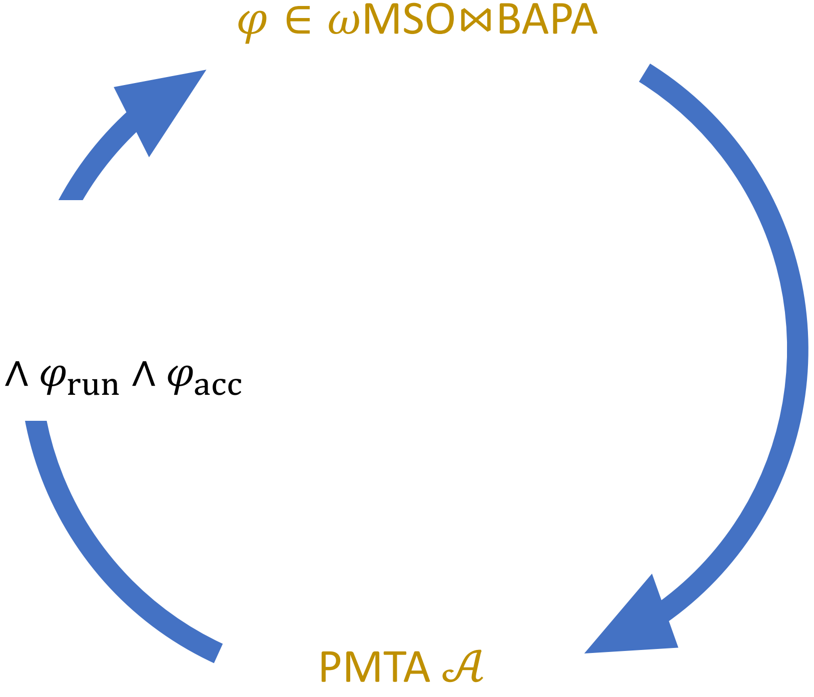

$\varphi \in \omega$MSO⋈BAPA

PMTA $\mathcal{A}$

Construction of formula φ

$\exists Z_1^0 \dots Z_1^K \dots Z_S^0 \dots Z_S^K.\, \exists X_1 \dots X_{|Q|} \forall P.\, \varphi_{\text{part}} \land \varphi_{\text{run}} \land \varphi_{\text{acc}}$

Z_i^k: counter i is increased by k

Induction on structure of

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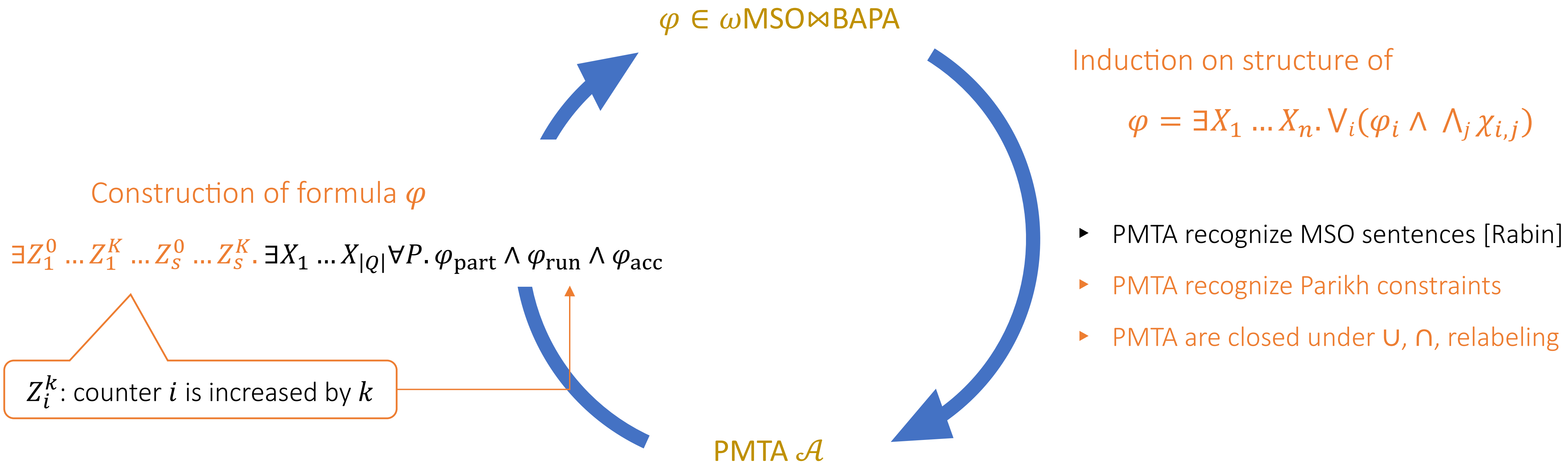

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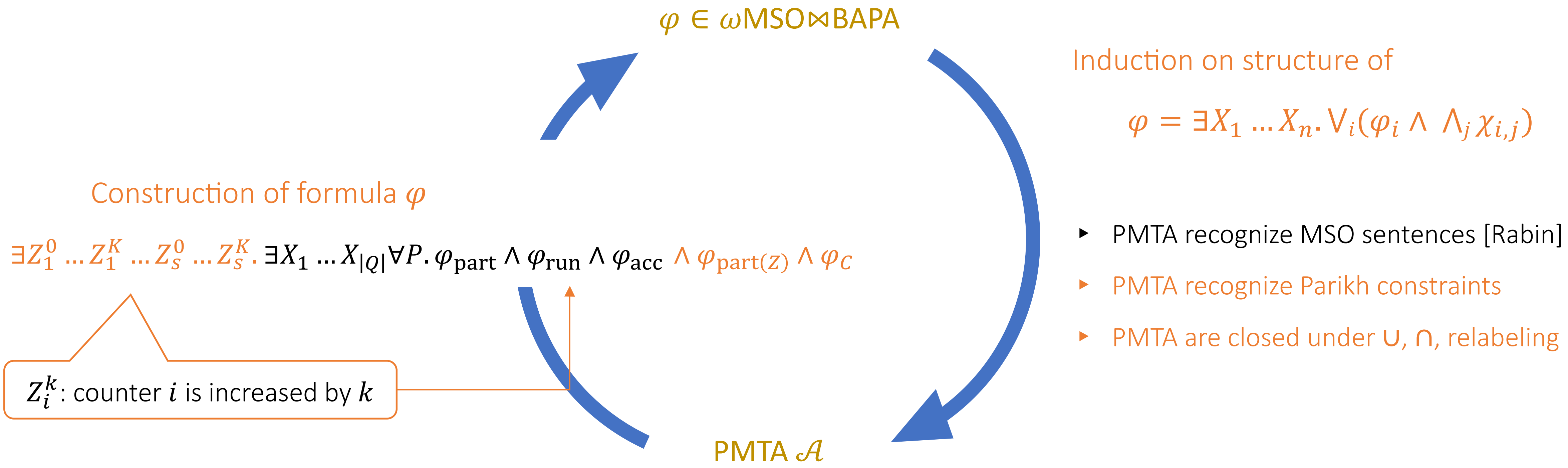
Correspondence of ωMSO⋈BAPA and PMTA



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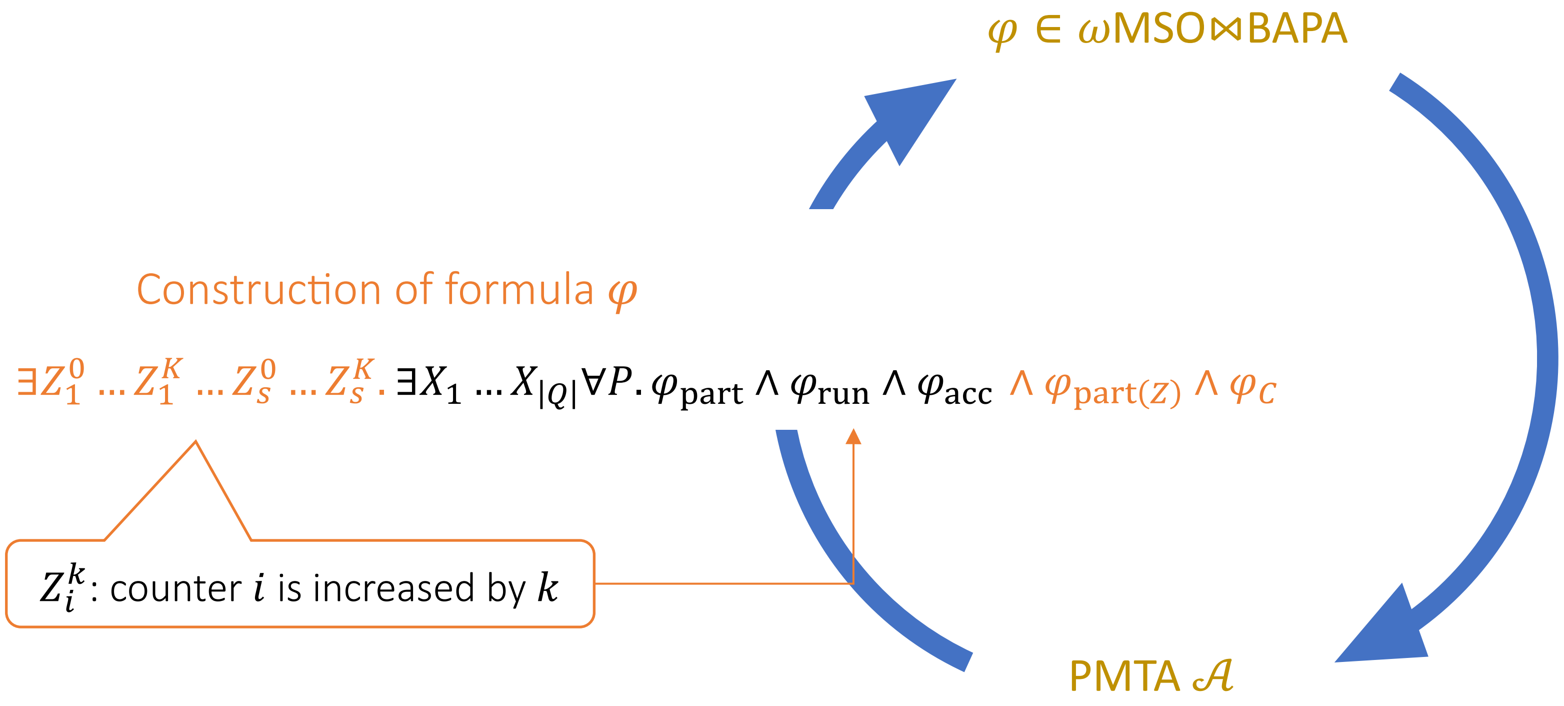

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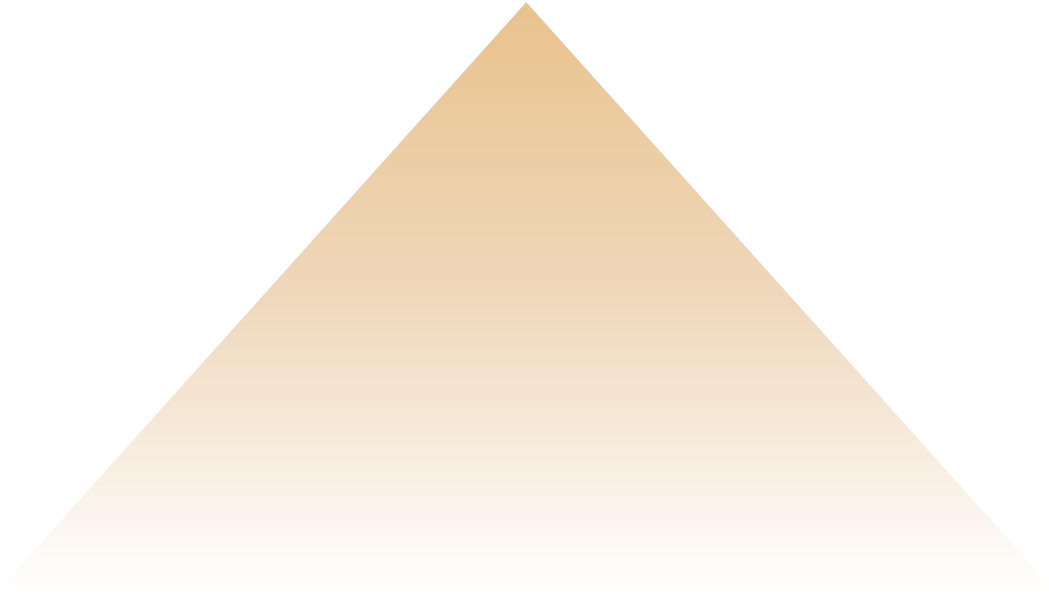
Theorem. ωMSO⋈BAPA = PMTA (on infinite labeled trees)

Emptiness of PMTA

Theorem. Given a PMTA $\mathcal{A}$, deciding $\mathcal{L}(\mathcal{A}) \neq \emptyset$ is PSPACE-complete.

Proof. Given PMTA $\mathcal{A} = (Q, \Xi, q_I, \Delta, \mathcal{F}, C)$ with $Q = Q_P \cup Q_\omega \cup \{q_I\}$.



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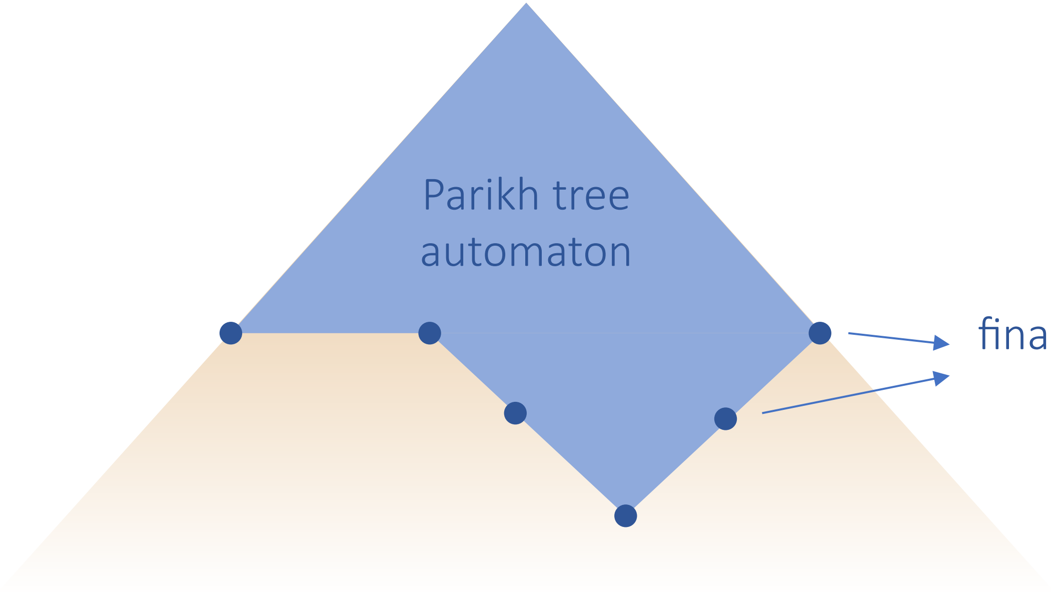

final states /

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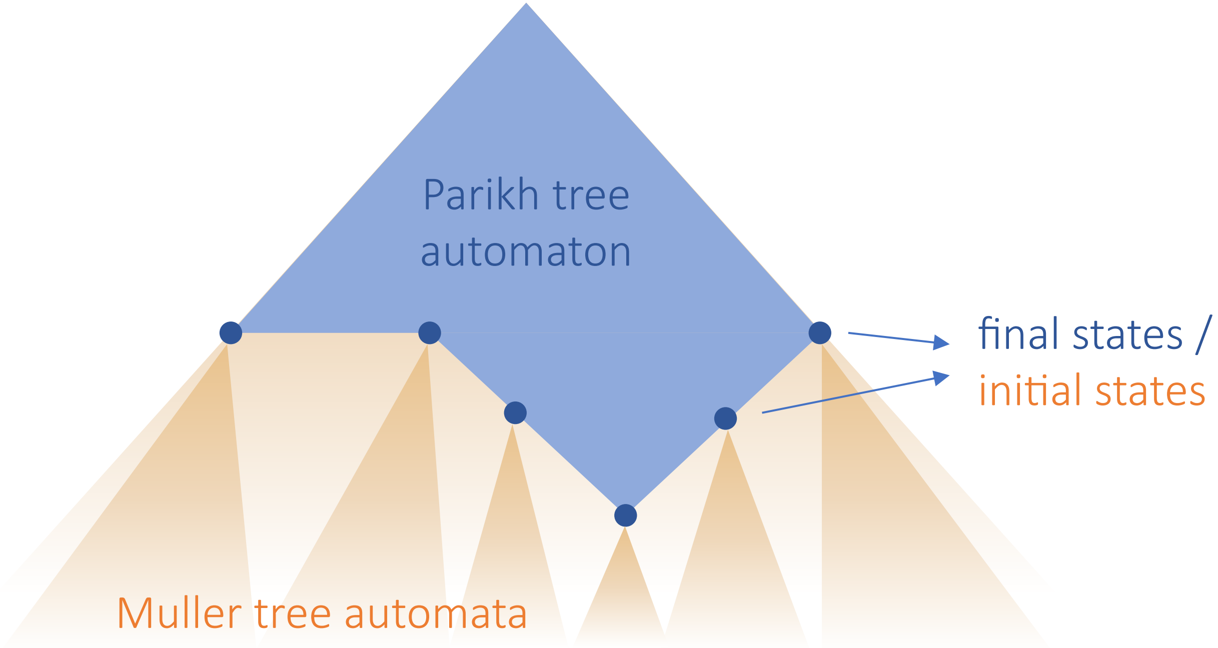

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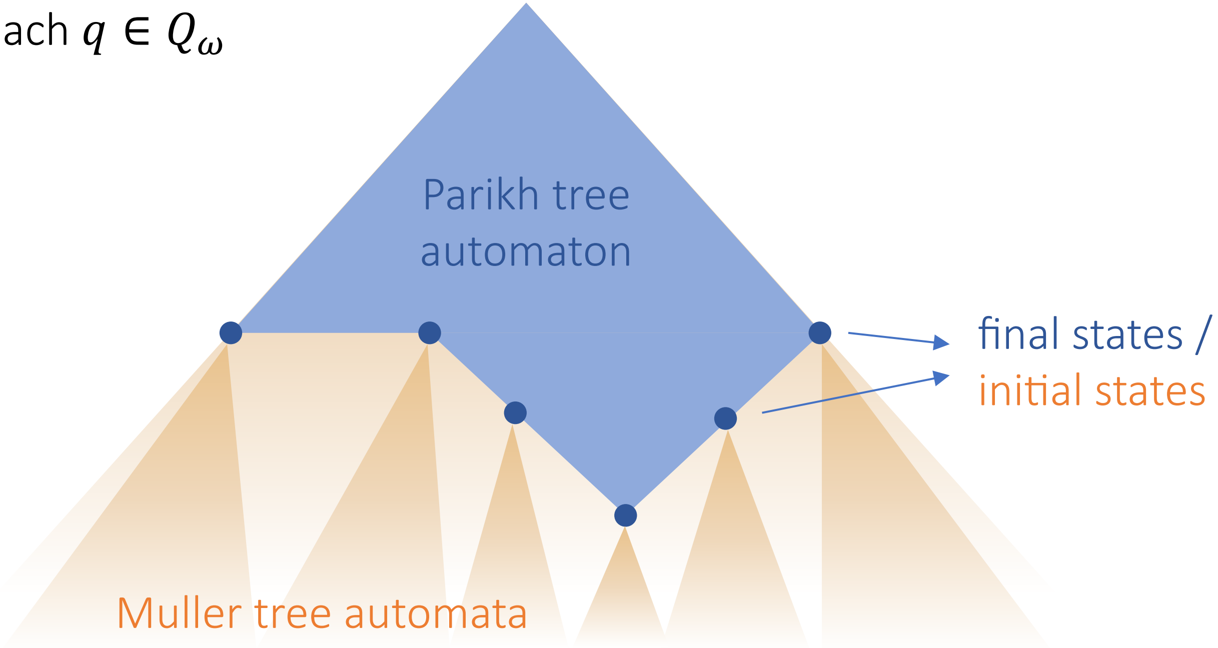
- $\mathcal{A}_q = (Q_\omega, q, \Delta_\omega, \mathcal{F})$ Muller tree automaton for each $q \in Q_\omega$

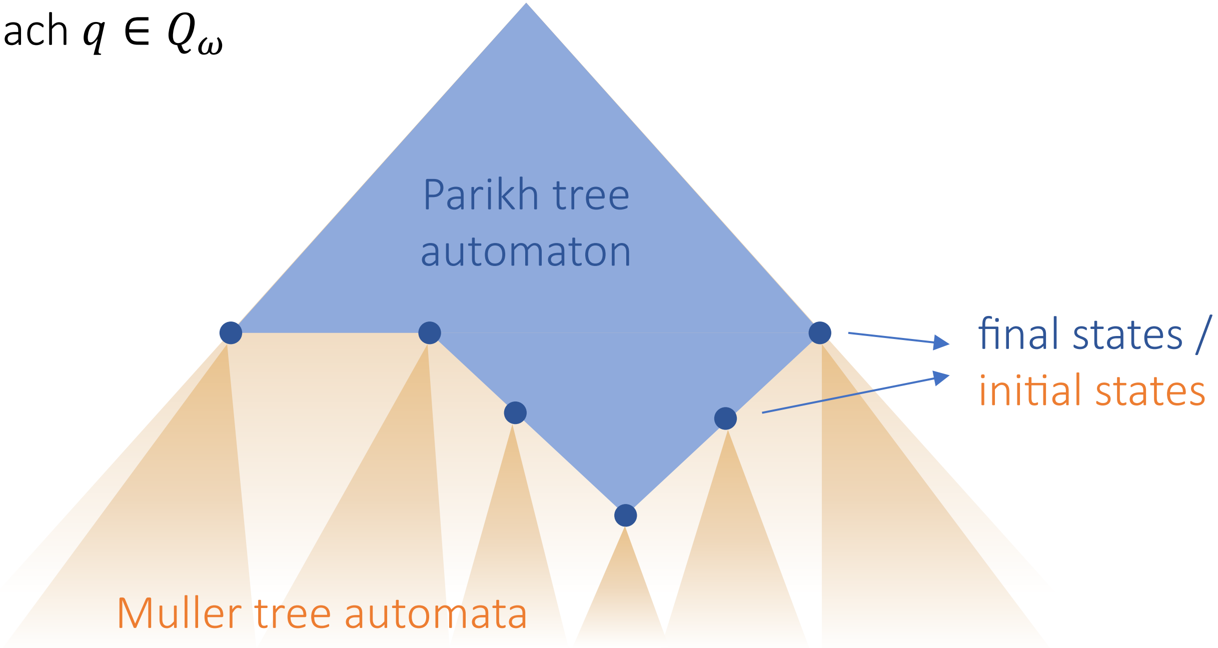


Emptiness of PMTA

Theorem. Given a PMTA $\mathcal{A}$, deciding $\mathcal{L}(\mathcal{A}) \neq \emptyset$ is PSPACE-complete.

Proof. Given PMTA $\mathcal{A} = (Q, \Xi, q_I, \Delta, \mathcal{F}, C)$ with $Q = Q_P \cup Q_\omega \cup \{q_I\}$.

- $\mathcal{A}_q = (Q_\omega, q, \Delta_\omega, \mathcal{F})$ Muller tree automaton for each $q \in Q_\omega$
- $F_P = \{q \in Q_\omega \mid \mathcal{L}(\mathcal{A}_q) \neq \emptyset\}$



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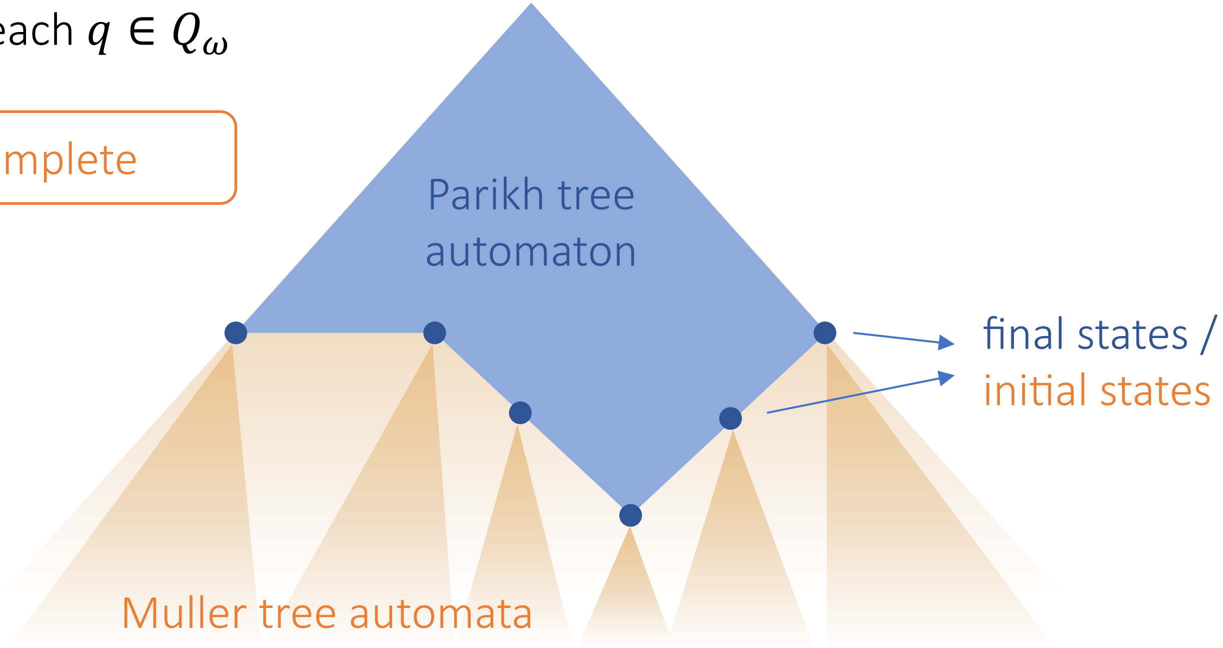

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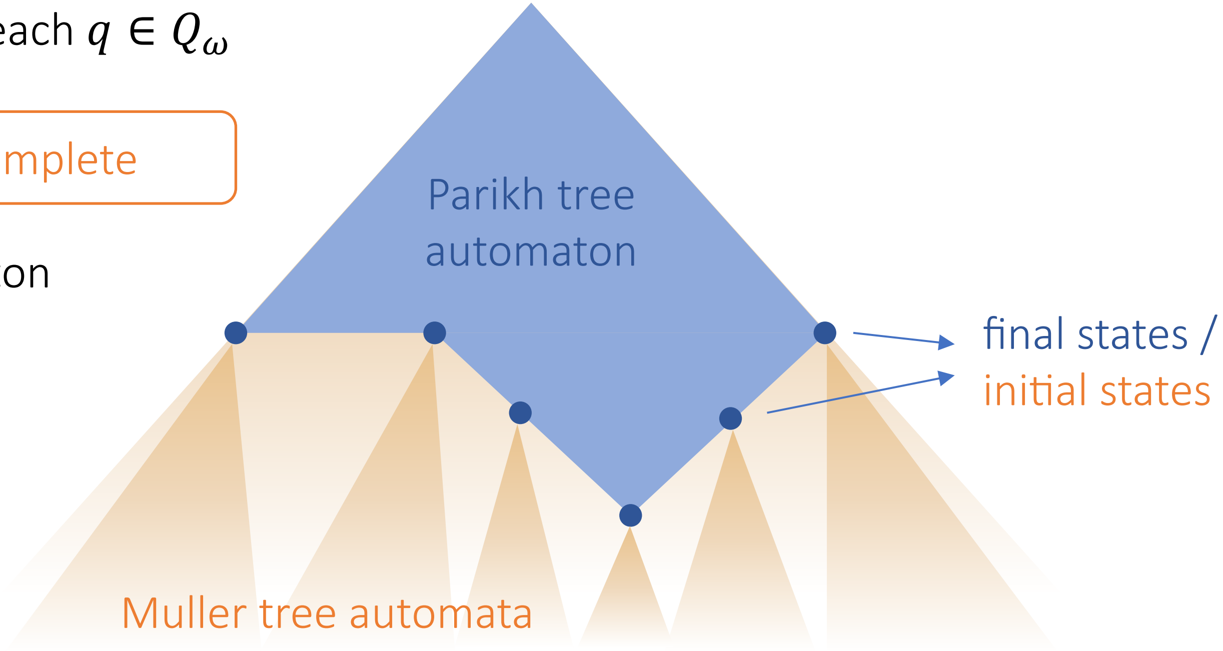
- $\mathcal{A}_P = (Q, \Sigma{\times}D, q_I, \Delta_P, F_p, C)$ Parikh tree automaton

$\mathcal{L}(\mathcal{A}) \neq \emptyset$ iff $\mathcal{L}(\mathcal{A}_P) \neq \emptyset$

NP-complete



Satisfiability of ωMSO⋈BAPA

Theorem. ωMSO⋈BAPA = PMTA (on infinite labeled trees)

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can be lifted with MSO-interpretations
to all tree-interpretable classes

Theorem. Satisfiability of ωMSO$\bowtie$BAPA is decidable over the classes of finite or countable $\mathbb{S}$-structures of bounded treewidth, cliquewidth, and partitionswidth.

Summary

- highly expressive logic ωMSO·BAPA for cardinality relationships → undecidable in general
- fragment ωMSO⋈BAPA: still expressive and admits normal form
- Parikh-Muller tree automata correspond to ωMSO⋈BAPA on infinite trees
- … and have a decidable emptiness problem
- satisfiability of ωMSO⋈BAPA on infinite trees and tree-interpretable classes is decidable
- decidability showcases: coupling with $\mathrm{FO}^2_{\mathrm{Pres}}$, µ-calculus with global Presburger constraints

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Thank you!