Sebastian Rudolph
International Center for Computational Logic
TU Dresden

Existential Rules – Lecture 3

Adapted from slides by Andreas Pieris and Michaël Thomazo
Winter Term 2025/2026

Syntax of Existential Rules

An existential rule is an expression

$$\forall \mathbf{X} \forall \mathbf{Y} \, (\underbrace{\varphi(\mathbf{X},\mathbf{Y})}_{\text{body}} \rightarrow \exists \mathbf{Z} \, \underbrace{\psi(\mathbf{X},\mathbf{Z})}_{\text{head}})$$

- $\mathbf{X}$,$\mathbf{Y}$ and $\mathbf{Z}$ are tuples of variables of $\mathbf{V}$
- $\varphi(\mathbf{X},\mathbf{Y})$ and $\psi(\mathbf{X},\mathbf{Z})$ are (constant-free) conjunctions of atoms

…a.k.a. tuple-generating dependencies, and Datalog$^{\pm}$ rules

Semantics of Existential Rules

- An instance J is a model of the rule

  $$\sigma = \forall \mathbf{X} \forall \mathbf{Y} \, (\varphi(\mathbf{X},\mathbf{Y}) \rightarrow \exists \mathbf{Z} \, \psi(\mathbf{X},\mathbf{Z}))$$

  written as $J \vDash \sigma$, if the following holds:

  whenever there exists a homomorphism h such that $h(\varphi(\mathbf{X},\mathbf{Y})) \subseteq J$,

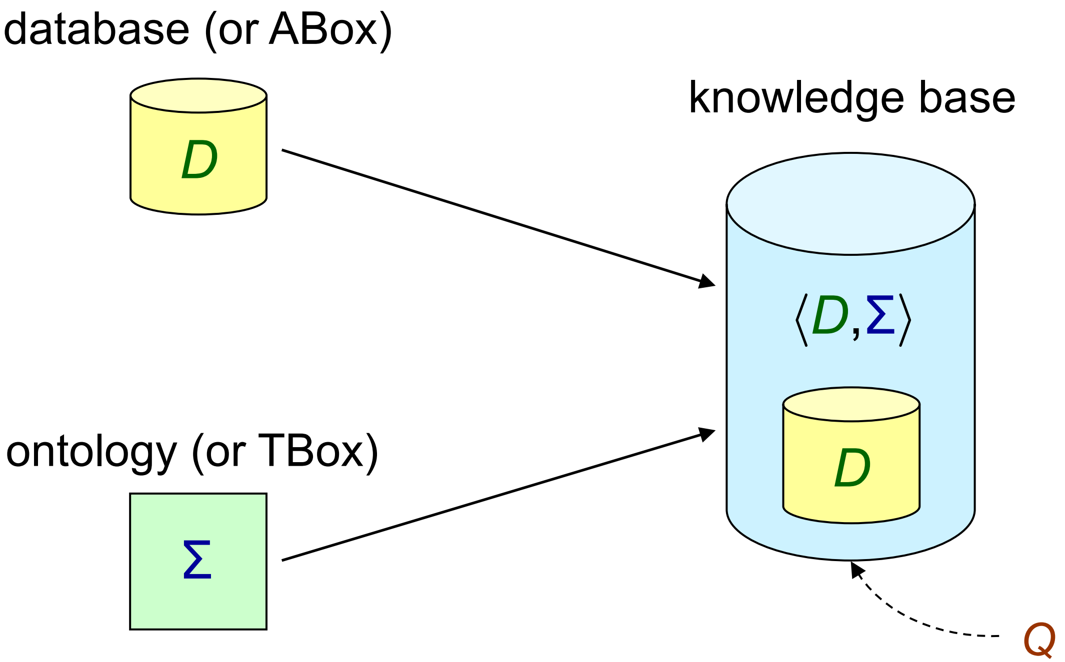
  then there exists $g \supseteq h_{|\mathbf{X}}$ such that $g(\psi(\mathbf{X},\mathbf{Z})) \subseteq J$

  $\{t \mapsto h(t) \mid t \in \mathbf{X}\}$ – the restriction of h to **X**

- Given a set Σ of existential rules, J is a model of Σ, written as $J \vDash \Sigma$, if the following holds: for each $\sigma \in \Sigma$, $J \vDash \sigma$

- It can be shown that $J \vDash \Sigma$ iff J is a model of the first-order theory $\bigwedge_{\sigma \in \Sigma} \sigma$

Ontology-Based Query Answering (OBQA)



existential rules

$$\forall \mathbf{X} \forall \mathbf{Y} \, (\varphi(\mathbf{X},\mathbf{Y}) \rightarrow \exists \mathbf{Z} \, \psi(\mathbf{X},\mathbf{Z}))$$

Syntax of Conjunctive Queries

A conjunctive query (CQ) is an expression

$$\exists \mathbf{Y}\ (\varphi(\mathbf{X},\mathbf{Y}))$$

- $\mathbf{X}$ and $\mathbf{Y}$ are tuples of variables of $\mathbf{V}$
- $\varphi(\mathbf{X},\mathbf{Y})$ is a conjunction of atoms (possibly with constants)

The most important query language used in practice

Forms the SELECT-FROM-WHERE fragment of SQL

Semantics of Conjunctive Queries

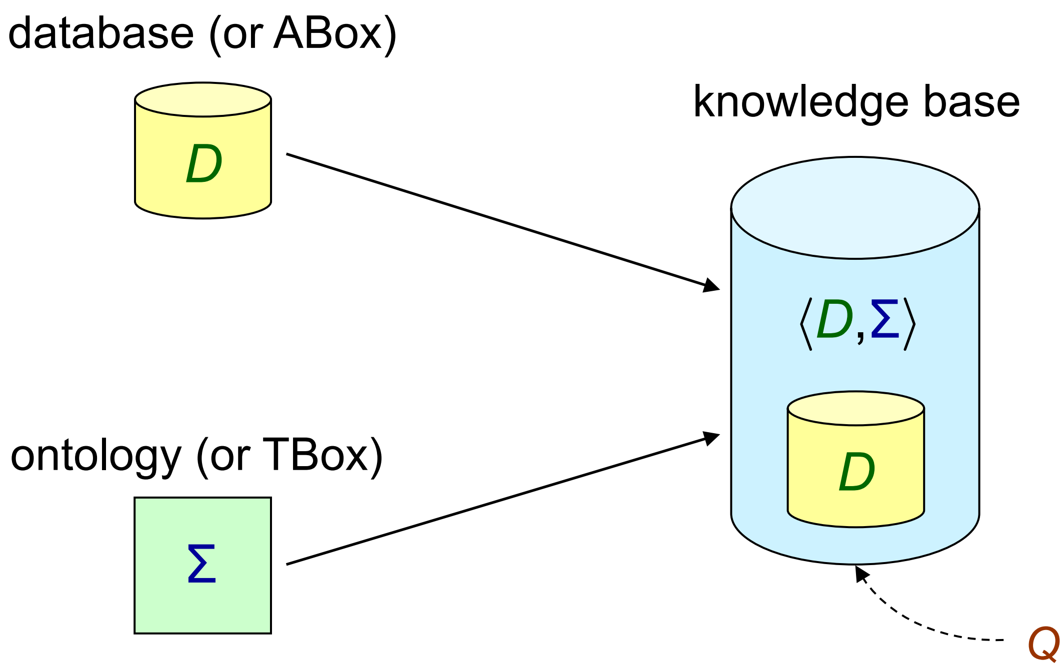
- A match of a CQ $\exists \mathbf{Y}\ (\varphi(\mathbf{X},\mathbf{Y}))$ in an instance J is a homomorphism h such that $h(\varphi(\mathbf{X},\mathbf{Y})) \subseteq J$ i.e., all the atoms of the query are satisfied

- The answer to $Q = \exists \mathbf{Y}\ (\varphi(\mathbf{X},\mathbf{Y}))$ over J is the set of tuples

$$Q(J) = \{h(\mathbf{X}) \mid h \text{ is a match of } Q \text{ in } J\}$$

- The answer consists of the witnesses for the free variables of the query

Ontology-Based Query Answering (OBQA)



existential rules

$\forall \mathbf{X} \forall \mathbf{Y} \; (\varphi(\mathbf{X},\mathbf{Y}) \rightarrow \exists \mathbf{Z} \; \psi(\mathbf{X},\mathbf{Z}))$

conjunctive queries

$\exists \mathbf{Y} \; (\varphi(\mathbf{X},\mathbf{Y}))$

OBQA: Formal Definition

active domain – constants occurring in D

CQ-Answering:

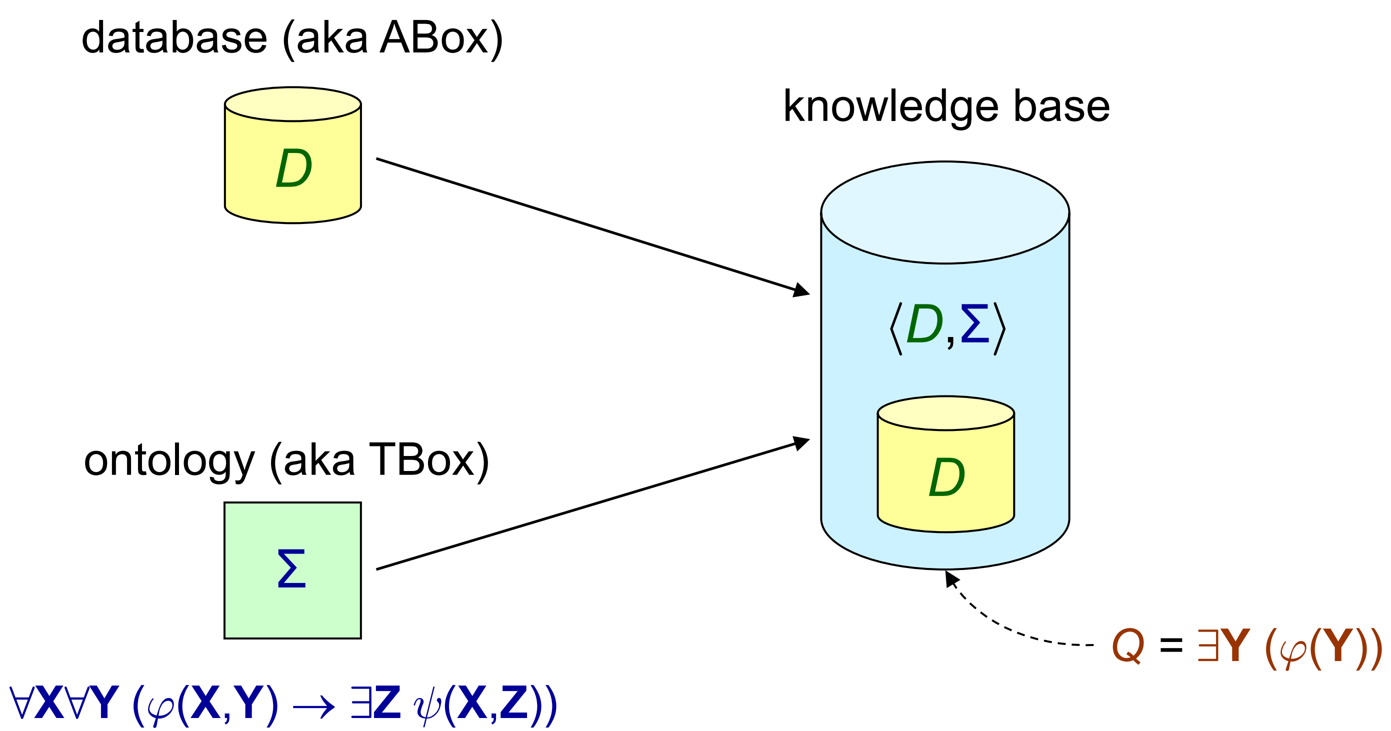
Input: database D, existential rules Σ, CQ $Q = \exists \mathbf{Y}\,(\varphi(\mathbf{X},\mathbf{Y}))$, tuple $t \in \mathrm{adom}(D)^{|\mathbf{X}|}$

Question: decide whether $\mathbf{t} \in \mathrm{certain}(Q,\langle D,\Sigma\rangle) = \bigcap_{J \in \mathrm{models}(D \wedge \Sigma)} Q(J)_{\downarrow}$

$$\begin{aligned}
\mathbf{t} \in \mathrm{certain}(Q,\langle D\ \Sigma\rangle) \quad & \text{iff} \quad \mathbf{t} \in \bigcap_{J \in \mathrm{models}(D \wedge \Sigma)} Q(J)_{\downarrow} \\
& \text{iff} \quad \forall J \in \mathrm{models}(D \wedge \Sigma),\ J \vDash \exists \mathbf{Y}\,(\varphi(\mathbf{t},\mathbf{Y})) \\
& \text{iff} \quad D \wedge \Sigma \vDash \exists \mathbf{Y}\,(\varphi(\mathbf{t},\mathbf{Y}))
\end{aligned}$$

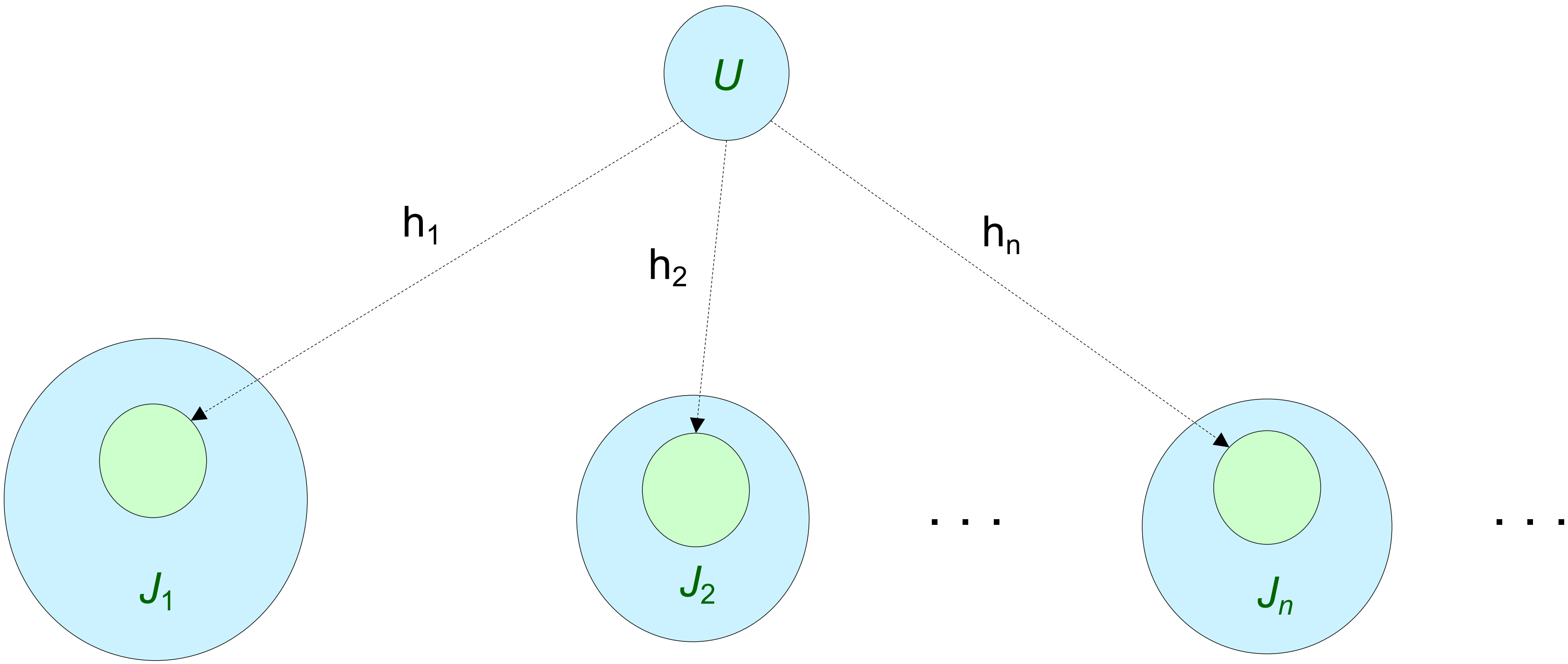
Boolean CQ (BCQ) – no free variables

BCQ-Answering: Our Main Decision Problem



decide whether $D \wedge \Sigma \models Q$

Universal Models (a.k.a. Canonical Models)



An instance U is a universal model of $D \wedge \Sigma$ if the following holds:

1. U is a model of $D \wedge \Sigma$

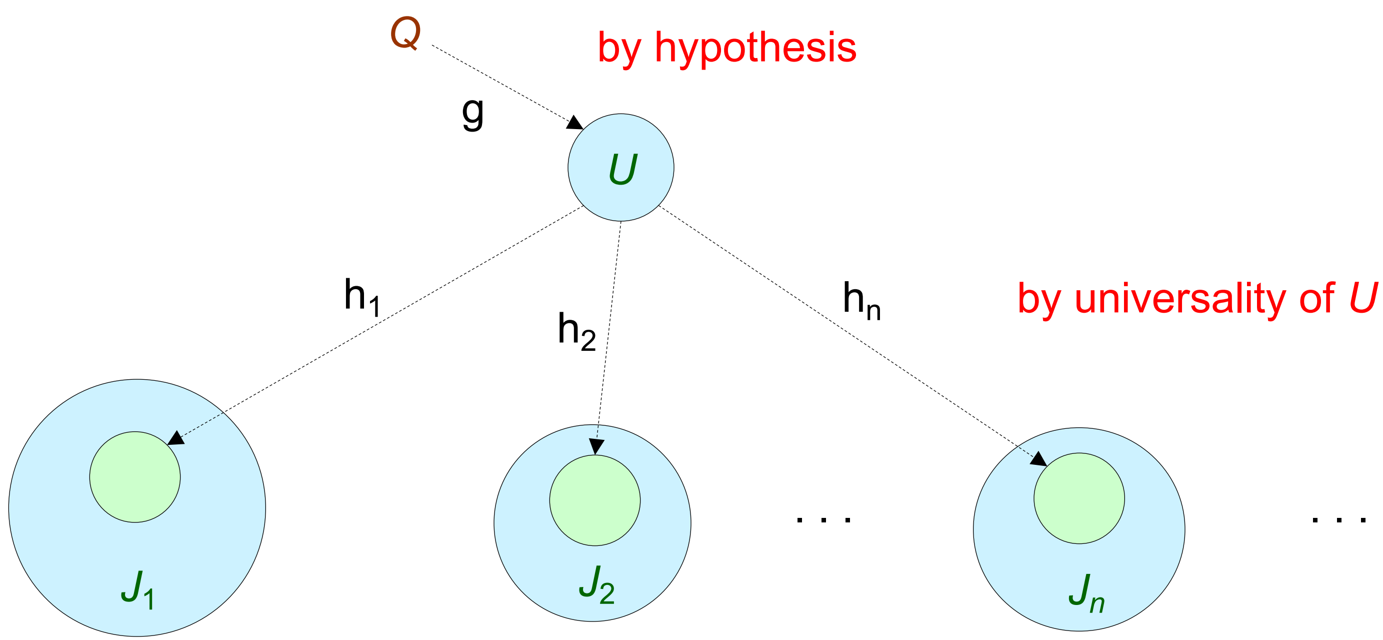
2. $\forall J \in \text{models}(D \wedge \Sigma)$, there exists a homomorphism h_J such that $h_J(U) \subseteq J$

Query Answering via Universal Models

Theorem: $D \wedge \Sigma \models Q$ iff $U \models Q$, where U is a universal model of $D \wedge \Sigma$

Proof: (⇒) Trivial since, for every $J \in$ models($D \wedge \Sigma$), $J \models Q$

(⇐) By exploiting the universality of U



$\forall J \in$ models($D \wedge \Sigma$), $\exists h_J$ such that $h_J(g(Q)) \subseteq J$ $\Rightarrow$ $\forall J \in$ models($D \wedge \Sigma$), $J \models Q$

$\Rightarrow$ $D \wedge \Sigma \models Q$

The Chase Procedure

- Fundamental algorithmic tool used in databases

- It has been applied to a wide range of problems:
    - Checking containment of queries under constraints
    - Computing data exchange solutions
    - Computing certain answers in data integration settings
    - …

… what's the reason for the ubiquity of the chase in databases?

it constructs universal models

The Chase Procedure

D

person(Alice)

Σ

$\forall X\ (\mathit{Person}(X) \rightarrow \exists Y\ (\mathit{hasParent}(X,Y) \wedge \mathit{Person}(Y)))$

$\mathrm{chase}(D,\Sigma) = D \cup$

The Chase Procedure

D

person(Alice)

Σ

$\forall X\ (\mathit{Person}(X) \rightarrow \exists Y\ (\mathit{hasParent}(X,Y) \wedge \mathit{Person}(Y)))$

$\mathrm{chase}(D,\Sigma) = D \cup \{\mathit{hasParent}(\mathrm{Alice}, z_1), \mathit{Person}(z_1)$

The Chase Procedure

D

person(Alice)

Σ

$\forall X\ (\mathit{Person}(X) \rightarrow \exists Y\ (\mathit{hasParent}(X,Y) \wedge \mathit{Person}(Y)))$

$\mathrm{chase}(D,\Sigma) = D \cup \{\mathit{hasParent}(\mathrm{Alice}, z_1),\ \mathit{Person}(z_1),$

$\mathit{hasParent}(z_1, z_2),\ \mathit{Person}(z_2)$

The Chase Procedure

D

person(Alice)

Σ

$\forall X\ (Person(X) \rightarrow \exists Y\ (hasParent(X,Y) \wedge Person(Y)))$

chase(D,Σ) = $D \cup \{hasParent(Alice, z_1), Person(z_1),$

$hasParent(z_1, z_2), Person(z_2),$

$hasParent(z_2, z_3), Person(z_3)$

The Chase Procedure

D

person(Alice)

Σ

$\forall X\,(\mathit{Person}(X) \rightarrow \exists Y\,(\mathit{hasParent}(X,Y) \wedge \mathit{Person}(Y)))$

$\text{chase}(D,\Sigma) = D \cup \{\mathit{hasParent}(\text{Alice}, z_1), \mathit{Person}(z_1),$

$\mathit{hasParent}(z_1, z_2), \mathit{Person}(z_2),$

$\mathit{hasParent}(z_2, z_3), \mathit{Person}(z_3), \ldots$

infinite instance

The Chase Procedure: Formal Definition

- Chase rule - the building block of the chase procedure

- A rule $\sigma = \forall \mathbf{X} \forall \mathbf{Y} \, (\varphi(\mathbf{X},\mathbf{Y}) \rightarrow \exists \mathbf{Z} \, \psi(\mathbf{X},\mathbf{Z}))$ is applicable to instance J if:
   1. There exists a homomorphism h such that $h(\varphi(\mathbf{X},\mathbf{Y})) \subseteq J$
   2. There is no $g \supseteq h_{|\mathbf{X}}$ such that $g(\psi(\mathbf{X},\mathbf{Z})) \subseteq J$

$J = \{R(a), P(a,b)\}$

$h = \{X \rightarrow a\}$ $\quad$ $g = \{X \rightarrow a, Y \rightarrow b\}$

$\forall X \, (R(X) \rightarrow \exists Y \, P(X,Y))$

✗

$J = \{R(a), P(b,a)\}$

$h = \{X \rightarrow a\}$ $\quad$ ✗

$\forall X \, (R(X) \rightarrow \exists Y \, P(X,Y))$

✓

The Chase Procedure: Formal Definition

- Chase rule - the building block of the chase procedure

- A rule $\sigma = \forall \mathbf{X} \forall \mathbf{Y} \, (\varphi(\mathbf{X},\mathbf{Y}) \rightarrow \exists \mathbf{Z} \, \psi(\mathbf{X},\mathbf{Z}))$ is applicable to instance J if:
    1. There exists a homomorphism h such that $h(\varphi(\mathbf{X},\mathbf{Y})) \subseteq J$
    2. There is no $g \supseteq h_{|\mathbf{X}}$ such that $g(\psi(\mathbf{X},\mathbf{Z})) \subseteq J$

- Let $J_{+} = J \cup \{g(\psi(\mathbf{X},\mathbf{Z}))\}$, where $g \supseteq h_{|\mathbf{X}}$ and $g(\mathbf{Z})$ are "fresh" nulls not in J

- The result of applying σ to J is J_{+}, denoted $J\langle\sigma,h\rangle J_{+}$ - single chase step

The Chase Procedure: Formal Definition

- A finite chase of D w.r.t. Σ is a finite sequence

$$D\langle\sigma_1,h_1\rangle J_1\langle\sigma_2,h_2\rangle J_2\langle\sigma_3,h_3\rangle J_3 \ \dots\ \langle\sigma_n,h_n\rangle J_n$$

  where no rule from Σ is applicable in J_n .

  Then, chase(D,Σ) is defined as the instance J_n

all applicable rules will eventually be applied

- An infinite chase of D w.r.t. Σ is a fair finite sequence

$$D\langle\sigma_1,h_1\rangle J_1\langle\sigma_2,h_2\rangle J_2\langle\sigma_3,h_3\rangle J_3 \ \dots\ \langle\sigma_n,h_n\rangle J_n \dots$$

  and chase(D,Σ) is defined as the instance $\cup_{k\geq 0} J_k$ (with $J_0 = D$)

least fixpoint of a monotonic operator - chase step

Chase: A Universal Model

Theorem: chase(D,Σ) is a universal model of $D \wedge \Sigma$

the result of the chase after k applications of the chase step

Proof:

- By construction, chase(D,Σ) $\in$ models($D \wedge \Sigma$)
- It remains to show that chase(D, Σ) can be homomorphically embedded into every other model of $D \wedge \Sigma$
- Fix an arbitrary instance $J \in$ models($D \wedge \Sigma$). We need to show that there exists h such that h(chase(D,Σ)) $\subseteq J$
- By induction on the number of applications of the chase step, we show that for every $k \geq 0$, there exists h_k such that $h_k(\text{chase}^{[k]}(D,\Sigma)) \subseteq J$, and h_k is compatible with h_{k-1}
- Clearly, $\cup_{k \geq 0}$ h_k is a well-defined homomorphism that maps chase(D,Σ) to J
- The claim follows with $h = \cup_{k \geq 0} h_k$

Chase: Uniqueness Property

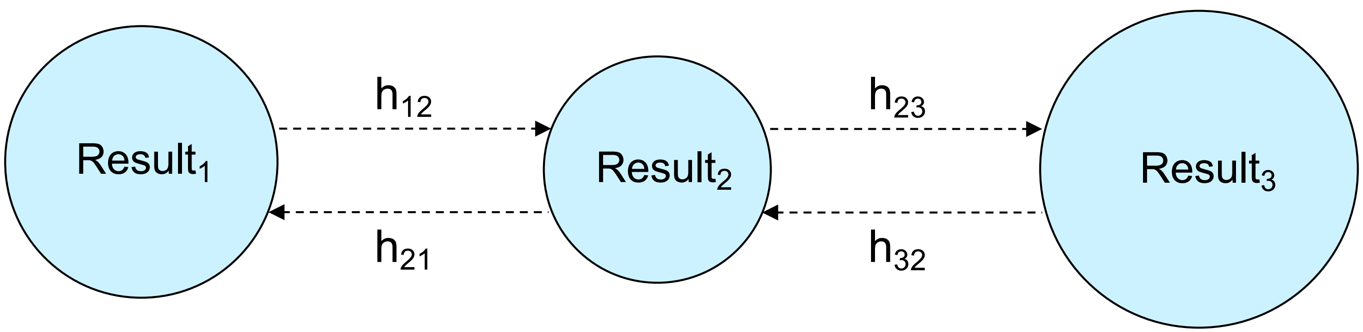
- The result of the chase is not unique - depends on the order of rule application

$D = \{P(a)\}$ $\sigma_1 = \forall X\,(P(X) \rightarrow \exists Y\, R(Y))$ $\sigma_2 = \forall X\,(P(X) \rightarrow R(X))$

$\text{Result}_1 = \{P(a), R(z), R(a)\}$ σ_1 then σ_2

$\text{Result}_2 = \{P(a), R(a)\}$ σ_2 then σ_1

- But, it is unique up to homomorphic equivalence



- Thus, it is unique for query answering purposes

Query Answering via the Chase

Theorem: $D \wedge \Sigma \vDash Q$ iff $U \vDash Q$, where U is a universal model of $D \wedge \Sigma$

+

Theorem: chase(D, Σ) is a universal model of $D \wedge \Sigma$

=

Corollary: $D \wedge \Sigma \vDash Q$ iff chase(D,Σ) $\vDash Q$

- We can tame the first dimension of infinity by exploiting the chase procedure
- But, what about the second dimension of infinity? - the chase may be infinite

Rest of the Lectrure

- Undecidability of BCQ-Answering
- Gaining decidability - terminating chase
- Full Existential Rules
- Acyclic Existential Rules

Undecidability of BCQ-Answering

Theorem: BCQ-Answering is undecidable

Proof : By simulating a deterministic Turing machine with an empty tape

Deterministic Turing Machine (DTM)

$S \setminus \{s_{acc}\} \times \Lambda \to S \times \Lambda \times \{-1,0,+1\}$

accepting state

$$M \;=\; (S, \Lambda, \sqcup, \delta, s_0, s_{acc})$$

states

tape symbols

blank symbol

initial state

$\delta(s_1, \alpha) = (s_2, \beta, +1)$

IF at some time instant τ the machine is in sate s_1, the cursor points to cell κ, and this cell contains α

THEN at instant τ+1 the machine is in state s_2, cell κ contains β, and the cursor points to cell κ+1

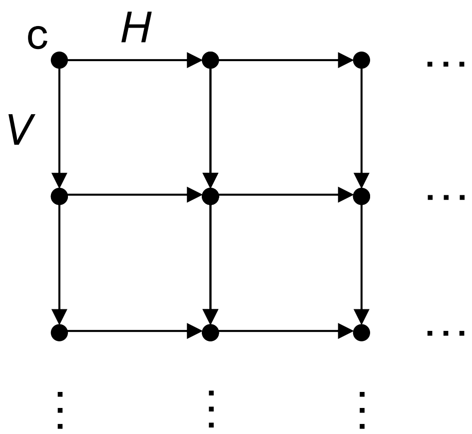
Undecidability of BCQ-Answering

Our Goal: Encode the computation of a DTM *M* with an empty tape

using a database *D*, a set Σ of existential rules, and a BCQ *Q* such that

$D \wedge \Sigma \models Q$ iff *M* accepts

Build an Infinite Grid



k-th horizontal line represents the

k-th configuration of the machine

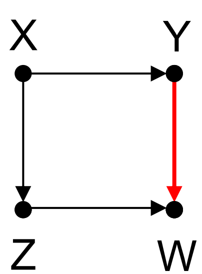
$\forall X\ (\mathit{Start}(X) \rightarrow \mathit{Node}(X) \wedge \mathit{Initial}(X))$

$D = \{\mathit{Start}(c)\}$

fixes the origin of the grid

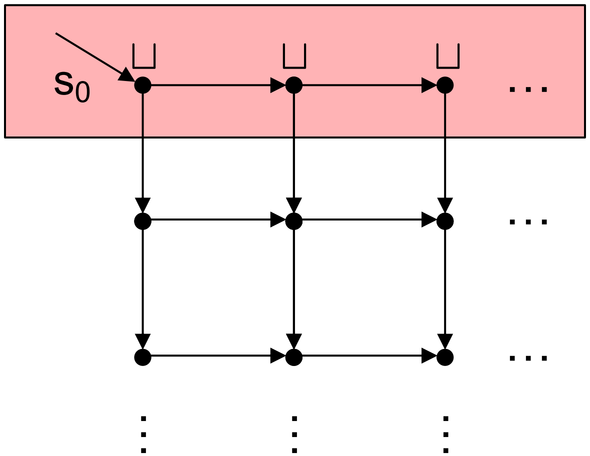
$\forall X\ (\mathit{Node}(X) \rightarrow \exists Y\ (H(X,Y) \wedge \mathit{Node}(Y)))$

$\forall X\ (\mathit{Node}(X) \rightarrow \exists Y\ (V(X,Y) \wedge \mathit{Node}(Y)))$



$\forall X \forall Y \forall Z \forall W\ (H(X,Y)\ H(Z,W)\ V(X,Z) \rightarrow V(Y,W))$

Initialization Rules

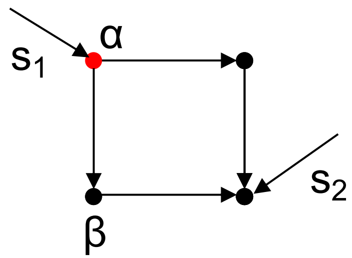


$\forall X \forall Y\ (\mathit{Initial}(X) \wedge H(X,Y) \rightarrow \mathit{Initial}(Y))$

$\forall X\ (\mathit{Start}(X) \rightarrow \mathit{Cursor}[s_0](X))$

$\forall X\ (\mathit{Initial}(X) \rightarrow \mathit{Symbol}[\sqcup](X))$

Transition Rules

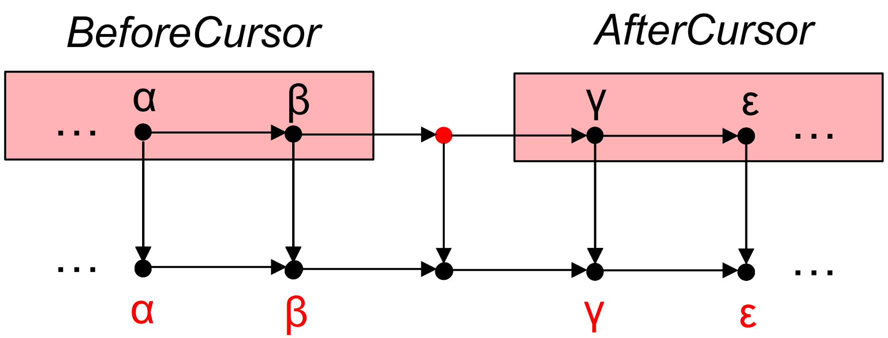


$\delta(s_1,\alpha) \;=\; (s_2,\beta,+1)$

$\forall X \forall Y \forall Z\ (\mathit{Cursor}[s_1](X) \wedge \mathit{Symbol}[\alpha](X) \wedge V(X,Y) \wedge H(Y,Z) \rightarrow$

$\mathit{Cursor}[s_2](Z) \wedge \mathit{Symbol}[\beta](Y) \wedge \mathit{Mark}(X))$

Inertia Rules



$$\forall X \forall Y\, (\mathit{Mark}(X) \wedge H(X,Y) \rightarrow \mathit{AfterCursor}(Y))$$

$$\forall X \forall Y\, (\mathit{AfterCursor}(X) \wedge H(X,Y) \rightarrow \mathit{AfterCursor}(Y))$$

$$\forall X \forall Y\, (\mathit{AfterCursor}(X) \wedge \mathit{Symbol}[\alpha](X) \wedge V(X,Y) \rightarrow \mathit{Symbol}[\alpha](Y))$$

…we have similar rules for the cells before the cursor

Accepting Rule

Once we reach the accepting state we accept

$$\forall X\ (\mathit{Cursor}[s_{acc}](X) \rightarrow \mathit{Accept}(X))$$

$D \wedge \Sigma \models \exists X\ \mathit{Accept}(X)$ iff the DTM M accepts

Undecidability of BCQ-Answering

Theorem: BCQ-Answering is undecidable

Proof : By simulating a deterministic Turing machine with an empty tape

…syntactic restrictions are needed!!!