Guidelines

for the collection of problems and solutions

Version from August 6, 2014

3 Propositional Logic

3.1 Syntax

3.1.1 Formulae

In the focus are the problems

3.3 Existence of $L(R)$
3.4 Construction of $L(R)$
3.5 Proof of $L(R) = K(R)$

You should have understood the problems

3.6 Do smallest sets always exist?
3.9 Infinite formulae?

3.1.2 Structural Induction and Recursion

This section comprises lots of problems for training proofs by mathematical induction and by structural induction, as well as for defining functions recursive functions.

You should be familiar with these techniques.

Problem

3.23 Fallacious ‘proofs’ by induction

is highly recommended to be studied carefully, because it teaches what may go wrong with a proof by induction when not working carefully.

3.1.3 Subformulae

Problem

3.27 Verifying a set of subformulae

shows what it means to work directly with the definition of subformulae.

In the focus are the problems

3.28 Existence of sets of subformulae
3.30 Construction of sets of subformulae
3.31 Constructability of sets of subformulae

which are straightforward analogues to the problems 3.3, 3.4 and 3.5.

The problems

3.29 Pseudosets of subformulae
3.35 Sets of pseudo-subformulae vs. set of subformulae

about the very artificial concept of pseudo-subformulae illustrate that it's not always straightforward to turn closure properties into a constructive view.

3.2 Semantics

3.2.1 The Structure of the Truth Values

You should be aware of the contents of problems

3.38 Unary and Binary Truth Functions
3.39 Example with the connectives $\swarrow$ and $\not\leftrightarrow$

3.2.2 Interpretations

You should be aware of the contents of problem

3.41 Proposition 3.11: F^I is determined by I on R_F

3.2.3 Truth Tables

In the focus is the problem

3.50 Completeness and correctness

3.2.4 Models

Just lots of easy problems which shall help to get acquainted with the concept of a model.

3.2.5 Logical Consequence

In the focus are the problems

3.65 Deduction theorem
3.67 Ex falso quodlibet seguitur
3.68 Monotonicity of propositional logic

The problems

3.56 Short Questions about $I \models F$ and $F \models G$
3.59 Questions about $F \models G$
3.60 Questions about $F \models G$
3.61 Concerning $F \models \neg G$ and $\neg H \models F$
3.62 Logical Consequence and FG
3.63 Disjoint sets of variables in $F \models G$
3.64 Satisfiability of sets of formulae

are relatively easy. But we observed again and again that many students have problems with them, also quite often due to problems with English.

3.3 Equivalence and Normal Forms

3.3.1 Semantic Equivalence

You should know about

3.74 A complete system of connectives: $\{\neg, \vee\}$
3.78 Semantical equivalence as equivalence relation

In the focus are the problems about the relationship between models and formulae

3.80 Formulae with p_1 and p_2
3.81 Formulae with $p_1, \ldots, p_n$ [1]

In the focus are the problems about the semantical consequences of replacement of subformulae

3.85 Formula replacement
3.86 Replacement theorem (Proof completion)
3.87 Inverse of the replacement theorem?
3.88 Relative replacement theorem
3.89 Variable replacement in a tautology

You should know about the following problems of complexity and the ways to get around it.

3.92 Length explosion due to $\leftrightarrow$ elimination
3.93 Definitional Transformation: Satisfiability
3.94 Definitional transformation: Linear growth

3.3.2 Negation Normal Form

You should be able to solve problems like

3.95 Examples of NNF transformation

3.3.3 Clausal Form

You should be able to transform a formula stepwise into clausal form according to the algorithm given in the lectures., as for instance in

3.104 Examples of CF transformations
3.115 Transformation into NNF, CF and DCF

and in several other problems.

In the focus are the problems on the theory of the CF transformation:

3.106 Proof of lemma 3.29
3.107 Multisets of CF formulae
3.108 Multisets with the CF transformation
3.110 Termination of the CF transformation

[1] The solution of subproblem (e) is buggy. Here is a corrected version of the last paragraph:
On the other hand, every full conjunction $K' = \langle L'_1, \ldots, L'_n \rangle$ (of length n), different from K, is false under the interpretations in $\mathcal{I}$, since K and K' must differ in at least one of the L_i resp. L'_i – i.e. $L_i = \neg L'_i$ resp. $L'_i = \neg L_i$ for at least one $i \in \{1, \ldots, n\}$ – and consequently, $[L'_i]^I = \bot$ holds for at least one $i \in \{1, \ldots, n\}$, which implies $[K']^I = \bot$.

3.3.4 A Prolog implementation

Not relevant since PROLOG is not part of this course.

3.4 Proof Methods

3.4.1 Resolution

You should be able to construct a resolution derivation/refutation and to prove a formula by resolution.

3.128 Resolution Derivations
3.129 Applications of the Resolution Method

You should be aware of the following ways to improve the resolution methods.

3.136 Tautology Elimination
3.137 Positive/negative clauses and satisfiability
3.138 Subsumption
3.139 Pure literals
3.140 (Un)satisfiable Sets of Clauses
3.141 Auto-resolvable Clauses

3.4.2 Semantic Tableaux

Not part of the lectures.

3.4.3 Calculus of natural deduction

You should be able to construct derivations by natural deduction. Train with the examples given in

3.149 Derivations with natural deduction

Additional examples on Reasoning with and about natural deduction

3.150 Derivations of conjunctions/disjunctions
3.151 An additional $\wedge$-introduction rule
3.152 Lemmata and theorems
3.153 From derivations to theorems

3.4.4 Other Proof Systems and Calculi

Outside of the course

3.5 Satisfiability Testing

Outside of the course

3.6 Properties

3.6.1 Compactness Theorem

3.171 Lemma from Proof of Compactness Theorem
3.172 Example: from models In to the model I

and look at some examples of the application of the Compactness Theorem.

3.173 Compactness Theorem: Example I
3.174 Compactness Theorem: Example II
3.175 Compactness Theorem: Example III
3.176 Compactness Theorem: Example IV
3.177 Satisfiable finite subsets
3.179 Consequences and a finite number of premises

3.6.2 Correctness and Completeness

The following problems complete important lemmata/theorems of the lectures.

3.182 Deletion of Clauses and Literals
3.183 Proof of Corollary 3.48

The following two problem link resolution and natural deduction

3.187 Natural deduction and resolution: I
3.188 Natural deduction and resolution: II

4 First Order Logic

4.1 Syntax

Just a couple of simple problems about the Syntax of First Order Logic, and about how to make proofs by structural induction and how to define recursive functions.

Of course, we can ask again the old questions concerning the existence and constructability of (the sets of) terms and subterms, and formulae and subformulae (see e.g. Problem 4.9).

4.2 Substitutions

In the focus are the problems

4.17 Substitution composition is a substitution
4.18 Substitution composition is not commutative
4.20 $t\widehat{\sigma\theta} = (t\hat{\sigma})\hat{\theta}$

Have a look at further problems to increase familiarity with substitutions.

4.3 Semantics

4.3.1 Interpretations

You should be able to evaluate *stepwise* a given formula on a given interpretation as exemplified by most of the examples in this section.

4.3.2 Herbrand-Interpretations

In the focus are the problems

4.56 Existence of Herbrand interpretations
4.57 How Herbrand interpretations differ
4.58 Existence of non-Herbrand interpretations
4.62 Substitutions vs. variable assignments

4.3.3 Models for Closed Formulas

In the focus are the problems

4.72 False in singleton domains
4.73 Formulae without finite models

4.3.4 Models for Non-Closed Formulas

4.4 Equivalence and Normal Forms

4.4.1 Semantic Equivalence

You should be able to prove semantical equivalences as exemplified by the first problems in this section.

4.4.2 Prenex Normal Form

You should be able to transform a formula *stepwise* into Prenex Normal Form.

4.4.3 Skolem Normal Form

You should be able to transform a formula *stepwise* into Skolem Normal Form.

Important are also:

4.93 Loss of model with Skolemization
4.96 Interpretations of Skolem functions

4.4.4 Clause Form

You should be able to transform a formula *stepwise* into Clause Form.

4.5 Unification

You should be able to apply *stepwise* the unification algorithm; see problem **4.101**: Application of the unification algorithm.

Important are also:

4.104 Most general and not most general unifiers
4.105 The relations $\geq$ and $\sim$ on substitutions
4.108 Ordering unifiers

4.111 Termination of the unification algorithm

4.6 Proof Methods

4.6.1 Resolution

You should be able to prove formulae *stepwise* by the resolution method; see:

4.114 Computing all resolvents
4.115 Resolution application in full detail
4.116 Resolution proofs stepwise
4.117 Validity proofs with resolution

See in addition

4.119 Derivability of the empty clause?
4.126 Examples about first order subsumption

Moreover, pay attention to

4.127 Applicability of 1-order subsumption
4.128 Necessity of factorization
4.129 Resolution comparison: Propositional vs. first order

4.6.2 Semantic Tableaux

4.6.3 Calculus of Natural Deduction

You should be able to make simple proofs of theorems by Natural Deduction; see e.g. problem

4.131 Natural deduction: first order examples

Moreover, pay attention to

4.132 Conditions on the $(\exists E)$ rule

4.6.4 Other Methods

4.8 Properties

4.8.1 Herbrand interpretations

Get a practical feeling for corresponding Herbrand models, e.g.:

4.141 Example of corresponding Herbrand model

4.8.2 Soundness and Completeness

In the focus are the problems

4.143 Proof completion: resolution lemma 4.63
4.144 Proof completion: lifting lemma 4.65
4.145 Proof completion: lemma 4.66

4.8.3 Compactness

4.8.4 Undecidability