Course Tutors

David Carral

Markus Krötzsch

David Carral, April 16, 2019  Existential Rules Seminar  slide 2 of 24
Acknowledgements

Content
Michaël Thomazo and Andreas Pieris
Student Session at ESSLLI
More info at http://esslli-stus-2015.phil.hhu.de/

Beamer Style
Markus Krötzsch

Tips on How to Run a Seminar
Lukas Schweizer
Organisation

Lectures
Tuesdays, DS 5 (14:50–16:20), APB E005

Web Page

Lecture Notes
All slides will be available online.
(Non-)Prerequisites

- First-order logic (syntax and semantics).
- Complexity theory (complexity classes, reductions...).
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- First-order logic (syntax and semantics).
- Complexity theory (complexity classes, reductions…).

Reading list

- Uwe Schöning: *Logic for Computer Scientists*; Birkhäuser.
Structure of the Seminar and Evaluation

Lectures

- **April 2, 2019**: Introductory lecture 1
- **April 9, 2019 (i.e., today)**: Introductory lecture 2
- **Afterwards**: Office hours in APB 3035 and presentations

Evaluation

- **Paper summary**: self-selected research paper; a 10 pages
- **Presentation**: 20 minutes + discussion

See the “Literature” tab at: https://iccl.inf.tu-dresden.de/web/Seminar_Existential_Rules_(SS2019).
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Motivation: Accessing Big Data

“Data is stored in various heterogeneous formats over many differently structured databases. As a result, the gathering of only relevant data spread over disparate sources becomes a very time consuming task.” – Jim Crompton, W3C Workshop on Semantic Web in Oil & Gas Industry, 2008

More info at: http://www.expertsytem.com/semantic-web-in-oil-gas-industry/
Experts in geology and geophysics develop stratigraphic models of unexplored areas on the basis of data acquired from previous operations at nearby geographical locations.

Facts:

- 1000 TB of relational data
- Using diverse schemata
- Spread over 2000 tables, over multiple individual data bases
A Possible Solution

- Achieve transparency in accessing data using logic – e.g., existential rules!
- Manage data by exploiting Knowledge Representation techniques.
- Provide a conceptual, high level representation of the domain of interest of terms of an ontology (i.e., a logical theory).
A Simple Example

Example 1.1:

\[ HasSon(x, y) \land HasSister(y, z) \rightarrow HasDaughter(x, z) \]  
\[ HasFather(x, y) \rightarrow HasSon(y, x) \]  
\[ Person(x) \rightarrow \exists y. HasFather(x, y) \]  

\[ Person(anakin) \]  
\[ HasFather(luke, anakin) \]  
\[ HasSister(luke, leia) \]
A Simple Example

Example 1.1:

\[ HasSon(x, y) \land HasSister(y, z) \rightarrow HasDaughter(x, z) \quad (1) \]

\[ HasFather(x, y) \rightarrow HasSon(y, x) \quad (2) \]

\[ Person(x) \rightarrow \exists y. HasFather(x, y) \quad (3) \]

\[ Person(anakin) \quad (4) \]

\[ HasFather(luke, anakin) \quad (5) \]

\[ HasSister(luke, leia) \quad (6) \]

Is leia the daughter of anakin?
I.e., does \( HasDaughter(anakin, leia) \) follow from (7-12)?
• A **signature** is a tuple \( \langle P, V, C, N \rangle \) with \( P \) a set of **predicates**, \( V \) a set of **variables**, \( C \) a set of **constants**, and \( N \) a set of **nulls**.

• Every predicate \( P \in P \) is associated to some arity \( \text{ar}(P) \geq 1 \).

• \( T = V \cup C \cup N \) is the set of **terms**.

• An **atom** is a formula of the form \( P(\vec{t}) \) with \( P \in P \), \( \text{ar}(P) = |\vec{t}| \), and \( t \in T \) for all \( t \in \vec{t} \).
A **signature** is a tuple $\langle P, V, C, N \rangle$ with $P$ a set of **predicates**, $V$ a set of **variables**, $C$ a set of **constants**, and $N$ a set of **nulls**.

Every predicate $P \in P$ is associated to some arity $ar(P) \geq 1$.

$T = V \cup C \cup N$ is the set of **terms**.

An **atom** is a formula of the form $P(\vec{t})$ with $P \in P$, $ar(P) = |\vec{t}|$, and $t \in T$ for all $t \in \vec{t}$.

**Example 1.2:** Entities and atoms.

\[
\text{Person}(x) \rightarrow \exists y. \text{HasFather}(x, y)
\]
\[
\text{HasSister}(luke, leia)
\]
**Definition 1.3:** An *(existential) rule* is a formula of the form

\[ \forall \vec{x}, \vec{z}. (\beta[\vec{x}, \vec{z}] \rightarrow \exists \vec{y}. \eta[\vec{x}, \vec{y}]) \]

with \( \beta \) and \( \eta \) conjunctions of null-free atoms and \( \vec{x}, \vec{y}, \) and \( \vec{z} \) mutually disjoint sequences of variables. A **fact** is a rule with an empty body that contains no occurrences of variables.
Definition 1.3: An (existential) rule is a formula of the form

$$\forall \vec{x}, \vec{z}. (\beta[\vec{x}, \vec{z}] \rightarrow \exists \vec{y}. \eta[\vec{x}, \vec{y}])$$

with $\beta$ and $\eta$ conjunctions of null-free atoms and $\vec{x}$, $\vec{y}$, and $\vec{z}$ mutually disjoint sequences of variables. A fact is a rule with an empty body that contains no occurrences of variables.

Formulas (7-12) from slide 9 are existential rules. Formulas (10-12) are also facts.
Definition 1.4: A homomorphism is a partial function over the set of terms with $h(c) = c$ for all $c \in C$. 
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Let $\phi$ be a formula, $h$ a homomorphism, and $\vec{x}$ a sequence of variables. Then,

- $h(\phi)$ is the formula that results from replacing every term $t$ in the domain of $h$ by $h(t)$, and
- $h_{\vec{x}} \subseteq h$ is the restriction of $h$ over $\vec{x} \cup C$. 

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**Definition 1.5:** A pair $\langle \rho, h \rangle$ with $\rho = \beta[\vec{x}, \vec{z}] \rightarrow \exists \vec{y}.\eta[\vec{x}, \vec{y}]$ a rule and $h$ a homomorphism is **applicable** to a set of facts $F$ if

1. $h(\beta) \subseteq F$, and
2. for all $h' \supseteq h_{\vec{x}}$, $h'(\eta) \not\subseteq F$.

Alternatively, we say that $\langle \rho, h \rangle$ is **not satisfied** by $F$.

If $\langle \rho, h \rangle$ is applicable to $F$, then we define $\rho_h(F) = F \cup h'(\eta)$ with $h' \supseteq h$ a homomorphism mapping every $y \in \vec{y}$ to a fresh null.
Definition 1.5: A pair $\langle \rho, h \rangle$ with $\rho = \beta[\vec{x}, \vec{z}] \rightarrow \exists \vec{y}.\eta[\vec{x}, \vec{y}]$ a rule and $h$ a homomorphism is **applicable** to a set of facts $F$ if

1. $h(\beta) \subseteq F$, and
2. for all $h' \supseteq h_{\vec{x}}$, $h'(\eta) \notin F$.

Alternatively, we say that $\langle \rho, h \rangle$ is not satisfied by $F$. If $\langle \rho, h \rangle$ is applicable to $F$, then we define $\rho_h(F) = F \cup h'(\eta)$ with $h' \supseteq h$ a homomorphism mapping every $y \in \vec{y}$ to a fresh null.

Often, we refer to $\rho_h(F)$ as the **application** of $\langle \rho, h \rangle$ to $F$. 
Definition 1.6: An **interpretation** $\mathcal{I}$ is a set of facts. $\mathcal{I}$ **satisfies** a rule $\rho = \beta \rightarrow \exists \vec{y}.\eta$ if $\langle \rho, h \rangle$ is satisfied by $\mathcal{I}$ for every homomorphism $h$. $\mathcal{I}$ is a **model** of a rule set $R$ if it satisfies every rule $\rho \in R$.

We write $\mathcal{I} \models \rho$ to indicate that $\mathcal{I}$ satisfies $\rho$. Analogously, we write $\mathcal{I} \models R$ to indicate that $\mathcal{I}$ is a model of $R$. 
Definition 1.6: An **interpretation** $\mathcal{I}$ is a set of facts. $\mathcal{I}$ **satisfies** a rule $\rho = \beta \rightarrow \exists \vec{y} \cdot \eta$ if $\langle \rho, h \rangle$ is satisfied by $\mathcal{I}$ for every homomorphism $h$. $\mathcal{I}$ is a **model** of a rule set $R$ if it satisfies every rule $\rho \in R$.

We write $\mathcal{I} \models \rho$ to indicate that $\mathcal{I}$ satisfies $\rho$. Analogously, we write $\mathcal{I} \models R$ to indicate that $\mathcal{I}$ is a model of $R$.

Definition 1.7: An interpretation $\mathcal{I}$ **entails** a query $q = \exists \vec{y} \cdot \beta$, written $\mathcal{I} \models q$, if $h(\beta) \subseteq \mathcal{I}$ for some homomorphism $h$. A rule set $R$ **entails** $q$, written $R \models q$, if $\mathcal{M} \models q$ for all $\mathcal{M} \models R$. 
Solving CQE: The two dimensions of infinity.

To determine if a query $q$ is entailed by a rule set $R$, we have to check that $\mathcal{M} \models q$ for every model $\mathcal{M}$ of $R$. Alas, this is not easy!

1. $R$ may accept an infinite number of models.
2. Each model may be of infinite size.

To address (1), we introduce the notion of universal models which can be used to solve conjunctive query entailment independently.
Solving CQE: The two dimensions of infinity.

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1. $R$ may accept an infinite number of models.
2. Each one models may be of infinite size.

To address (1), we introduce the notion of universal models which can be used to solve conjunctive query entailment independently.
Definition 1.8: An interpretation $U$ is a universal model of a rule set $R$ if

1. $U$ is a model of $R$, and
2. for all $M \models R$, $h(U) \subseteq M$ for some homomorphism $h$. 

Proposition 1.9: If $U \models R$ with $U$ a universal model of a rule set $R$ and $q$ a query, then $R \models q$.

Proof:

1. Let $M \models R$. Then, there is some $h$ with $h(U) \subseteq M$.
2. There is some $h'$ with $h'(\beta) \subseteq U$ and $\beta$ the body of $q$.
3. By (1) and (2): $h \circ h'(\beta) \subseteq M$. Hence, $M \models q$.
4. By (1) and (3): $R \models q$.
**Definition 1.8:** An interpretation $U$ is a **universal model** of a rule set $R$ if

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**Definition 1.8:** An interpretation $\mathcal{U}$ is a **universal model** of a rule set $\mathcal{R}$ if

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**Proof:**

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Solving CQE with Universal Models

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**Proof:**

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2. There is some $h'$ with $h'(\beta) \subseteq \mathcal{U}$ and $\beta$ the body of $q$.
3. By (1) and (2): $h \circ h'(\beta) \subseteq \mathcal{M}$. Hence, $\mathcal{M} \models q$.
4. By (1) and (3): $\mathcal{R} \models q$. 
Definition 1.10: An interpretation $\mathcal{U}$ is a universal model of a rule set $R$ if

1. $\mathcal{U}$ is a model of $R$, and
2. for all $\mathcal{M} \models R$, $h(\mathcal{U}) \subseteq \mathcal{M}$ for some homomorphism $h$. 

Theorem 1.11: Consider a rule set $R$, a query $q$, and a universal model $\mathcal{U}$ for $R$. Then, $R \models q$ if and only if $\mathcal{U} \models q$.

Proof:

$\Rightarrow$ Trivial, since $\mathcal{U}$ is a model of $R$.

$\Leftarrow$ By Proposition 9.
Definition 1.10: An interpretation \( \mathcal{U} \) is a universal model of a rule set \( R \) if

1. \( \mathcal{U} \) is a model of \( R \), and
2. for all \( \mathcal{M} \models R \), \( h(\mathcal{U}) \subseteq \mathcal{M} \) for some homomorphism \( h \).

Theorem 1.11: Consider a rule set \( R \), a query \( q \), and a universal model \( \mathcal{U} \) for \( R \). Then, \( R \models q \) if and only if \( \mathcal{U} \models q \).
**Definition 1.10:** An interpretation $U$ is a universal model of a rule set $R$ if

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**Theorem 1.11:** Consider a rule set $R$, a query $q$, and a universal model $U$ for $R$. Then, $R \models q$ if and only if $U \models q$.

**Proof:**

$\Rightarrow$ Trivial, since $U$ is a model of $R$. 
Definition 1.10: An interpretation $\mathcal{U}$ is a \textbf{universal model} of a rule set $R$ if

1. $\mathcal{U}$ is a model of $R$, and
2. for all $\mathcal{M} \models R$, $h(\mathcal{U}) \subseteq \mathcal{M}$ for some homomorphism $h$.

Theorem 1.11: Consider a rule set $R$, a query $q$, and a universal model $\mathcal{U}$ for $R$. Then, $R \models q$ if and only if $\mathcal{U} \models q$.

Proof:

$\Rightarrow$ Trivial, since $\mathcal{U}$ is a model of $R$.

$\Leftarrow$ By Proposition 9.
Definition 1.12: A chase sequence of a rule set $R$ is a (possibly infinite) sequence of sets of facts $F^0, F^1, \ldots$ such that

1. $F^0$ is the empty set,
2. for all $i \geq 1$, $F^i = F^{i-1} \cup \rho_h(F^{i-1})$ for some $\rho \in R$ and homomorphism $h$, and
3. if, for some $i \geq 1$, a pair $\langle \rho, h \rangle$ with $\rho \in R$ is applicable to $F^i$; then there is some $j \geq i$ such that $\langle \rho, h \rangle$ is not applicable to $F^j$ (fairness).

A chase for $R$ is some (possibly infinite) set that results from the union of all sets of facts in some chase sequence.
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A chase for $R$ is some (possibly infinite) set that results from the union of all sets of facts in some chase sequence.

Given a rule set $R$, let $C(R)$ be some arbitrarily chosen chase of $R$. 
Example 1.13:

\[\text{HasSon}(x, y) \land \text{HasSister}(y, z) \rightarrow \text{HasDaughter}(x, z)\]  
(7)

\[\text{HasFather}(x, y) \rightarrow \text{HasSon}(y, x)\]  
(8)

\[\text{Person}(x) \rightarrow \exists y. \text{HasFather}(x, y)\]  
(9)

\[\text{Person}(\text{anakin})\]  
(10)

\[\text{HasFather}(\text{luke}, \text{anakin})\]  
(11)

\[\text{HasSister}(\text{luke}, \text{leia})\]  
(12)
Lemma 1.14: For a rule set $R$, we have that $C(R) \models R$. 

Proof:

Proof by contradiction:

1. Let us assume that $C(R) \not\models R$.

2. By (1): There is a chase sequence $F_0, F_1, \ldots$ with $C(R) = \bigcup_{i \geq 0} F_i$.

3. By (1): There is some pair $\langle \rho, h \rangle$ with $\rho \in R$ applicable to $C(R)$.

4. By (2) and (3): For all $i \geq 0$, $\langle \rho, h \rangle$ is applicable to $F_i$.

5. By (2) and (4): The sequence $F_0, F_1, \ldots$ does not satisfy the fairness requirement introduced in Definition 12.

Lemma 1.14: For a rule set R, we have that $C(R) \models R$.

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3. By (1): There is some pair $\langle \rho, h \rangle$ with $\rho \in R$ applicable to $C(R)$.

4. By (2) and (3): For all $i \geq 0$, $\langle \rho, h \rangle$ is applicable to $F_i$.

5. By (2) and (4): The sequence $F_0, F_1, \ldots$ does not satisfy the fairness requirement introduced in Definition 12.

Lemma 1.14: For a rule set $R$, we have that $C(R) \models R$.

Proof: Proof by contradiction:

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3. By (1): There is some pair $\langle \rho, h \rangle$ with $\rho \in R$ applicable to $C(R)$.
4. By (2) and (3): For all $i \geq 0$, $\langle \rho, h \rangle$ is applicable to $F^i$. 

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Solving CQE with the Chase

**Lemma 1.14:** For a rule set R, we have that $C(R) \models R$.

**Proof:** Proof by contradiction:

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Solving CQE with the Chase

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4. By (2) and (3): For all $i \geq 0$, $\langle \rho, h \rangle$ is applicable to $F^i$.
5. By (2) and (4): The sequence $F^0, F^1, \ldots$ does not satisfy the fairness requirement introduced in Definition 12.
Theorem 1.15: A rule set R entails a query $q$ iff $C(R) \models q$.

Proof: The theorem follows from Theorem 11 and the fact that $C(R)$ is a universal model of R (proven below).
**Theorem 1.15:** A rule set $R$ entails a query $q$ iff $C(R) \models q$.

**Proof:** The theorem follows from Theorem 11 and the fact that $C(R)$ is a universal model of $R$ (proven below).

1. By Lemma 14, $C(R) \models R$. 

By Lemma 14, $C(R) \models R$. 

By (1), (2) and (5): $C(R)$ is a universal model.
Theorem 1.15: A rule set R entails a query q iff $C(R) \models q$.

Proof: The theorem follows from Theorem 11 and the fact that $C(R)$ is a universal model of R (proven below).

1. By Lemma 14, $C(R) \models R$.
2. Let $\mathcal{M}$ be some model of R.
Theorem 1.15: A rule set $R$ entails a query $q$ iff $C(R) \models q$.

Proof: The theorem follows from Theorem 11 and the fact that $C(R)$ is a universal model of $R$ (proven below).

1. By Lemma 14, $C(R) \models R$.
2. Let $M$ be some model of $R$.
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Theorem 1.15: A rule set R entails a query $q$ iff $C(R) \models q$.

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2. Let $\mathcal{M}$ be some model of R.
3. There is some chase sequence $F^0, F^1, \ldots$ with $C(R) = \bigcup_{i \geq 1} F^i$.
4. In the following slide, we show that there is a sequence of homomorphisms $h_0, h_1, \ldots$ such that $h_i(F^i) \subseteq \mathcal{M}$ and $h_{i+1} \supseteq h_i$ for all $i \geq 0$. 
Theorem 1.15: A rule set $R$ entails a query $q$ iff $C(R) \models q$.

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5. Let $h = \bigcup h_i$ for all $i \geq 0$. Then, $h(C(R)) \subseteq M$ by (4).
Theorem 1.15: A rule set R entails a query q iff $C(R) \models q$.

Proof: The theorem follows from Theorem 11 and the fact that $C(R)$ is a universal model of R (proven below).

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5. Let $h = \bigcup h_i$ for all $i \geq 0$. Then, $h(C(R)) \subseteq \mathcal{M}$ by (4).
6. By (1), (2) and (5): $C(R)$ is a universal model.
Solving CQE with the Chase

**Theorem 1.16:** A rule set \( R \) entails a query \( q \) iff \( C(R) \models q \).

**Proof:** We show via induction that, for every \( i \geq 0 \), there is a homomorphism \( h_i \) with \( h_i(F_i) \subseteq \mathcal{M} \) and \( h_i \supseteq h_{i+1} \).
Theorem 1.16: A rule set R entails a query $q$ iff $C(R) \models q$.

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1 Base case: $h_0 = \emptyset$. 
Solving CQE with the Chase

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1. **Base case:** $h_0 = \emptyset$.
2. **Induction step:** Let $i \geq 1$
   1. Let $\langle \rho, h' \rangle$ be some pair with $\rho = \beta[\vec{x}, \vec{z}] \rightarrow \exists \vec{y}. \eta[\vec{x}, \vec{y}] \in R$ and $F^i = \rho_{h'}(F^{i-1}) \cup F^{i-1}$. Note that, $h'(\beta) \subseteq F^{i-1}$.
   2. By (1) and IH: $h_{i-1}(h'(\beta)) \subseteq M$.
   3. By (2) and $M \models R$: $h''(\eta) \subseteq F^{i-1}$ for some $h'' \supseteq h_{i-1} \circ h'_x$.
   4. $h_i \supseteq h_{i-1}$ is the smallest function mapping $h'(y)$ to $h''(y)$ for all $y \in \vec{y}$.
Brief recap

• Syntax and semantics
• Universal models
• The chase algorithm

What’s next?
• Select your own papers!