EXERCISE 1

Logic

Steffen Hölldobler, Emmanuelle Dietz Saldanha

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Problem 1.1

In definition 3.5 of the course the set $\mathcal{L}(\mathcal{R})$ of strings over a given propositional alphabet $\Sigma_{\mathcal{R}}$ (with $\mathcal{R}$ a set of propositional variables) was defined as the smallest set of strings over $\Sigma_{\mathcal{R}}$ which satisfies certain closure properties, to which we will refer in the following as conditions 1–3.
However, the proof of existence of this set $\mathcal{L}(\mathcal{R})$ was just sketched.

Prove *in full detail* the existence of the set $\mathcal{L}(\mathcal{R})$. Proceed stepwise as follows:

1. Show at first that for every propositional alphabet $\Sigma_{\mathcal{R}}$ exists a set $\mathcal{Z}$ of strings, which fulfills conditions 1–3.
2. We define $\mathcal{F} := \{N \mid N$ is a set and fulfills conditions 1–3$\}$.
   Show that $\bigcap \mathcal{F}$ is a set of strings over $\Sigma_{\mathcal{R}}$.
3. Show that $\bigcap \mathcal{F} := \{H \mid H \in N$ for all $N \in \mathcal{F}\}$ fulfills conditions 1–3 as well.
4. Prove the following more general lemma:

   For a set $\mathcal{N}$ of sets holds:
   If $\bigcap \mathcal{N} \in \mathcal{N}$, then $\bigcap \mathcal{N}$ is the smallest element of $\mathcal{N}$ w.r.t. the $\subseteq$ relation.
5. Show by means of the lemmata (1)–(4) given above that the set $\mathcal{L}(\mathcal{R})$ exists.

Problem 1.2

Motivated by definition 3.5 of propositional formulae we make the following definitions:

A sequence $[z_1, \ldots, z_n]$ of strings z_j over a propositional alphabet $\Sigma_{\mathcal{R}}$ is called a *construction* of length n of the string $F \in \Sigma_{\mathcal{R}}^*$ iff

1. $z_n = F$
2. for all z_j $(j \in \{1, \ldots, n\})$ one of the following alternatives holds:
   (a) $z_j \in \mathcal{R}$
   (b) it exists a $k < j$ such that $z_j = \neg z_k$
   (c) there exist $k, l < j$ such that $z_j = (z_k \circ z_l)$.

Solve the following problems concerning the concept of a construction just defined.

1. Find constructions of the following formulae.

   (a) $\neg\neg p$
   (b) $(p \to q)$
   (c) $\neg(p \wedge p)$

2. Are the following sequences of strings correct constructions ?

   (a) $[p_1]$
   (b) $[(p_4 \vee p_2)]$
   (c) $[p_4, p_2, (p_2 \vee p_4)]$
   (d) $[p_2, (p_4 \vee p_2)]$
   (e) $[\neg p_2, p_2]$
   (f) $[p_2, p_5, \neg p_2]$
   (g) $[p_2, \neg p_2, p_2]$

Problem 1.3 (9 points)

We denote by $\mathcal{K}_n(\mathcal{R})$ the set of all strings over $\Sigma_{\mathcal{R}}$ for which exists a construction of length n according to the definitions given in problem 1.2 above, and we define $\mathcal{K}(\mathcal{R}) := \bigcup_{n=1}^{\infty} \mathcal{K}_n(\mathcal{R})$ as the set of all strings over $\Sigma_{\mathcal{R}}$ for which exists a construction of any length.

Show that the equation $\mathcal{L}(\mathcal{R}) = \mathcal{K}(\mathcal{R})$ holds.