Technische Universität Dresden

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COMPLEXITY THEORY

Lecture 14: P vs. NP: Ladner's Theorem

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Review

Review: Hierarchies and Gaps

Hierarchy theorems tell us that more time/space leads to more power:

$$
\begin{tikzcd}
\mathsf{L} \arrow[r, phantom, "\subseteq"] & \mathsf{NL} \arrow[r, phantom, "\subseteq"] \arrow[rrr, bend left, no head, "\neq"] & \mathsf{P} \arrow[r, phantom, "\subseteq"] \arrow[rrr, bend right, no head, "\neq"'] & \mathsf{NP} \arrow[r, phantom, "\subseteq"] \arrow[rrr, bend left, no head, "\neq"] & \mathsf{PSpace} \arrow[r, phantom, "\subseteq"] \arrow[rrr, bend right, no head, "\neq"'] & \mathsf{ExpTime} \arrow[r, phantom, "\subseteq"] & \mathsf{NExpTime} \arrow[r, phantom, "\subseteq"] & \mathsf{ExpSpace}
\end{tikzcd}
$$

Gap theorems tell us that, for non-constructible functions as time/space bounds, arbitrary (constructible or not) boosts in resources may not lead to more power

Any natural problems in the hierarchy?

To show that complexity classes are different

- we have defined concrete diagonalisation languages that can show the difference (i.e., our argument was constructive),
- but these diagonalisation languages are rather artificial (i.e., not natural).

Are there, e.g., any natural ExpTime problems that are not in P?

Yes, many:

Theorem 14.1: If **L** is ExpTime-hard, then **L** $\notin$ P.

Proof: We have shown that there is a language **D** $\in$ ExpTime $\setminus$ P. If **L** is ExpTime-hard, then there is a polynomial many-one reduction **D** $\leq_p$ **L**. Therefore, if **L** were in P, then so would **D** – contradiction. □

Similar results hold for other classes we separated: A problem that is hard for the larger class cannot be included in the smaller.

Ladner's Theorem

P vs. NP revisited

We have seen that a great variety of difficult problems in NP turn out to be NP-complete.

A natural question to ask is whether this apparent dichotomy is a law of nature:

Hypothesis: Every problem in NP is either in P or NP-complete.

In 1975, Richard E. Ladner showed that this is wrong, unless P = NP

(in the latter case, uninterestingly, the non-trivial problems in P would turn out to be exactly the set of NP-complete problems)

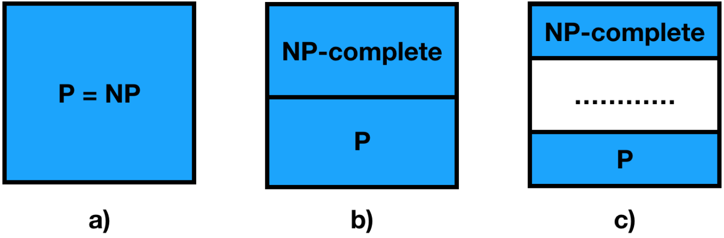
Theorem 14.2 (Ladner, 1975): If P $\neq$ NP, then there are problems in NP that are neither in P nor NP-complete.

Such problems are called NP-intermediate.

Illustration

Theorem 14.2 (Ladner, 1975): If P $\neq$ NP, then there are problems in NP that are neither in P nor NP-complete.

In other words, given the following illustrations of the possible relationships between P and NP:



Ladner tells us that the middle cannot be correct.

Proving the Theorem

Theorem 14.2 (Ladner, 1975): If P $\neq$ NP, then there are problems in NP that are neither in P nor NP-complete.

Proof idea: We will directly define an NP-intermediate language by defining an NTM $\mathcal{K}$ that recognises it.

We want to construct $\mathbf{L}(\mathcal{K})$ to be:

(1) different from all problems in P
(2) different from all problems that **SAT** can be reduced to

Observation: This is similar to two concurrent diagonalisation arguments

Moreover, the sets we diagonalise against are effectively enumerable:

- There is an effective enumeration $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2, \ldots$ of all polynomially time-bounded DTMs, each together with a suitable bounding function
  For example, enumerate all pairs of TMs and polynomials, and make the enumeration consist of the TMs obtained by artificially restricting the run of a TM with a suitable countdown.
- There is an effective enumeration $\mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_2, \ldots$ of all polynomial many-one reductions, each together with a suitable bounding function
  This is similar to enumerating polytime TMs; we can restrict to one input alphabet that we also use for **SAT**

The problem with diagonalisation

How can we do two diagonalisations at once? — Simply interleave the enumerations:

- In each even "step" $2i$, show that the ith polytime TM $\mathcal{M}_i$ is not equivalent to $\mathcal{K}$: there is w such that $\mathcal{M}_i(w) \neq \mathcal{K}(w)$
- In each odd "step" $2i + 1$, show that the ith reduction $\mathcal{R}_i$ does not reduce **Sat** to $\mathcal{K}$: there is w such that $\mathcal{K}(\mathcal{R}_i(w)) \neq \textbf{Sat}(w)$

Nevertheless, there is a problem: How can we flip the output of **Sat**?

- $\mathcal{K}$ is required to run in NP
- Computing the actual result of **Sat** is NP-hard
- To show $\mathcal{K}(\mathcal{R}_i(w)) \neq \textbf{Sat}(w)$, one might have to show $w \notin \textbf{Sat}$, which is presumably not in NP

$\leadsto$ the required computation seems too hard!

Solution: Lazy diagonalisation

Idea: Do not attempt to show too much on small inputs, but wait patiently until inputs are large enough to show the required differences

Main ingredients:

- A very slow growing but polynomially computable function f
- A problem in NP that is NP-hard: **Sat**
- A problem in NP that is not NP-hard: $\emptyset$

We will define a TM $\mathcal{K}$ that does the following on input w:

(1) Compute the value $f(|w|)$

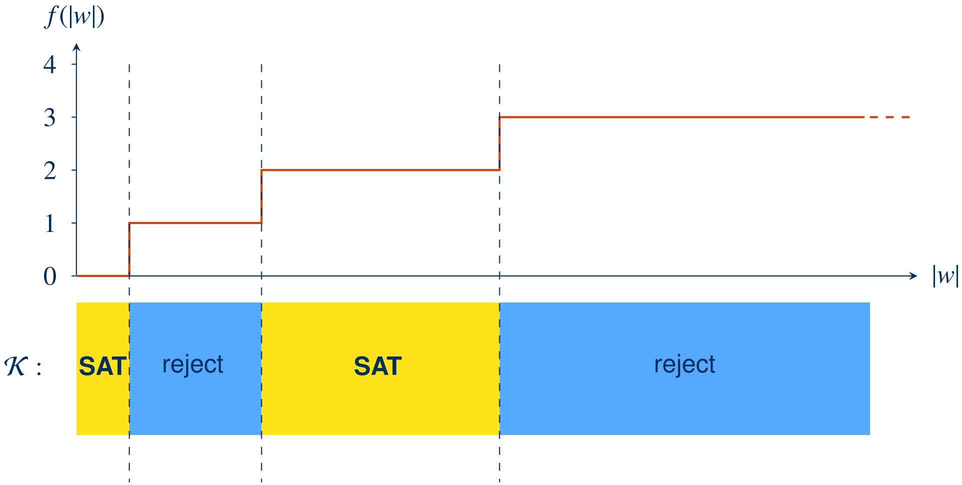
(2) If $f(|w|)$ is even: return whether $w \in$ **Sat**

(3) If $f(|w|)$ is odd: return whether $w \in \emptyset$, i.e., reject

Intuition: the NP-intermediate language $\mathbf{L}(\mathcal{K})$ is **Sat** with "holes punched out of it" (namely for all inputs where f is odd)

Illustration of $\mathcal{K}$'s behaviour

We can sketch the behaviour of $\mathcal{K}$ as follows:



What is f?

Reminder: $\mathcal{K}(w)$ is $\textbf{Sat}(w)$ if $f(|w|)$ is even, and *false* if $f(|w|)$ is odd.

The key to the proof is the definition of f – this is where the diagonalisation happens.

Intuition: Keep the current value of f until progress has been made in diagonalisation

- Keep an even value $f(|w|) = 2i$ until you can show in polynomial time (in $|w|$) that there is v such that $\mathcal{M}_i(v) \neq \mathcal{K}(v)$
- Keep an odd value $f(|w|) = 2i + 1$ until you can show in polynomial time (in $|w|$) that there is v such that $\mathcal{K}(\mathcal{R}_i(v)) \neq \textbf{Sat}(v)$

If we can do this in NP, it will be enough already:

- If $\mathcal{K}$ were equivalent to any $\mathcal{M}_i$, then f would eventually become an even constant, and $\mathcal{K}$ would solve **Sat** on all but finitely many instances
  $\rightsquigarrow$ $\mathbf{L}(\mathcal{K})$ would be NP-hard, and $\mathcal{K}$ equivalent to a polytime TM $\rightsquigarrow$ P = NP
- If $\mathbf{L}(\mathcal{K})$ would allow **Sat** to be reduced to it by some reduction $\mathcal{R}_i$, then f would eventually become an odd constant, and $\mathbf{L}(\mathcal{K})$ would be a finite language
  $\rightsquigarrow$ $\mathbf{L}(\mathcal{K})$ would be in P, and **Sat** would reduce to it $\rightsquigarrow$ P = NP

In either case, this contradicts our assumption that P $\neq$ NP

What is f?

We consider some fixed deterministic TM $\mathcal{S}$ with $\mathbf{L}(\mathcal{S}) = \textsc{Sat}$, and an enumeration $v_0, v_1, \ldots$ of all words ordered by length, and lexicographic for words of equal length.

Reminder: $\mathcal{K}(w)$ is $\mathcal{S}(w)$ if $f(|w|)$ is even, and false if $f(|w|)$ is odd.

Definition: The value of f on input w with $|w| = n$ is defined recursively

(1) Perform the computations of $f(0), f(1), f(2), \ldots$ in order until n computing steps have been performed in total. Store the largest value $f(\ell) = k$ that could be computed in this time (set $k = 0$ if no value was computed).

(2) Determine if $f(n)$ should remain k or increase to $k + 1$:

(2.a) If $k = 2i$ is even: Iterate over all words v, simulate $\mathcal{M}_i(v)$, $\mathcal{S}(v)$, and (recursively) compute $f(|v|)$. Terminate this effort after n steps. If a word is found such that $\mathcal{K}(v) \neq \mathcal{M}_i(v)$, then return $k + 1$; else return k

(2.b) If $k = 2i + 1$ is odd: Iterate over all words v, simulate $\mathcal{R}_i(v)$ (this produces a word), $\mathcal{S}(v)$, $\mathcal{S}(\mathcal{R}_i(v))$, and (recursively) compute $f(|\mathcal{R}_i(v)|)$. Terminate this effort after n steps. If a word is found such that $\mathcal{K}(\mathcal{R}_i(v)) \neq \mathcal{S}(v)$, then return $k + 1$; else return k.

Is f well-defined?

Our definition of f computes values for f recursively. Is this ok?

- Yes, the computation that needs to be done for each $f(n)$ is fully defined
- All the simulated TMs are known or computable
- Since computation is time-limited to the input value n, there is no danger of endless recursion
- For example, $f(0) = 0$: nothing will be achieved in 0 steps

Indeed, f grows very slowly!

- A large input n might be needed to find the next counterexample word v needed in diagonalisation
- Even if such v was found in n steps (making progress from n to $n + 1$), it will be only much later that $f(n)$ can be computed in step (1) and f will even start to look for a way of getting to $n + 2$.
- In fact, already the requirement to recompute all previous values of f before considering an increase ensures that $f \in O(\log \log n)$.

Concluding the Proof

Theorem 14.2 (Ladner, 1975): If P $\neq$ NP, then there are problems in NP that are neither in P nor NP-complete.

Proof: Let $\mathcal{K}$ be defined as before.

$\mathcal{K}$ runs in nondeterministic polynomial time:

- The computation of f is in deterministic polynomial time (since it is artificially bounded to a short time)
- The computation of **Sat** for the cases where $f(|w|)$ is even is possible in NP

L$(\mathcal{K})$ is not in P: As argued before: if it were in P, it would be equivalent to some polytime TM $\mathcal{M}_i$, and f would eventually be constant at $2i$, making **L**$(\mathcal{K})$ equivalent to **Sat** (up to finite variations), which contradicts P $\neq$ NP.

L$(\mathcal{K})$ is not NP-hard: As argued before: if it were NP-hard, there would be a polynomial many-one reduction $\mathcal{R}_i$ from **Sat**, and f would eventually be constant at $2i + 1$, making **L**$(\mathcal{K})$ equivalent to $\emptyset$ (up to finite variations), which contradicts P $\neq$ NP. $\square$

Discussion: Proof of Ladner's Theorem

Note 1: It is interesting to meditate on the following facts:

- We have defined a rather "busy" computation of f that checks that diagonalisation (over two different sets) must happen
- This definition of computation is essential to prove the result
- Nevertheless, diagonalisation remained "internal": from the outside, $\mathcal{K}$ is just a TM that sometimes solves **Sat** (for a long range of inputs), and at other times just rejects every input (again for very long ranges of inputs)

Note 2: The constructed language is very artificial

- It is very "non-uniform" in terms of how hard it is, alternating between long stretches of NP-hardness and long stretches of triviality

Note 3: Are there any natural problems that are known to be NP-intermediate?

- No: finding one would prove P $\neq$ NP
- Candidate problems (link) include, e.g., **Graph Isomorphism** and **Factoring**
  Beware: the latter is not about deciding if a number is prime, but about checking something specific about its factors, e.g., whether the largest factor contains at least one 7 when written in decimal

Summary and Outlook

Ladner's theorem tells us that, in the intuitive case that P $\neq$ NP, there must (counterintuitively?) be many problems in NP that are neither polynomially solvable nor NP-complete

The proof is based on a technique of lazy diagonalisation

What's next?

- Generalising Ladner's Theorem
- Computing with oracles (reprise)
- The limits of diagonalisation, proved by diagonalisation