Exercise 8: Datalog

Database Theory
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Exercise 1

**Exercise.** A graph is *planar* if it can be drawn on the plane without intersections of edges. For example, the following graph A is planar, while graph B is not:

![Figure: A](image1.png)

![Figure: B](image2.png)

1. Can the graphs A and B be distinguished by a FO query?
2. Show that planarity is not FO-definable by using locality.
Exercise 1

Exercise. A graph is **planar** if it can be drawn on the plane without intersections of edges. For example, the following graph A is planar, while graph B is not:

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2. Show that planarity is not FO-definable by using locality.

Solution.
Exercise 1

Exercise. A graph is planar if it can be drawn on the plane without intersections of edges. For example, the following graph A is planar, while graph B is not:

Figure: A

Figure: B

1. Can the graphs A and B be distinguished by a FO query?
2. Show that planarity is not FO-definable by using locality.

Solution.
1. This query matches B but not A:
   \[ \exists x, y, z, w, v. \ E(x, y) \land E(y, z) \land E(z, w) \land E(w, x) \land E(x, v) \land E(y, v) \land E(z, v) \land E(w, v) \land E(x, z) \land E(y, w) \]
Exercise 1

**Exercise.** A graph is *planar* if it can be drawn on the plane without intersections of edges. For example, the following graph A is planar, while graph B is not:

![Figure: A](image1.png) ![Figure: B](image2.png)

1. Can the graphs A and B be distinguished by a FO query?
2. Show that planarity is not FO-definable by using locality.

**Solution.**

1. This query matches B but not A:

   \[ \exists x, y, z, w, v. E(x, y) \land E(y, z) \land E(z, w) \land E(w, x) \land E(x, v) \land E(y, v) \land E(z, v) \land E(w, v) \land E(x, z) \land E(y, w) \]

2. For \( \varphi \) with quantifier rank \( r \), consider counterexamples of size \( d = 3^r \):

![Counterexample 1](counterexample1.png) ![Counterexample 2](counterexample2.png)
Exercise 2

Exercise. Consider the example Datalog program from the lecture:

(a) Father(alice, bob)
(b) Mother(alice, carla)
(c) Mother(evan, carla)
(d) Father(carla, david)

Parent(x, y) ← Father(x, y)  
(1)

Parent(x, y) ← Mother(x, y)  
(2)

Ancestor(x, y) ← Parent(x, y)  
(3)

Ancestor(x, z) ← Parent(x, y) ∧ Ancestor(y, z)  
(4)

SameGeneration(x, y) ←  
(5)

SameGeneration(x, y) ← Parent(x, v) ∧ Parent(y, w) ∧ SameGeneration(v, w)  
(6)

1. Give a proof tree for SameGeneration(evan, alice).
2. Compute the sets $T_0^P$, $T_1^P$, $T_2^P$, ... When is the fixed point reached?
Exercise 2

**Exercise.** Consider the example Datalog program from the lecture:

(a) Father(alice, bob)  
(b) Mother(alice, carla)  
(c) Mother(evan, carla)  
(d) Father(carla, david)

Parent(x, y) ← Father(x, y)  \[1\]  
Parent(x, y) ← Mother(x, y)  \[2\]  
Ancestor(x, y) ← Parent(x, y)  \[3\]

\[\text{Ancestor}(x, z) \leftarrow \text{Parent}(x, y) \land \text{Ancestor}(y, z)\]  \[4\]

\[\text{SameGeneration}(x, y) \leftarrow\]  \[5\]

\[\text{SameGeneration}(x, y) \leftarrow \text{Parent}(x, v) \land \text{Parent}(y, w) \land \text{SameGeneration}(v, w)\]  \[6\]

1. Give a proof tree for SameGeneration(evan, alice).
2. Compute the sets \(T_P^0, T_P^1, T_P^2, \ldots\) When is the fixed point reached?

**Solution.**
Exercise 2

Exercise. Consider the example Datalog program from the lecture:

(a) Father(alice, bob)

(b) Mother(alice, carla)

(c) Mother(evan, carla)

(d) Father(carla, david)

Parent(x, y) ← Father(x, y)  \hspace{1cm} (1)

Parent(x, y) ← Mother(x, y) \hspace{1cm} (2)

Ancestor(x, y) ← Parent(x, y)  \hspace{1cm} (3)

Ancestor(x, z) ← Parent(x, y) \land Ancestor(y, z) \hspace{1cm} (4)

SameGeneration(x, x) ← \hspace{1cm} (5)

SameGeneration(x, y) ← Parent(x, v) \land Parent(y, w) \land SameGeneration(v, w) \hspace{1cm} (6)

1. Give a proof tree for SameGeneration(evan, alice).
2. Compute the sets $T^0_P$, $T^1_P$, $T^2_P$, ... When is the fixed point reached?

Solution.

1. 

\[
\begin{array}{c}
\text{SameGeneration(evan, alice)} \\
\text{Parent(evan, carla)} \\
\text{Mother(evan, carla)}
\end{array}
\hspace{1cm}
\begin{array}{c}
\text{Parent(alice, carla)} \\
\text{Mother(alice, carla)}
\end{array}
\hspace{1cm}
\begin{array}{c}
\text{SameGeneration(carla, carla)} \\
\text{SameGeneration(\{ x \mapsto evan, v \mapsto carla, y \mapsto alice, w \mapsto carla \})}
\end{array}
\]

\[
\begin{array}{c}
\text{(c)} \\
\text{(5); \{ x \mapsto carla \}}
\end{array}
\hspace{1cm}
\begin{array}{c}
\text{(2); \{ x \mapsto evan, y \mapsto carla \}}
\end{array}
\hspace{1cm}
\begin{array}{c}
\text{(6); \{ x \mapsto evan, v \mapsto carla, y \mapsto alice, w \mapsto carla \}}
\end{array}
\]

\[
\begin{array}{c}
\text{(2); \{ x \mapsto alice, y \mapsto carla \}}
\end{array}
\]

\[
\text{(b)}
\]

\[
\text{SameGeneration(evan, alice)} \leftarrow
\]

\[
\text{SameGeneration(carla, carla)} \leftarrow
\]

\[
\text{SameGeneration(evan, alice)} \leftarrow
\]

\[
\text{SameGeneration(carla, carla)} \leftarrow
\]

\[
\text{SameGeneration(evan, alice)} \leftarrow
\]

\[
\text{SameGeneration(carla, carla)} \leftarrow
\]

\[
\text{SameGeneration(evan, alice)} \leftarrow
\]

\[
\text{SameGeneration(carla, carla)} \leftarrow
\]

\[
\text{SameGeneration(evan, alice)} \leftarrow
\]

\[
\text{SameGeneration(carla, carla)} \leftarrow
\]

\[
\text{SameGeneration(evan, alice)} \leftarrow
\]

\[
\text{SameGeneration(carla, carla)} \leftarrow
\]

\[
\text{SameGeneration(evan, alice)} \leftarrow
\]

\[
\text{SameGeneration(carla, carla)} \leftarrow
\]
Exercise 2

Exercise. Consider the example Datalog program from the lecture:

(a) Father(alice,bob)
(b) Mother(alice,carla)
(c) Mother(evan,carla)
(d) Father(carla,david)

1. Give a proof tree for SameGeneration(evan, alice).
2. Compute the sets $T^0_P$, $T^1_P$, $T^2_P$, ... When is the fixed point reached?

Solution.

2. $T^0_P = \emptyset$
Exercise 2

Exercise. Consider the example Datalog program from the lecture:

(a) Father(alice,bob)
(b) Mother(alice,carla)
(c) Mother(evan,carla)
(d) Father(carla,david)

\[
\begin{align*}
\text{Parent}(x,y) & \leftarrow \text{Father}(x,y) \\
\text{Parent}(x,y) & \leftarrow \text{Mother}(x,y) \\
\text{Ancestor}(x,y) & \leftarrow \text{Parent}(x,y) \\
\text{Ancestor}(x,y) & \leftarrow \text{Parent}(x,y) \land \text{Ancestor}(y,z) \\
\text{SameGeneration}(x,x) & \leftarrow \\
\text{SameGeneration}(x,y) & \leftarrow \text{Parent}(x,v) \land \text{Parent}(y,w) \land \text{SameGeneration}(v,w) \\
\end{align*}
\]

1. Give a proof tree for SameGeneration(evan, alice).
2. Compute the sets $T^0_P, T^1_P, T^2_P, \ldots$ When is the fixed point reached?

Solution.

2. 

\[
T^0_P = \emptyset \\
T^1_P = \{ \text{Father(alice,bob), Mother(alice,carla), Mother(evan,carla), Father(carla,david) } \}
\]
Exercise 2

Exercise. Consider the example Datalog program from the lecture:

(a) Father(alice,bob)
(b) Mother(alice,carla)
(c) Mother(evan,carla)
(d) Father(carla,david)

\[
\text{Parent}(x,y) \leftarrow\text{Father}(x,y) \quad (1)
\]
\[
\text{Parent}(x,y) \leftarrow\text{Mother}(x,y) \quad (2)
\]
\[
\text{Ancestor}(x,y) \leftarrow \text{Parent}(x,y) \quad (3)
\]
\[
\text{Ancestor}(x,y) \leftarrow \text{Parent}(x,y) \land \text{Ancestor}(y,z) \quad (4)
\]
\[
\text{SameGeneration}(x,x) \leftarrow\quad (5)
\]
\[
\text{SameGeneration}(x,y) \leftarrow \text{Parent}(x,v) \land \text{Parent}(y,w) \land \text{SameGeneration}(v,w) \quad (6)
\]

1. Give a proof tree for SameGeneration(evan, alice).
2. Compute the sets $T_0^P$, $T_1^P$, $T_2^P$, ... When is the fixed point reached?

Solution.

2. $T_0^P = \emptyset$

$T_1^P = \{ \text{Father(alice,bob)}, \text{Mother(alice,carla)}, \text{Mother(evan,carla)}, \text{Father(carla,david)} \}$

$T_2^P = T_1^P \cup \{ \text{Parent(alice,bob)}, \text{Parent(alice,carla)}, \text{Parent(evan,carla)}, \text{Parent(carla,david)},$

\begin{align*}
&\quad \text{SameGeneration(alice,alice)}, \text{SameGeneration(bob,bob)}, \text{SameGeneration(carla,carla)}, \text{SameGeneration(david,david)}, \text{SameGeneration(evan,even)} \} \\
\end{align*}$
Exercise 2

Exercise. Consider the example Datalog program from the lecture:

(a) Father(alice, bob)
(b) Mother(alice, carla)
(c) Mother(evan, carla)
(d) Father(carla, david)

\[
\begin{align*}
\text{Parent}(x,y) &\leftarrow \text{Father}(x,y) \\
\text{Parent}(x,y) &\leftarrow \text{Mother}(x,y) \\
\text{Ancestor}(x,y) &\leftarrow \text{Parent}(x,y) \\
\text{SameGeneration}(x,y) &\leftarrow \text{Parent}(x,v) \land \text{Parent}(y,w) \\
\text{Ancestor}(x,y) &\leftarrow \text{Parent}(x,y) \land \text{Ancestor}(y,z)
\end{align*}
\]

1. Give a proof tree for SameGeneration(evan, alice).
2. Compute the sets \( T_P^0, T_P^1, T_P^2, \ldots \) When is the fixed point reached?

Solution.

2. 

\[
T_P^0 = \emptyset \\
T_P^1 = \{ \text{Father(alice,bob)}, \text{Mother(alice,carla)}, \text{Mother(evan,carla)}, \text{Father(carla,david)} \} \\
T_P^2 = T_P^1 \cup \{ \text{Parent(alice,bob)}, \text{Parent(alice,carla)}, \text{Parent(evan,carla)}, \text{Parent(carla,david)}, \text{SameGeneration(alice,alice)}, \text{SameGeneration(bob,bob)}, \text{SameGeneration(carla,carla)}, \text{SameGeneration(david,david)}, \text{SameGeneration(evan,evan)} \} \\
T_P^3 = T_P^2 \cup \{ \text{Ancestor(alice,bob)}, \text{Ancestor(alice,carla)}, \text{Ancestor(evan,carla)}, \text{Ancestor(carla,david)}, \text{SameGeneration(alice,evan)}, \text{SameGeneration(evan,alice)} \}
\]
Exercise 2

Exercise. Consider the example Datalog program from the lecture:

(a) Father(alice, bob)
(b) Mother(alice, Carla)
(c) Mother(evan, Carla)
(d) Father(carla, david)

Parent(x, y) ← Father(x, y) (1)
Parent(x, y) ← Mother(x, y) (2)
Ancestor(x, y) ← Parent(x, y) (3)
Ancestor(x, y) ← Parent(x, y) ∧ Ancestor(y, z) (4)

1. Give a proof tree for SameGeneration(evan, alice).
2. Compute the sets $T_P^0, T_P^1, T_P^2, \ldots$ When is the fixed point reached?

Solution.

2.

$T_P^0 = \emptyset$

$T_P^1 = \{ \text{Father(alice,bob), Mother(alice,carla), Mother(evan,carla), Father(carla,david)} \}$

$T_P^2 = T_P^1 \cup \{ \text{Parent(alice,bob), Parent(alice,carla), Parent(evan,carla), Parent(carla,david),}$

$\text{SameGeneration(alice,alice), SameGeneration(carla,carla), SameGeneration(david,david)}$,

$\text{SameGeneration(evan,evan)} \}$

$T_P^3 = T_P^2 \cup \{ \text{Ancestor(alice,bob), Ancestor(alice,carla), Ancestor(evan,carla), Ancestor(carla,david),}$

$\text{SameGeneration(alice,evan), SameGeneration(evan,alice)} \}$

$T_P^4 = T_P^3 \cup \{ \text{Ancestor(alice,david), Ancestor(evan,david)} \} = T_P^5$

$\cdots = T_P^\infty$
Exercise 2

Exercise. Consider the example Datalog program from the lecture:

\[
\begin{align*}
(a) & \quad \text{Father}(\text{alice, bob}) \\
(b) & \quad \text{Mother}(\text{alice, carla}) \\
(c) & \quad \text{Mother}(\text{evan, carla}) \\
(d) & \quad \text{Father}(\text{carla, david}) \\
\end{align*}
\]

\[
\begin{align*}
\text{Parent}(x, y) & \leftarrow \text{Father}(x, y) & (1) \\
\text{Parent}(x, y) & \leftarrow \text{Mother}(x, y) & (2) \\
\text{Ancestor}(x, y) & \leftarrow \text{Parent}(x, y) & (3) \\
\text{Ancestor}(x, z) & \leftarrow \text{Parent}(x, y) \land \text{Ancestor}(y, z) & (4) \\
\text{SameGeneration}(x, x) & \leftarrow & (5) \\
\text{SameGeneration}(x, y) & \leftarrow \text{Parent}(x, v) \land \text{Parent}(y, w) \land \text{SameGeneration}(v, w) & (6)
\end{align*}
\]

1. Give a proof tree for \text{SameGeneration}(\text{evan, alice}).
2. Compute the sets \( T_P^0, T_P^1, T_P^2, \ldots \) When is the fixed point reached?

Solution.

2. 
\[
\begin{align*}
T_P^0 &= \emptyset \\
T_P^1 &= \{ \text{Father}(\text{alice, bob}), \text{Mother}(\text{alice, carla}), \text{Mother}(\text{evan, carla}), \text{Father}(\text{carla, david}) \} \\
T_P^2 &= T_P^1 \cup \{ \text{Parent}(\text{alice, bob}), \text{Parent}(\text{alice, carla}), \text{Parent}(\text{evan, carla}), \text{Parent}(\text{carla, david}), \text{SameGeneration}(\text{alice, alice}), \text{SameGeneration}(\text{bob, bob}), \text{SameGeneration}(\text{carla, carla}), \text{SameGeneration}(\text{david, david}), \text{SameGeneration}(\text{evan, evan}) \} \\
T_P^3 &= T_P^2 \cup \{ \text{Ancestor}(\text{alice, bob}), \text{Ancestor}(\text{alice, carla}), \text{Ancestor}(\text{evan, carla}), \text{Ancestor}(\text{carla, david}), \text{SameGeneration}(\text{alice, evan}), \text{SameGeneration}(\text{evan, alice}) \} \\
T_P^4 &= T_P^3 \cup \{ \text{Ancestor}(\text{alice, david}), \text{Ancestor}(\text{evan, david}) \} = T_P^5 = T_P^\infty
\end{align*}
\]
Exercise 3

**Exercise.** Consider databases that encode a labelled, directed graph by means of a ternary EDB predicate $e$ ("edge"). The first two parameters are the source and target nodes of the edge, while the third parameter is its label. For example, the edge $m \xrightarrow{a} n$ would be represented by the fact $e(m, n, a)$. Moreover, assume that only constants $a$ and $b$ are used as labels. Can you express the following queries using Datalog?

1. “Which nodes in the graph are reachable from the node $n$?”
2. “Are all nodes of the graph reachable from the node $n$?”
3. “Does the graph have a directed cycle?”
4. “Does the graph have a path that is labelled by a palindrome?”
   (a palindrome is a word that reads the same forwards and backwards)
5. “Is the connected component that contains the node $n$ 2-colourable?”
6. “Is the graph 2-colourable?”
7. “Which pairs of nodes are connected by a path with an even number of $a$ labels?”
8. “Which pairs of nodes are connected by a path with the same number of $a$ and $b$ labels?”
9. “Is there a pair of nodes that is connected by two distinct paths?”
Exercise 3

Exercise. Consider databases that encode a labelled, directed graph by means of a ternary EDB predicate $e$ ("edge"). The first two parameters are the source and target nodes of the edge, while the third parameter is its label. For example, the edge $m \xrightarrow{a} n$ would be represented by the fact $e(m, n, a)$. Moreover, assume that only constants $a$ and $b$ are used as labels. Can you express the following queries using Datalog?

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5. “Is the connected component that contains the node $n$ 2-colourable?”
6. “Is the graph 2-colourable?”
7. “Which pairs of nodes are connected by a path with an even number of $a$ labels?”
8. “Which pairs of nodes are connected by a path with the same number of $a$ and $b$ labels?”
9. “Is there a pair of nodes that is connected by two distinct paths?”

Solution.
Exercise 3

Exercise. Consider databases that encode a labelled, directed graph by means of a ternary EDB predicate e (“edge”). The first two parameters are the source and target nodes of the edge, while the third parameter is its label. For example, the edge $m \rightarrow a n$ would be represented by the fact $e(m, n, a)$. Moreover, assume that only constants $a$ and $b$ are used as labels. Can you express the following queries using Datalog?

1. “Which nodes in the graph are reachable from the node $n$?”
2. “Are all nodes of the graph reachable from the node $n$?”
3. “Does the graph have a directed cycle?”
4. “Does the graph have a path that is labelled by a palindrome?”
   (a palindrome is a word that reads the same forwards and backwards)
5. “Is the connected component that contains the node $n$ 2-colourable?”
6. “Is the graph 2-colourable?”
7. “Which pairs of nodes are connected by a path with an even number of $a$ labels?”
8. “Which pairs of nodes are connected by a path with the same number of $a$ and $b$ labels?”
9. “Is there a pair of nodes that is connected by two distinct paths?”

Solution.

1. 

\[
\text{Reachable}(x, y) \leftarrow e(x, y, v) \\
\text{Reachable}(x, z) \leftarrow e(x, y, v) \land \text{Reachable}(y, z) \\
\text{Ans}(x) \leftarrow \text{Reachable}(n, x)
\]
Exercise 3

Exercise. Consider databases that encode a labelled, directed graph by means of a ternary EDB predicate e ("edge"). The first two parameters are the source and target nodes of the edge, while the third parameter is its label. For example, the edge $m \xrightarrow{a} n$ would be represented by the fact $e(m, n, a)$. Moreover, assume that only constants $a$ and $b$ are used as labels. Can you express the following queries using Datalog?

1. “Which nodes in the graph are reachable from the node $n$?”
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   (a palindrome is a word that reads the same forwards and backwards)
5. “Is the connected component that contains the node $n$ 2-colourable?”
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7. “Which pairs of nodes are connected by a path with an even number of $a$ labels?”
8. “Which pairs of nodes are connected by a path with the same number of $a$ and $b$ labels?”
9. “Is there a pair of nodes that is connected by two distinct paths?”

Solution.

2. Not expressible, since Datalog is monotone: any query that is true for some set of ground facts $I$ is also true for every set of ground facts $J \supseteq I$, but the query is true on $I = \{ e(n, n, a) \}$, but not on $J = I \cup \{ e(m, m, b) \}$. 


Exercise 3

Exercise. Consider databases that encode a labelled, directed graph by means of a ternary EDB predicate $e$ ("edge"). The first two parameters are the source and target nodes of the edge, while the third parameter is its label. For example, the edge $m \rightarrow n$ would be represented by the fact $e(m, n, a)$. Moreover, assume that only constants $a$ and $b$ are used as labels. Can you express the following queries using Datalog?

1. “Which nodes in the graph are reachable from the node $n$?”
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3. “Does the graph have a directed cycle?”
4. “Does the graph have a path that is labelled by a palindrome?”
   (a palindrome is a word that reads the same forwards and backwards)
5. “Is the connected component that contains the node $n$ 2-colourable?”
6. “Is the graph 2-colourable?”
7. “Which pairs of nodes are connected by a path with an even number of $a$ labels?”
8. “Which pairs of nodes are connected by a path with the same number of $a$ and $b$ labels?”
9. “Is there a pair of nodes that is connected by two distinct paths?”

Solution.

3.

Reachable($x, y$) ← $e(x, y, v)$
Reachable($x, z$) ← $e(x, y, v) \land$ Reachable($y, z$)
Ans() ← Reachable($x, x$)
Exercise 3

Exercise. Consider databases that encode a labelled, directed graph by means of a ternary EDB predicate e (“edge”). The first two parameters are the source and target nodes of the edge, while the third parameter is its label. For example, the edge $m \rightarrow n$ would be represented by the fact $e(m, n, a)$. Moreover, assume that only constants $a$ and $b$ are used as labels. Can you express the following queries using Datalog?

1. “Which nodes in the graph are reachable from the node $n$?”
2. “Are all nodes of the graph reachable from the node $n$?”
3. “Does the graph have a directed cycle?”
4. “Does the graph have a path that is labelled by a palindrome?”
   (a palindrome is a word that reads the same forwards and backwards)
5. “Is the connected component that contains the node $n$ 2-colourable?”
6. “Is the graph 2-colourable?”
7. “Which pairs of nodes are connected by a path with an even number of $a$ labels?”
8. “Which pairs of nodes are connected by a path with the same number of $a$ and $b$ labels?”
9. “Is there a pair of nodes that is connected by two distinct paths?”

Solution.

4.

\[
\text{Reachable}(x, y) \leftarrow e(x, y, v) \quad \text{Reachable}(x, z) \leftarrow e(x, y, a), \text{Reachable}(y, w), e(w, z, a) \\
\text{Reachable}(x, z) \leftarrow e(x, y, b), \text{Reachable}(y, w), e(w, z, b) \quad \text{Ans}() \leftarrow \text{Reachable}(x, y)
\]
Exercise 3

Exercise. Consider databases that encode a labelled, directed graph by means of a ternary EDB predicate \(e\) ("edge"). The first two parameters are the source and target nodes of the edge, while the third parameter is its label. For example, the edge \(m \xrightarrow{a} n\) would be represented by the fact \(e(m, n, a)\). Moreover, assume that only constants \(a\) and \(b\) are used as labels. Can you express the following queries using Datalog?

1. “Which nodes in the graph are reachable from the node \(n\)?”
2. “Are all nodes of the graph reachable from the node \(n\)?”
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   (a palindrome is a word that reads the same forwards and backwards)
5. “Is the connected component that contains the node \(n\) 2-colourable?”
6. “Is the graph 2-colourable?”
7. “Which pairs of nodes are connected by a path with an even number of \(a\) labels?”
8. “Which pairs of nodes are connected by a path with the same number of \(a\) and \(b\) labels?”
9. “Is there a pair of nodes that is connected by two distinct paths?”

Solution.

5. Not expressible; consider \(I = \{ e(n, 1, a), e(1, 2, a) \}\) and \(J = I \cup \{ e(2, n, a) \}\).
Exercise 3

**Exercise.** Consider databases that encode a labelled, directed graph by means of a ternary EDB predicate \( e \) ("edge"). The first two parameters are the source and target nodes of the edge, while the third parameter is its label. For example, the edge \( m \xrightarrow{a} n \) would be represented by the fact \( e(m, n, a) \). Moreover, assume that only constants \( a \) and \( b \) are used as labels. Can you express the following queries using Datalog?

1. “Which nodes in the graph are reachable from the node \( n \)?”
2. “Are all nodes of the graph reachable from the node \( n \)?”
3. “Does the graph have a directed cycle?”
4. “Does the graph have a path that is labelled by a palindrome?” (a palindrome is a word that reads the same forwards and backwards)
5. “Is the connected component that contains the node \( n \) 2-colourable?”
6. “Is the graph 2-colourable?”
7. “Which pairs of nodes are connected by a path with an even number of \( a \) labels?”
8. “Which pairs of nodes are connected by a path with the same number of \( a \) and \( b \) labels?”
9. “Is there a pair of nodes that is connected by two distinct paths?”

**Solution.**

6. Not expressible; consider \( I = \{ e(n, 1, a), e(1, 2, a) \} \) and \( J = I \cup \{ e(2, n, a) \} \).
Exercise 3

Exercise. Consider databases that encode a labelled, directed graph by means of a ternary EDB predicate e (“edge”). The first two parameters are the source and target nodes of the edge, while the third parameter is its label. For example, the edge $m \rightarrow a n$ would be represented by the fact $e(m, n, a)$. Moreover, assume that only constants $a$ and $b$ are used as labels. Can you express the following queries using Datalog?

1. “Which nodes in the graph are reachable from the node $n$?”
2. “Are all nodes of the graph reachable from the node $n$?”
3. “Does the graph have a directed cycle?”
4. “Does the graph have a path that is labelled by a palindrome?”
   (a palindrome is a word that reads the same forwards and backwards)
5. “Is the connected component that contains the node $n$ 2-colourable?”
6. “Is the graph 2-colourable?”
7. “Which pairs of nodes are connected by a path with an even number of $a$ labels?”
8. “Which pairs of nodes are connected by a path with the same number of $a$ and $b$ labels?”
9. “Is there a pair of nodes that is connected by two distinct paths?”

Solution.

7.

Reachable($x, y$) ← $e(x, y, b)$
Reachable($x, z$) ← $e(x, y, a)$, Reachable($y, w$), $e(w, z, a)$
Ans($x, y$) ← Reachable($x, y$)

Reachable($x, z$) ← $e(x, y, a)$, $e(y, z, a)$
Reachable($x, z$) ← Reachable($x, y$), Reachable($y, z$)
Exercise 3

**Exercise.** Consider databases that encode a labelled, directed graph by means of a ternary EDB predicate \( e \) ("edge"). The first two parameters are the source and target nodes of the edge, while the third parameter is its label. For example, the edge \( m \rightarrow n \) would be represented by the fact \( e(m, n, a) \). Moreover, assume that only constants \( a \) and \( b \) are used as labels. Can you express the following queries using Datalog?

1. “Which nodes in the graph are reachable from the node \( n \)?”
2. “Are all nodes of the graph reachable from the node \( n \)?”
3. “Does the graph have a directed cycle?”
4. “Does the graph have a path that is labelled by a palindrome?”
   (a palindrome is a word that reads the same forwards and backwards)
5. “Is the connected component that contains the node \( n \) 2-colourable?”
6. “Is the graph 2-colourable?”
7. “Which pairs of nodes are connected by a path with an even number of \( a \) labels?”
8. “Which pairs of nodes are connected by a path with the same number of \( a \) and \( b \) labels?”
9. “Is there a pair of nodes that is connected by two distinct paths?”

**Solution.**

8.

\[
\text{Reachable}(x, z) \leftarrow e(x, y, a), e(y, z, b) \\
\text{Reachable}(x, z) \leftarrow e(x, y, b), e(y, z, a) \\
\text{Reachable}(x, z) \leftarrow \text{Reachable}(x, y), \text{Reachable}(y, z) \\
\text{Ans}(x, y) \leftarrow \text{Reachable}(x, y)
\]
Exercise 3

**Exercise.** Consider databases that encode a labelled, directed graph by means of a ternary EDB predicate e ("edge"). The first two parameters are the source and target nodes of the edge, while the third parameter is its label. For example, the edge \( m \rightarrow n \) would be represented by the fact \( e(m, n, a) \). Moreover, assume that only constants \( a \) and \( b \) are used as labels. Can you express the following queries using Datalog?

1. “Which nodes in the graph are reachable from the node \( n \)?”
2. “Are all nodes of the graph reachable from the node \( n \)?”
3. “Does the graph have a directed cycle?”
4. “Does the graph have a path that is labelled by a palindrome?”
   (a palindrome is a word that reads the same forwards and backwards)
5. “Is the connected component that contains the node \( n \) 2-colourable?”
6. “Is the graph 2-colourable?”
7. “Which pairs of nodes are connected by a path with an even number of \( a \) labels?”
8. “Which pairs of nodes are connected by a path with the same number of \( a \) and \( b \) labels?”
9. “Is there a pair of nodes that is connected by two distinct paths?”

**Solution.**

9. Not expressible, since Datalog is *homomorphism-closed*; consider \( I = \{ e(n, 1, a), e(1, m, a), e(n, 2, a), e(2, m, a) \} \) and \( J = \{ e(n, 1, a), e(1, m, a) \} \) and the homomorphism \( \varphi : I \rightarrow J = \{ 2 \mapsto 1 \} \).
Exercise 4

**Exercise.** Consider a UCQ of the following form

\[(r_{11}(x) \land r_{12}(x)) \lor \ldots \lor (r_{1\ell}(x) \land r_{2\ell}(x))\]

Find a Datalog query that expresses this UCQ. How many rules and how many additional IDB predicates does your solution use (depending on \(\ell\))? 

**Solution.**

\[
\text{Ans}(x) \leftarrow r_{11}(x), r_{12}(x), \ldots, r_{1\ell}(x), r_{2\ell}(x)
\]

This solution uses \(\ell\) rules and one additional IDB predicate.
Exercise 4

**Exercise.** Consider a UCQ of the following form

\[(r_{11}(x) \land r_{12}(x)) \lor \ldots \lor (r_{\ell 1}(x) \land r_{\ell 2}(x))\]

Find a Datalog query that expresses this UCQ. How many rules and how many additional IDB predicates does your solution use (depending on \(\ell\))? 

**Solution.**
Exercise 4

**Exercise.** Consider a UCQ of the following form

$$(r_{11}(x) \land r_{12}(x)) \lor \ldots \lor (r_{\ell 1}(x) \land r_{\ell 2}(x))$$

Find a Datalog query that expresses this UCQ. How many rules and how many additional IDB predicates does your solution use (depending on $\ell$)?

**Solution.**

\[
\begin{align*}
\text{Ans}(x) & \leftarrow r_{11}(x), r_{12}(x) \\
\text{Ans}(x) & \leftarrow r_{21}(x), r_{22}(x) \\
& \vdots \\
\text{Ans}(x) & \leftarrow r_{\ell 1}(x), r_{\ell 2}(x)
\end{align*}
\]
Exercise 4

Exercise. Consider a UCQ of the following form

\[(r_{11}(x) \land r_{12}(x)) \lor \ldots \lor (r_{\ell 1}(x) \land r_{\ell 2}(x))\]

Find a Datalog query that expresses this UCQ. How many rules and how many additional IDB predicates does your solution use (depending on \(\ell\))? 

Solution.

\[
\text{Ans}(x) \leftarrow r_{11}(x), r_{12}(x) \\
\text{Ans}(x) \leftarrow r_{21}(x), r_{22}(x) \\
\vdots \quad \vdots \quad \vdots \\
\text{Ans}(x) \leftarrow r_{\ell 1}(x), r_{\ell 2}(x)
\]

This solution uses \(\ell\) rules and one additional IDB predicate.
Exercise 5

**Exercise.** Consider a Datalog query of the following form:

\[
\begin{align*}
A_1(x) & \leftarrow r_{11}(x) & \ldots & A_\ell(x) & \leftarrow r_{\ell 1}(x) \\
A_1(x) & \leftarrow r_{12}(x) & \ldots & A_\ell(x) & \leftarrow r_{\ell 2}(x) \\
\text{Ans}(x) & \leftarrow A_1(x), \ldots, A_\ell(x)
\end{align*}
\]

Find a UCQ that expresses this Datalog query. How many CQs does your solution contain (depending on \(\ell\))?
Exercise 5

Exercise. Consider a Datalog query of the following form:

\[
\begin{align*}
A_1(x) & \leftarrow r_{11}(x) \quad \ldots \quad A_{\ell}(x) & \leftarrow r_{\ell 1}(x) \\
A_1(x) & \leftarrow r_{12}(x) \quad \ldots \quad A_{\ell}(x) & \leftarrow r_{\ell 2}(x) \\
\text{Ans}(x) & \leftarrow A_1(x), \ldots, A_{\ell}(x)
\end{align*}
\]

Find a UCQ that expresses this Datalog query. How many CQs does your solution contain (depending on \(\ell\))? 

Solution.
Exercise. Consider a Datalog query of the following form:

\[ A_1(x) \leftarrow r_{11}(x) \quad \ldots \quad A_\ell(x) \leftarrow r_{\ell_1}(x) \]
\[ A_1(x) \leftarrow r_{12}(x) \quad \ldots \quad A_\ell(x) \leftarrow r_{\ell_2}(x) \]
\[ \text{Ans}(x) \leftarrow A_1(x), \ldots, A_\ell(x) \]

Find a UCQ that expresses this Datalog query. How many CQs does your solution contain (depending on \( \ell \))?

Solution.

\[ \varphi_{11\ldots 1}(x) = r_{11}(x) \land r_{21}(x) \land \cdots \land r_{\ell_1}(x) \]
\[ \varphi_{21\ldots 1}(x) = r_{12}(x) \land r_{21}(x) \land \cdots \land r_{\ell_1}(x) \]
\[ \varphi_{12\ldots 1}(x) = r_{11}(x) \land r_{22}(x) \land \cdots \land r_{\ell_1}(x) \]
\[ \varphi_{22\ldots 1}(x) = r_{12}(x) \land r_{22}(x) \land \cdots \land r_{\ell_1}(x) \]
\[ \vdots \quad \vdots \quad \vdots \quad \vdots \]
\[ \varphi_{22\ldots 2}(x) = r_{12}(x) \land r_{22}(x) \land \cdots \land r_{\ell_2}(x) \]
\[ \varphi = \bigvee_{i \in \{11\ldots 1,21\ldots 1,\ldots,22\ldots 2\}} \varphi_i \]
**Exercise 5**

**Exercise.** Consider a Datalog query of the following form:

\[
\begin{align*}
A_1(x) &\leftarrow r_{11}(x) &\ldots& A_\ell(x) &\leftarrow r_{\ell 1}(x) \\
A_1(x) &\leftarrow r_{12}(x) &\ldots& A_\ell(x) &\leftarrow r_{\ell 2}(x) \\
{\text{Ans}}(x) &\leftarrow A_1(x), \ldots, A_\ell(x)
\end{align*}
\]

Find a UCQ that expresses this Datalog query. How many CQs does your solution contain (depending on \(\ell\))?

**Solution.**

\[
\begin{align*}
\varphi_{1\ldots 1}(x) &= r_{11}(x) \land r_{21}(x) \land \cdots \land r_{\ell 1}(x) \\
\varphi_{2\ldots 1}(x) &= r_{12}(x) \land r_{21}(x) \land \cdots \land r_{\ell 1}(x) \\
\varphi_{1\ldots 2}(x) &= r_{11}(x) \land r_{22}(x) \land \cdots \land r_{\ell 1}(x) \\
\varphi_{2\ldots 2}(x) &= r_{12}(x) \land r_{22}(x) \land \cdots \land r_{\ell 2}(x)
\end{align*}
\]

\[
\varphi = \bigvee_{i \in \{1\ldots 1, 2\ldots 1, \ldots, 2\ldots 2\}} \varphi_i
\]

This solution uses \(2^\ell\) CQs.
Exercise 6

Exercise. Show that $T_P^\infty$ is the least fixed point of the $T_P$ operator.

1. Show that it is a fixed point, i.e., that $T_P(T_P^\infty) = T_P^\infty$.
2. Show that every fixed point of $T_P$ must contain every fact in $T_P^\infty$. 
Exercise 6

Exercise. Show that $T^\infty_P$ is the least fixed point of the $T_P$ operator.

1. Show that it is a fixed point, i.e., that $T_P(T^\infty_P) = T^\infty_P$.
2. Show that every fixed point of $T_P$ must contain every fact in $T^\infty_P$.

Solution.
**Exercise 6**

**Exercise.** Show that $T_P^\infty$ is the least fixed point of the $T_P$ operator.

1. Show that it is a fixed point, i.e., that $T_P(T_P^\infty) = T_P^\infty$.
2. Show that every fixed point of $T_P$ must contain every fact in $T_P^\infty$.

**Solution.**

1. ▶ We first show that $T_P$ is extensive, i.e., that $I \subseteq T_P(I)$ for any set of ground facts $I$: Clearly $\emptyset \subseteq T_P(\emptyset)$.
Exercise 6

**Exercise.** Show that $T^\infty_P$ is the least fixed point of the $T_P$ operator.

1. Show that it is a fixed point, i.e., that $T_P(T^\infty_P) = T^\infty_P$.
2. Show that every fixed point of $T_P$ must contain every fact in $T^\infty_P$.

**Solution.**

1. We first show that $T_P$ is **extensive**, i.e., that $I \subseteq T_P(I)$ for any set of ground facts $I$: Clearly $\emptyset \subseteq T_P(\emptyset)$.
   - Assume that $I \subseteq T_P(I)$ for some set of ground facts $I$, and consider a ground fact $H \in T_P(I)$. Then there is some ground rule $H \leftarrow B_1, \ldots, B_n \in \text{ground}(P)$ with $B_1, \ldots, B_n \in I$. Since $I \subseteq T_P(I)$, we have $B_1, \ldots, B_n \in T_P(I)$, and hence $H \in T_P(T_P(I))$. 

2. ▶ First, note that $T_P$ is clearly **monotone**, i.e., that for sets $I \subseteq J$ of ground facts, we have $T_P(I) \subseteq T_P(J)$.
   ▶ Consider some fixed point $F$ of $T_P$. We show $T_i P \subseteq F$ for all $i \geq 0$.
     ▶ Clearly, $T_0 P = \emptyset \subseteq F$.
     ▶ Assume that $T_i P \subseteq F$ for some $i \geq 0$. Then $T_{i+1} P = T_P(T_i P) \subseteq T_P(F)$, by monotonicity and since $F$ is a fixed point.
     ▶ But then $T_i P \subseteq F$ for all $i \geq 0$, and hence also $T^\infty_P = \bigcup_{i \geq 0} T_i P \subseteq F$. 

Exercise. Show that $T_P^\infty$ is the least fixed point of the $T_P$ operator.

1. Show that it is a fixed point, i.e., that $T_P(T_P^\infty) = T_P^\infty$.
2. Show that every fixed point of $T_P$ must contain every fact in $T_P^\infty$.

Solution.

1. We first show that $T_P$ is extensive, i.e., that $I \subseteq T_P(I)$ for any set of ground facts $I$: Clearly $\emptyset \subseteq T_P(\emptyset)$.
   Assume that $I \subseteq T_P(I)$ for some set of ground facts $I$, and consider a ground fact $H \in T_P(I)$. Then there is some ground rule $H \leftarrow B_1, \ldots, B_n \in \text{ground}(P)$ with $B_1, \ldots, B_n \in I$. Since $I \subseteq T_P(I)$, we have $B_1, \ldots, B_n \in T_P(I)$, and hence $H \in T_P(T_P(I))$.
   Thus, we have $T_P^{-1} \subseteq T_P$, and, in particular, $T_P(T_P^\infty) \supseteq T_P^\infty$.
Exercise 6

Exercise. Show that $T_P^\infty$ is the least fixed point of the $T_P$ operator.

1. Show that it is a fixed point, i.e., that $T_P(T_P^\infty) = T_P^\infty$.
2. Show that every fixed point of $T_P$ must contain every fact in $T_P^\infty$.

Solution.

1. We first show that $T_P$ is extensive, i.e., that $I \subseteq T_P(I)$ for any set of ground facts $I$: Clearly $\emptyset \subseteq T_P(\emptyset)$.
   - Assume that $I \subseteq T_P(I)$ for some set of ground facts $I$, and consider a ground fact $H \in T_P(I)$. Then there is some ground rule $H \leftarrow B_1, \ldots, B_n \in \text{ground}(P)$ with $B_1, \ldots, B_n \in I$. Since $I \subseteq T_P(I)$, we have $B_1, \ldots, B_n \in T_P(I)$, and hence $H \in T_P(T_P(I))$.
   - Thus, we have $T_P^{-1} \subseteq T_P$, and, in particular, $T_P(T_P^\infty) \supseteq T_P^\infty$.
   - Assume that we have some ground fact $H \in T_P(T_P^\infty)$, but $H \notin T_P^\infty$. 

2. ▶ First, note that $T_P$ is clearly monotone, i.e., that for sets $I \subseteq J$ of ground facts, we have $T_P(I) \subseteq T_P(J)$.
   - Consider some fixed point $F$ of $T_P$. We show $T_P^i \subseteq F$ for all $i \geq 0$.
     ▶ Clearly, $T_P^0 = \emptyset \subseteq F$.
     ▶ Assume that $T_P^i \subseteq F$ for some $i \geq 0$. Then $T_P^{i+1} \subseteq T_P(F) = F$, by monotonicity and since $F$ is a fixed point.
     ▶ But then $T_P^i \subseteq F$ for all $i \geq 0$, and hence also $T_P^\infty = \bigcup_{i \geq 0} T_P^i \subseteq F$. 

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Exercise 6

Exercise. Show that $T_P^\infty$ is the least fixed point of the $T_P$ operator.

1. Show that it is a fixed point, i.e., that $T_P(T_P^\infty) = T_P^\infty$.
2. Show that every fixed point of $T_P$ must contain every fact in $T_P^\infty$.

Solution.

1. We first show that $T_P$ is extensive, i.e., that $I \subseteq T_P(I)$ for any set of ground facts $I$: Clearly $\emptyset \subseteq T_P(\emptyset)$.

Assume that $I \subseteq T_P(I)$ for some set of ground facts $I$, and consider a ground fact $H \in T_P(I)$. Then there is some ground rule $H \leftarrow B_1, \ldots, B_n \in \text{ground}(P)$ with $B_1, \ldots, B_n \in I$. Since $I \subseteq T_P(I)$, we have $B_1, \ldots, B_n \in T_P(I)$, and hence $H \in T_P(T_P(I))$.

Thus, we have $T_P^i \subseteq T_P$, and, in particular, $T_P(T_P^\infty) \supseteq T_P^\infty$.

Assume that we have some ground fact $H \in T_P(T_P^\infty)$, but $H \notin T_P^\infty$.

Then there is a ground rule $H \leftarrow B_1, \ldots, B_n \in \text{ground}(P)$ with $B_1, \ldots, B_n \in T_P^\infty$. 


Exercise 6

Exercise. Show that $T_P^\infty$ is the least fixed point of the $T_P$ operator.

1. Show that it is a fixed point, i.e., that $T_P(T_P^\infty) = T_P^\infty$.

2. Show that every fixed point of $T_P$ must contain every fact in $T_P^\infty$.

Solution.

1. We first show that $T_P$ is extensive, i.e., that $I \subseteq T_P(I)$ for any set of ground facts $I$: Clearly $\emptyset \subseteq T_P(\emptyset)$.

   Assume that $I \subseteq T_P(I)$ for some set of ground facts $I$, and consider a ground fact $H \in T_P(I)$. Then there is some ground rule $H \leftarrow B_1, \ldots, B_n \in \text{ground}(P)$ with $B_1, \ldots, B_n \in I$. Since $I \subseteq T_P(I)$, we have $B_1, \ldots, B_n \in T_P(I)$, and hence $H \in T_P(T_P(I))$.

   Thus, we have $T_{P}^{-1} \subseteq T_{P}$, and, in particular, $T_{P}(T_{P}^{\infty}) \supseteq T_{P}^{\infty}$.

   Assume that we have some ground fact $H \in T_{P}(T_{P}^{\infty})$, but $H \notin T_{P}^{\infty}$.

   Then there is a ground rule $H \leftarrow B_1, \ldots, B_n \in \text{ground}(P)$ with $B_1, \ldots, B_n \in T_{P}^{\infty}$.

   Since $T_{P}^{\infty} = \bigcup_{i \geq 0} T_{P}^{i}$, there are $i_1, \ldots, i_n$ with $B_{j}^{i_j} \in T_{P}^{i_j}$, and thus $B_1, \ldots, B_n \in T_{P}^{k}$ with $k = \max\{i_1, \ldots, i_n\}$.
Exercise 6

Exercise. Show that $T_P^\infty$ is the least fixed point of the $T_P$ operator.

1. Show that it is a fixed point, i.e., that $T_P(T_P^\infty) = T_P^\infty$.
2. Show that every fixed point of $T_P$ must contain every fact in $T_P^\infty$.

Solution.

1. ▶ We first show that $T_P$ is extensive, i.e., that $I \subseteq T_P(I)$ for any set of ground facts $I$: Clearly $\emptyset \subseteq T_P(\emptyset)$.
   ▶ Assume that $I \subseteq T_P(I)$ for some set of ground facts $I$, and consider a ground fact $H \in T_P(I)$. Then there is some ground rule $H \leftarrow B_1, \ldots, B_n \in \text{ground}(P)$ with $B_1, \ldots, B_n \in I$. Since $I \subseteq T_P(I)$, we have $B_1, \ldots, B_n \in T_P(I)$, and hence $H \in T_P(T_P(I))$.
   ▶ Thus, we have $T_P^{i-1} \subseteq T_P^i$, and, in particular, $T_P(T_P^\infty) \supseteq T_P^\infty$.
   ▶ Assume that we have some ground fact $H \in T_P(T_P^\infty)$, but $H \notin T_P^\infty$.
   ▶ Then there is a ground rule $H \leftarrow B_1, \ldots, B_n \in \text{ground}(P)$ with $B_1, \ldots, B_n \in T_P^\infty$.
   ▶ Since $T_P^\infty = \bigcup_{i \geq 0} T_P^i$, there are $i_1, \ldots, i_n$ with $B_{i_j} \in T_P^{i_j}$, and thus $B_1, \ldots, B_n \in T_P^k$ with $k = \max\{i_1, \ldots, i_n\}$.
   ▶ But then $H \in T_P(T_P^k) = T_P^{k+1} \subseteq T_P^\infty$, which contradicts $H \notin T_P^\infty$.
Exercise 6

Exercise. Show that $T_P^\infty$ is the least fixed point of the $T_P$ operator.

1. Show that it is a fixed point, i.e., that $T_P(T_P^\infty) = T_P^\infty$.

2. Show that every fixed point of $T_P$ must contain every fact in $T_P^\infty$.

Solution.

1. We first show that $T_P$ is extensive, i.e., that $I \subseteq T_P(I)$ for any set of ground facts $I$: Clearly $\emptyset \subseteq T_P(\emptyset)$.

   Assume that $I \subseteq T_P(I)$ for some set of ground facts $I$, and consider a ground fact $H \in T_P(I)$. Then there is some ground rule $H \leftarrow B_1, \ldots, B_n \in \text{ground}(P)$ with $B_1, \ldots, B_n \in I$. Since $I \subseteq T_P(I)$, we have $B_1, \ldots, B_n \in T_P(I)$, and hence $H \in T_P(T_P(I))$.

   Thus, we have $T_P^i \subseteq T_P$, and, in particular, $T_P(T_P^\infty) \supseteq T_P^\infty$.

   Assume that we have some ground fact $H \in T_P(T_P^\infty)$, but $H \notin T_P^\infty$.

   Then there is a ground rule $H \leftarrow B_1, \ldots, B_n \in \text{ground}(P)$ with $B_1, \ldots, B_n \in T_P^\infty$.

   Since $T_P^\infty = \bigcup_{i \geq 0} T_P^i$, there are $i_1, \ldots, i_n$ with $B_{ij} \in T_P^i$, and thus $B_1, \ldots, B_n \in T_P^k$ with $k = \max\{i_1, \ldots, i_n\}$.

   But then $H \in T_P(T_P^k) = T_P^{k+1} \subseteq T_P^\infty$, which contradicts $H \notin T_P^\infty$.

2. First, note that $T_P$ is clearly monotone, i.e., that for sets $I \subseteq J$ of ground facts, we have $T_P(I) \subseteq T_P(J)$. 
Exercise 6

Exercise. Show that $T_P^\omega$ is the least fixed point of the $T_P$ operator.

1. Show that it is a fixed point, i.e., that $T_P(T_P^\omega) = T_P^\omega$.
2. Show that every fixed point of $T_P$ must contain every fact in $T_P^\omega$.

Solution.

1. ▶ We first show that $T_P$ is extensive, i.e., that $I \subseteq T_P(I)$ for any set of ground facts $I$: Clearly $\emptyset \subseteq T_P(\emptyset)$.
   ▶ Assume that $I \subseteq T_P(I)$ for some set of ground facts $I$, and consider a ground fact $H \in T_P(I)$. Then there is some ground rule $H \leftarrow B_1, \ldots, B_n \in \text{ground}(P)$ with $B_1, \ldots, B_n \in I$. Since $I \subseteq T_P(I)$, we have $B_1, \ldots, B_n \in T_P(I)$, and hence $H \in T_P(T_P(I))$.
   ▶ Thus, we have $T_P^{i-1} \subseteq T_P^i$, and, in particular, $T_P(T_P^\omega) \supseteq T_P^\omega$.
   ▶ Assume that we have some ground fact $H \in T_P(T_P^\omega)$, but $H \notin T_P^\omega$.
   ▶ Then there is a ground rule $H \leftarrow B_1, \ldots, B_n \in \text{ground}(P)$ with $B_1, \ldots, B_n \in T_P^\omega$.
   ▶ Since $T_P^\omega = \bigcup_{i \geq 0} T_P^i$, there are $i_1, \ldots, i_n$ with $B_{ij}^i \in T_P^i$, and thus $B_1, \ldots, B_n \in T_P^k$ with $k = \max\{i_1, \ldots, i_n\}$.
   ▶ But then $H \in T_P(T_P^k) = T_P^{k+1} \subseteq T_P^\omega$, which contradicts $H \notin T_P^\omega$.
2. ▶ First, note that $T_P$ is clearly monotone, i.e., that for sets $I \subseteq J$ of ground facts, we have $T_P(I) \subseteq T_P(J)$.
   ▶ Consider some fixed point $F$ of $T_P$. We show $T_P^i \subseteq F$ for all $i \geq 0$. 

Exercise 6

Exercise. Show that $T_P^∞$ is the least fixed point of the $T_P$ operator.

1. Show that it is a fixed point, i.e., that $T_P(T_P^∞) = T_P^∞$.

2. Show that every fixed point of $T_P$ must contain every fact in $T_P^∞$.

Solution.

1. We first show that $T_P$ is extensive, i.e., that $I ⊆ T_P(I)$ for any set of ground facts $I$. Clearly $∅ ⊆ T_P(∅)$.
   Assume that $I ⊆ T_P(I)$ for some set of ground facts $I$, and consider a ground fact $H ∈ T_P(I)$. Then there is some ground rule $H ← B_1, \ldots, B_n ∈ ground(P)$ with $B_1, \ldots, B_n ∈ I$. Since $I ⊆ T_P(I)$, we have $B_1, \ldots, B_n ∈ T_P(I)$, and hence $H ∈ T_P(T_P(I))$.
   Thus, we have $T_P^{-1}(I) ⊆ I$, and, in particular, $T_P(T_P^∞) ⊇ T_P^∞$.

2. First, note that $T_P$ is clearly monotone, i.e., that for sets $I ⊆ J$ of ground facts, we have $T_P(I) ⊆ T_P(J)$.
   Consider some fixed point $F$ of $T_P$. We show $T_P(I) ⊆ F$ for all $i ≥ 0$.
   Clearly, $T_P^0 = ∅ ⊆ F$. 
Exercise 6

Exercise. Show that $T_P^\infty$ is the least fixed point of the $T_P$ operator.

1. Show that it is a fixed point, i.e., that $T_P(T_P^\infty) = T_P^\infty$.

2. Show that every fixed point of $T_P$ must contain every fact in $T_P^\infty$.

Solution.

1. We first show that $T_P$ is extensive, i.e., that $I \subseteq T_P(I)$ for any set of ground facts $I$: Clearly $\emptyset \subseteq T_P(\emptyset)$.

   Assume that $I \subseteq T_P(I)$ for some set of ground facts $I$, and consider a ground fact $H \in T_P(I)$. Then there is some ground rule $H \leftarrow B_1, \ldots, B_n \in \text{ground}(P)$ with $B_1, \ldots, B_n \in I$. Since $I \subseteq T_P(I)$, we have $B_1, \ldots, B_n \in T_P(I)$, and hence $H \in T_P(T_P(I))$.

   Thus, we have $T_P^{i-1} \subseteq T_P^i$, and, in particular, $T_P(T_P^\infty) \supseteq T_P^\infty$.

   Assume that we have some ground fact $H \in T_P(T_P^\infty)$, but $H \not< T_P^\infty$.

   Then there is a ground rule $H \leftarrow B_1, \ldots, B_n \in \text{ground}(P)$ with $B_1, \ldots, B_n \in T_P^\infty$.

   Since $T_P^\infty = \bigcup_{i \geq 0} T_P^i$, there are $i_1, \ldots, i_n$ with $B_{ij} \in T_P^{i_j}$, and thus $B_1, \ldots, B_n \in T_P^k$ with $k = \max\{i_1, \ldots, i_n\}$.

   But then $H \in T_P(T_P^k) = T_P^{k+1} \subseteq T_P^\infty$, which contradicts $H \not< T_P^\infty$.

2. First, note that $T_P$ is clearly monotone, i.e., that for sets $I \subseteq J$ of ground facts, we have $T_P(I) \subseteq T_P(J)$.

   Consider some fixed point $F$ of $T_P$. We show $T_P^i \subseteq F$ for all $i \geq 0$.

   Clearly, $T_P^0 = \emptyset \subseteq F$.

   Assume that $T_P^i \subseteq F$ for some $i \geq 0$. Then $T_P^{i+1} = T_P(T_P^i) \subseteq T_P(F) = F$, by monotonicity and since $F$ is a fixed point.
Exercise 6

Exercise. Show that $T_P^\infty$ is the least fixed point of the $T_P$ operator.

1. Show that it is a fixed point, i.e., that $T_P(T_P^\infty) = T_P^\infty$.

2. Show that every fixed point of $T_P$ must contain every fact in $T_P^\infty$.

Solution.

1. ▶ We first show that $T_P$ is extensive, i.e., that $I \subseteq T_P(I)$ for any set of ground facts $I$: Clearly $\emptyset \subseteq T_P(\emptyset)$.
   ▶ Assume that $I \subseteq T_P(I)$ for some set of ground facts $I$, and consider a ground fact $H \in T_P(I)$. Then there is some ground rule $H \leftarrow B_1, \ldots, B_n \in \text{ground}(P)$ with $B_1, \ldots, B_n \in I$. Since $I \subseteq T_P(I)$, we have $B_1, \ldots, B_n \in T_P(I)$, and hence $H \in T_P(T_P(I))$.
   ▶ Thus, we have $T_P^{i-1} \subseteq T_P^i$, and, in particular, $T_P(T_P^\infty) \supseteq T_P^\infty$.
   ▶ Assume that we have some ground fact $H \in T_P(T_P^\infty)$, but $H \not< T_P^\infty$.
   ▶ Then there is a ground rule $H \leftarrow B_1, \ldots, B_n \in \text{ground}(P)$ with $B_1, \ldots, B_n \in T_P^\infty$.
   ▶ Since $T_P^\infty = \bigcup_{i \geq 0} T_P^i$, there are $i_1, \ldots, i_n$ with $B_{ij} \in T_P^{i_j}$, and thus $B_1, \ldots, B_n \in T_P^k$ with $k = \max\{i_1, \ldots, i_n\}$.
   ▶ But then $H \in T_P(T_P^k) = T_P^{k+1} \subseteq T_P^\infty$, which contradicts $H \not< T_P^\infty$.

2. ▶ First, note that $T_P$ is clearly monotone, i.e., that for sets $I \subseteq J$ of ground facts, we have $T_P(I) \subseteq T_P(J)$.
   ▶ Consider some fixed point $F$ of $T_P$. We show $T_P^i \subseteq F$ for all $i \geq 0$.
   ▶ Clearly, $T_P^0 = \emptyset \subseteq F$.
   ▶ Assume that $T_P^i \subseteq F$ for some $i \geq 0$. Then $T_P^{i+1} = T_P(T_P^i) \subseteq T_P(F) = F$, by monotonicity and since $F$ is a fixed point.
   ▶ But then $T_P^i \subseteq F$ for all $i \geq 0$, and hence also $T_P^\infty = \bigcup_{i \geq 0} T_P^i \subseteq F$. 