Finite and Algorithmic Model Theory

Lecture 3 (Dresden 26.10.22, Short version)

Lecturer: Bartosz "Bart" Bednarczyk

Technische Universität Dresden & Uniwersytet Wrocławski

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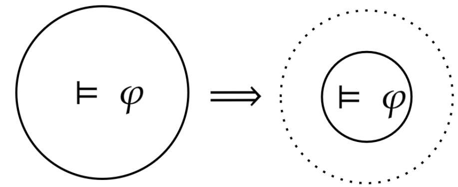
Today's agenda

Goal: Investigate important properties of FO and see whether they stay true in the finite.

1. Diagrams and embeddings.
2. Preservation Theorem of Łoś-Tarski.
3. Failure of Łoś-Tarski in the finite.
4. Discussion of related preservation theorems.
5. ~~Robinson's Joint-Consistency (without a proof).~~
6. ~~Craig Interpolation Property (CIP).~~
7. ~~Projective Beth's Definability Property (PBDP).~~

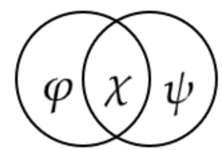


$\Longleftrightarrow \quad \varphi \equiv \forall^* \varphi'$

Based on Chapters 0.1, 0.2.1–0.2.3, 1.2 by [Otto]

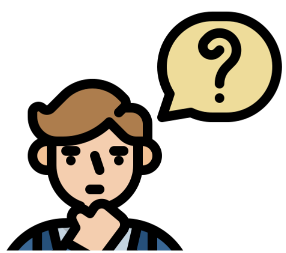
Chapters 1.9–1.11 by [Väänänen]

+ recent research papers.



$sig(\chi) \subseteq sig(\varphi) \cap sig(\psi)$

$$\varphi \models \psi \Longrightarrow \exists \chi \quad \varphi \models \chi \models \psi$$



Feel free to ask questions and interrupt me!

Don't be shy! If needed send me an email (bartosz.bednarczyk@cs.uni.wroc.pl) or approach me after the lecture!

Reminder: this is an advanced lecture. Target: people that had fun learning logic during BSc studies!

Algebraic Diagrams and Embeddings

Goal: Describe a τ-structure $\mathfrak{A}$ up to isomorphism with a (possibly infinite) FO theory $\mathcal{T}_{\mathfrak{A}}$

1. Start with $\mathcal{T}_{\mathfrak{A}} := \emptyset$.

2. With each domain element $a \in A$ we associate a constant symbol "a".

Let τ_A be the extended signature, and let $\mathfrak{A}_A := \mathfrak{A} +$ the interpretation of each a as the corresponding $a \in A$.

3. Append $\bigwedge_{\mathrm{a} \neq \mathrm{b} \in \tau_A \setminus \tau} \mathrm{a} \neq \mathrm{b}$ to $\mathcal{T}_{\mathfrak{A}}$.

4. For all $n \in \mathbb{N}$, all n-tuples of constant symb. $\bar{\mathrm{a}}$ from $\tau_A \setminus \tau$, and relational symb. $\mathrm{R} \in \tau$ of arity n:

- append $\mathrm{R}(\bar{\mathrm{a}})$ to $\mathcal{T}_{\mathfrak{A}}$ iff the corresponding n-tuple belongs to $\mathrm{R}^{\mathfrak{A}}$.
- proceed similarly with $\neg\mathrm{R}(\bar{\mathrm{a}})$ and n-tuples outside $\mathrm{R}^{\mathfrak{A}}$.

5. Close $\mathcal{T}_{\mathfrak{A}}$ under $\wedge, \vee$. We denote it $\mathrm{D}(\mathfrak{A})$ and call it the algebraic diagram of $\mathfrak{A}$.

Alternative definition: $\mathrm{D}(\mathfrak{A}) := \{\varphi \in \mathrm{FO}[\tau_A] \mid \mathfrak{A}_A \models \varphi,\ \varphi \text{ is quantifier free}\}$

fresh constants | make them different | iterate through τ | positive facts | negative facts

Preservation Theorems

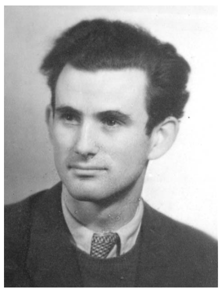
Common theme: Formulae having certain properties are precisely these of a certain fragment of FO

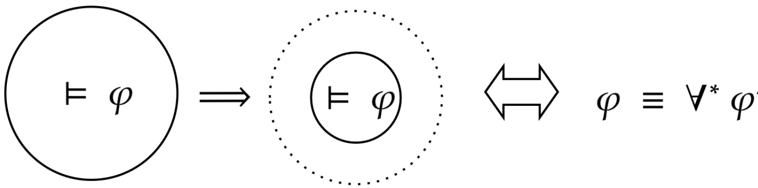
Theorem (Łoś-Tarski 1954)

An FO formula is preserved under substructures[a] iff it is equivalent to a universal[b] formula.

[a] i.e. $\mathfrak{A} \models \varphi$ and $\mathfrak{B} \subseteq \mathfrak{A}$ then $\mathfrak{B} \models \varphi$

[b] (possibly negated) atomic symbols $+ \wedge, \vee$ and $\forall$





- Finitary analogous of Łoś-Tarski fails in the finite, c.f. [`Tait 1933`].
- Finitary generalisations of Łoś-Tarski by Abhisekh Sankaran [`MFCS 2014`].
- There are $\mathcal{L} \subseteq$ FO with Łoś-Tarski (also in the finite), e.g. the Guarded Neg. Frag. [`B.B.tC. 2018`]
- Open problem: Is there a non-trivial $\mathcal{L} \subseteq$ FO (without equality) without Łoś-Tarski? [`B. 2022`]

Theorem (Łoś-Tarski 1954)

An FO formula is preserved under substructures[a] iff it is equivalent to a universal[b] formula.

[a] i.e. $\mathfrak{A} \models \varphi$ and $\mathfrak{B} \subseteq \mathfrak{A}$ then $\mathfrak{B} \models \varphi$

[b] (possibly negated) atomic symbols $+ \wedge, \vee$ and $\forall$

Proof

Every universal formula is preserved under substructures, so let us focus on the other direction.

Assume that φ is preserved under substructures, and consider the set

$$\Psi := \left\{ \psi \mid \varphi \models \psi, \psi \text{ is universal} \right\}.$$

Note that $\varphi \models \Psi$. It suffices to show that $\Psi \models \varphi$. Why?

By compactness there would be a finite subset $\Psi_0 \subseteq_{\text{fin}} \Psi$ such that $\Psi_0 \models \varphi$.

But then $\bigwedge_{\psi \in \Psi_0} \psi$ is the desired universal formula equivalent to φ.

collect universal consequences

compactness

universal formulae are closed under $\wedge$

Proof of Łoś-Tarski Preservation Theorem: Part II

Recall that: φ is preserved under substructures, $\Psi := \{\psi \mid \varphi \models \psi, \psi \text{ is universal}\}$ and our goal is: $\Psi \models \varphi$.

Let $\mathfrak{A} \models \Psi$. We want to show $\mathfrak{A} \models \varphi$. It suffices to find a model $\mathfrak{B}$ of φ containing $\mathfrak{A}$ as a substructure.

Indeed, as φ is preserved under substructures, from $\mathfrak{B} \models \varphi$ we conclude $\mathfrak{A} \models \varphi$.

How to find such $\mathfrak{B}$? Show that $\mathrm{D}(\mathfrak{A}) \cup \{\varphi\}$ is satisfiable!

Ad absurdum, assume that $\mathrm{D}(\mathfrak{A}) \cup \{\varphi\}$ has no model. So $\varphi \models \neg\mathrm{D}(\mathfrak{A})$ holds, i.e. $\varphi \models \neg \bigwedge_{\psi(\bar{a}) \in \mathrm{D}(\mathfrak{A})} \psi(\bar{a})$.

By compactness there is a finite $D_0 \subseteq_{\text{fin}} \mathrm{D}(\mathfrak{A})$ such that $\varphi \models \neg \bigwedge_{\psi(\bar{a}) \in D_0} \psi(\bar{a})$.

But as diagrams are closed under conjunction, we get a single formula $\xi(\bar{a}) \in \mathrm{D}(\mathfrak{A})$ s.t. $\varphi \models \neg\xi(\bar{a})$.

Note that φ does not use extra constants from τ_A. Thus actually $\varphi \models \forall\bar{x}\ \neg\xi(\bar{x})$ holds.

As $\forall\bar{x}\ \neg\xi(\bar{x})$ is universal and follows from φ, we know that $\forall\bar{x}\ \neg\xi(\bar{x}) \in \Psi$.

From $\xi(\bar{a}) \in \mathrm{D}(\mathfrak{A})$ we infer $\mathfrak{A} \models \exists\bar{x}\xi(\bar{x})$. A contradiction with $\mathfrak{A} \models \Psi$. $\square$

Strengthen $\varphi \models \neg\xi(\bar{a})$ and use Ψ.

def of $\models$ | magic | assumption φ | diagrams | contradiction | def of $\models$ | compactness | $\mathrm{D}(\mathfrak{A})$ clos.u.$\wedge$ | Shape of ξ/φ

Failure of Łoś-Tarski in the finite. (Part I)

Theorem (Tait 1933)

There is an FO formula that is preserved under substructures of finite structures but it is not equivalent (in the finite) to any universal formula.

Proof

Consider $\tau = \{\mathtt{min}^{(0)}, \mathtt{max}^{(0)}, <^{(2)}, \mathrm{Next}^{(2)}, \mathrm{P}^{(1)}\}$. Let φ_0 be a universal stating that

$\mathfrak{A} \models \varphi_0$ iff $<^{\mathfrak{A}}$ is a strict linear order with the minimal/maximal elements $\mathtt{min}^{\mathfrak{A}}$, $\mathtt{max}^{\mathfrak{A}}$, and $\mathrm{Next}^{\mathfrak{A}} \subseteq <^{\mathfrak{A}}$.

Moreover, take $\varphi_1 := \forall x \forall y\ \mathrm{Next}(x, y) \leftrightarrow (x < y \land \neg(\exists z\ x < z \land z < y))$.

Note: if $\mathfrak{A} \models \varphi_0 \land \varphi_1$, then $\mathrm{Next}^{\mathfrak{A}}$ is the induced successor of $<^{\mathfrak{A}}$. Finally, let $\varphi := \varphi_0 \land (\varphi_1 \rightarrow \exists x\ \mathrm{P}(x))$.

Observation (The set of finite models of φ is closed under substructures.)

Take a finite $\mathfrak{A} \models \varphi$ and $\mathfrak{B} \subseteq \mathfrak{A}$. Observe that $\mathfrak{B} \models \varphi_0$ (because φ_0 is universal). If $\mathfrak{B} \not\models \varphi_1$ we are done.

If $\mathfrak{B} \models \varphi_1$ then $\mathfrak{A} = \mathfrak{B}$, concluding $\mathfrak{B} \models \varphi$. $\square$

universals are preserved under $\subseteq$

finiteness

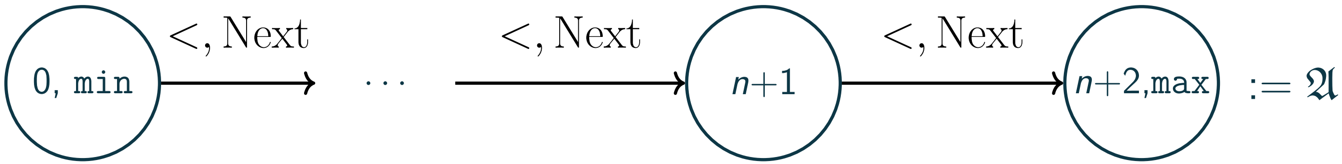
Failure of Łoś-Tarski in the finite. (Part II)

$\mathfrak{A} \models \varphi_0$ iff $<^{\mathfrak{A}}$ is a strict linear order with the minimal/maximal elements $\mathtt{min}^{\mathfrak{A}}$, $\mathtt{max}^{\mathfrak{A}}$, and $\mathrm{Next}^{\mathfrak{A}} \subseteq <^{\mathfrak{A}}$.

$\varphi_1 := \forall x \forall y \, \mathrm{Next}(x, y) \leftrightarrow (x < y \land \neg(\exists z \; x < z \land z < y))$ and $\varphi := \varphi_0 \land (\varphi_1 \rightarrow \exists x \; \mathrm{P}(x))$.

Lemma (φ is not equivalent (in the finite) to any universal formula.)

Ad absurdum, there exists quantifier-free $\chi(\bar{x})$ with n variables so that $\varphi \equiv_{\mathsf{fin}} \forall \bar{x} \; \chi(\bar{x})$. Take $\mathfrak{A}$ as below.



By construction $\mathfrak{A} \models \varphi_0 \land \varphi_1$. Moreover, observe that $(\mathfrak{A}, \mathrm{P}^{\mathfrak{A}}) \models \varphi$ iff $\mathrm{P}^{\mathfrak{A}} \neq \emptyset$.

Then $(\mathfrak{A}, \emptyset) \not\models \varphi$ implies $(\mathfrak{A}, \emptyset) \not\models \forall \bar{x} \; \chi(\bar{x})$. Thus $(\mathfrak{A}, \emptyset) \models \neg\chi(\bar{a})$ for suitable $\bar{a}$.

Take b to be different from $\bar{a}$, $\mathtt{max}^{\mathfrak{A}}$ and $\mathtt{min}^{\mathfrak{A}}$ (we have enough elements!). Then $(\mathfrak{A}, \{b\}) \models \varphi$.

But $(\mathfrak{A}, \{b\}) \models \neg\chi(\bar{a})$ ($\mathfrak{A} \restriction \bar{a}$ was not touched!). But it means $(\mathfrak{A}, \{b\}) \not\models \forall \bar{x} \; \chi(\bar{x}) \equiv \varphi$. A contradiction!

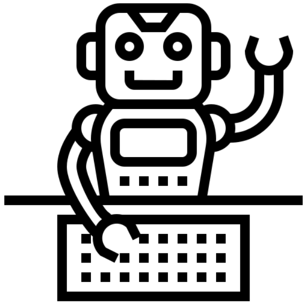
contradiction — def of P — when $\mathrm{P}^{\mathfrak{A}} = \emptyset$ — witness — select suitable b and make it satisfy P — def of φ

Can we make Łoś-Tarski theorem computable?

Input: First-Order φ closed under substructures (in the general setting).

Output: the equivalent universal formula.

Is this problem solvable?: **YES!** Ask Gödel for help!



Unfortunately, the finitary analogue is unsolvable. [Chen and Flum 2021]

Other preservation theorems?

Theorem (Lyndon–Tarski 1956, Rossmann 2005)

An FO formula is preserved under homomorphic images[a] iff it is equivalent to a positive existential[b] formula.

[a] i.e. $\mathfrak{A} \models \varphi$ and there is a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ then $\mathfrak{B} \models \varphi$

[b] atomic symbols + $\wedge$, $\vee$ and $\exists$

- A notable example of classical MT theorem that works in the finite, c.f. [Rossmann's paper]

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