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Cooperative Games: Stable Sets and Shapley Value

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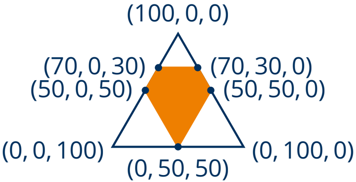
Previously . . .

- In **cooperative** games, players P form explicit **coalitions** $C \subseteq P$.
- Coalitions receive payoffs, which are distributed among its members.
- We concentrate on **superadditive** games, where disjoint coalitions can never decrease their payoffs by joining together.
- Of particular interest is the **grand coalition** $\{P\}$ and whether it is *stable*.
- An **imputation** is an outcome that is efficient and individually rational.
- Various solution concepts formalise stability of the grand coalition:
  - the **core** contains all imputations where no coalition has an incentive to leave;
  - the **ε-core** disincentivises leaving the grand coalition via a fine of ε;
  - the **cost of stability** subsidises staying in the grand coalition via a bonus γ.
- The core is a convex set.

Consider the game $G = (P, v)$ with:

- $P = \{A, B, C\}$,
- $v(P) = 100$ and $v(\{i\}) = 0$ for $i \in P$,
- $v(\{A, B\}) = v(\{A, C\}) = 50$, and $v(\{B, C\}) = 30$.



Overview

Solution Concept: Stable Sets

Solution Concept: Shapley Value

Solution Concept: Stable Sets

Stable Sets

Definition [von Neumann and Morgenstern, 1941]

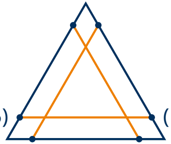
Let $G = (P, v)$ be a cooperative game, and let $\mathbf{a}$ and $\mathbf{b}$ be imputations for G.

- **a dominates b via a coalition** C with $\emptyset \subsetneq C \subseteq P$, written $\mathbf{a} \succ_C \mathbf{b}$, iff
  - $a_i > b_i$ for all $i \in C$, and
  - $\sum_{i \in C} a_i \leq v(C)$.
- **a dominates b**, written $\mathbf{a} \succ \mathbf{b}$, iff $\mathbf{a}$ dominates $\mathbf{b}$ via some coalition $C \subseteq P$.
- A set $S \subseteq Imp(G)$ is a **stable set of** G iff both of the following hold:
  - Internal stability: For any two $\mathbf{a}, \mathbf{b} \in S$, we have $\mathbf{a} \not\succ \mathbf{b}$.
  - External stability: For every $\mathbf{b} \in Imp(G) \setminus S$, there is some $\mathbf{a} \in S$ with $\mathbf{a} \succ \mathbf{b}$.

- If $a_i > b_i$ for all $i \in C$, then every member of C is better off in $\mathbf{a}$ than in $\mathbf{b}$.
- If $\sum_{i \in C} a_i \leq v(C)$, then C can plausibly threaten to leave the grand coalition.
- Internal stability: No imputations need to be removed from S.
- External stability: No imputations can be added to S.

Stable Sets: Example and Visualisation

Recall Hospitals and X-Ray Machines with $P = \{1, 2, 3\}$ and $v(P) = 6$, $v(C) = 5$ if $|C| = 2$, and $v(C) = 0$ otherwise.



$S = \{(1, x, 5 - x) \mid x \in [0, 5]\}$ is a stable set of $G = (P, v)$:

- Internal stability:
  - Consider $(1, x, 5 - x) \in S$ and $(1, y, 5 - y) \in S$.
  - If $x > y$, then $5 - x < 5 - y$, thus $(1, x, 5 - x) \not\succ_{\{2,3\}} (1, y, 5 - y)$.
- External stability:
  - Consider $\mathbf{b} = (b_1, b_2, b_3) \in Imp(G) \setminus S$. Then $b_1 + b_2 + b_3 = 6$ and $b_1 \neq 1$.
  - If $b_1 < 1$, then $\min\{b_2, b_3\} \leq 3$ whence $(1, 4, 1) \succ_{\{1,2\}} \mathbf{b}$ or $(1, 1, 4) \succ_{\{1,3\}} \mathbf{b}$.
  - If $b_1 > 1$, then $b_2 + b_3 < 5$, whence we can choose $\mathbf{a} \in S$ such that $\mathbf{a} \succ_{\{2,3\}} \mathbf{b}$.

The Core vs. Stable Sets (1)

Proposition

Let $G = (P, v)$ be a cooperative game.

1. $Core(G)$ is contained in every (if any) stable set of G.
2. If $Core(G)$ is a stable set of G, then it is the only stable set of G.

Proof.

1. – Let $\mathbf{a} \in Core(G)$ and $\mathbf{b} \in Imp(G)$.
   – Assume (for contradiction) that for some $C \subseteq P$, we have $\mathbf{b} \succ_C \mathbf{a}$.
   – Then $a_i < b_i$ for all $i \in C$ and $\sum_{i \in C} b_i \leq v(C)$.
   – But then $\sum_{i \in C} a_i < \sum_{i \in C} b_i \leq v(C)$.
   – But $\mathbf{a} \in Core(G)$ means that $\sum_{i \in C} a_i \geq v(C)$. Contradiction.
   – Thus $\mathbf{b} \not\succ \mathbf{a}$ and $\mathbf{a}$ is contained in every (if any) stable set of G.
2. – No stable set can be a proper subset of another stable set:
   – If $S_1 \subsetneq S_2$ and both are stable then $\mathbf{b} \in S_2 \setminus S_1$ is dominated by some $\mathbf{a} \in S_1$.
   – But then $\mathbf{a} \in S_2$ and S_2 does not satisfy internal stability, contradiction. □

The Core vs. Stable Sets (2)

Proposition

For any superadditive cooperative game $G = (P, v)$, we have
$Core(G) = \{\mathbf{a} \in Imp(G) \mid \text{there is no } \mathbf{b} \in Imp(G) \text{ with } \mathbf{b} \succ \mathbf{a}\}$.

Proof.

- Direction $\subseteq$ follows from the previous slide, so it remains to show $\supseteq$.
- Let $\mathbf{b} \in Imp(G) \setminus Core(G)$. Then $\sum_{i \in P} b_i = v(P)$ and $b_i \geq v(\{i\})$ for all $i \in P$.
- Since $\mathbf{b} \notin Core(G)$, there is a $C \subseteq P$ such that $v(C) > \sum_{i \in C} b_i$, whence $C \neq \emptyset$.
- Denote $\delta := v(C) - \sum_{i \in C} b_i$ and define $\mathbf{a} \in Imp(G)$ with $\mathbf{a} \succ_C \mathbf{b}$ by setting

$$a_i := \begin{cases} b_i + \frac{1}{|C|} \cdot \delta & \text{if } i \in C, \\ b_i - \frac{d_i}{\sum_{j \in P \setminus C} d_j} \cdot \delta & \text{otherwise,} \end{cases} \quad \text{where } d_j := b_j - v(\{j\}) \text{ for each } j \in P \setminus C.$$

- Note that $\sum_{j \in P \setminus C} d_j = \sum_{j \in P \setminus C} b_j - \sum_{j \in P \setminus C} v(\{j\}) \geq \delta = v(C) - \sum_{i \in C} b_i$ because v is superadditive: $\sum_{j \in P \setminus C} b_j + \sum_{i \in C} b_i = v(P) \geq v(C) + \sum_{j \in P \setminus C} v(\{j\})$. □

The Core vs. Stable Sets: Example

G^1

$$P = \{1,2,3\}$$

$$v(C) = \begin{cases} 1 & \text{if } 1 \in C \text{ and } |C| \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

- The core of G^1, $Core(G^1) = \{(1,0,0)\}$, is not a stable set of G:
- We have $(1,0,0) \not\succ (0,0.5,0.5)$ since $(1,0,0) \not\succ_{\{1\}} (0,0.5,0.5)$.

$\rightsquigarrow$ The core does not necessarily satisfy external stability.

- One stable set of G^1 is $S_{1,2} = \{(x, 1-x, 0) \mid x \in [0,1]\}$:
  - If $(x, 1-x, 0), (y, 1-y, 0) \in S_{1,2}$, then $x > y$ would imply $1-x < 1-y$.
  - If $(x,y,z) \in Imp(G^1)$ with $z > 0$, then $\left(x + \frac{z}{2}, y + \frac{z}{2}, 0\right) \succ_{\{1,2\}} (x,y,z)$.
- Likewise, $S_{1,3} = \{(x, 0, 1-x) \mid x \in [0,1]\}$ is a stable set of G^1.

Exercise: Find additional stable sets, if any.

Convex Games

Definition

1. A function $v\colon 2^P \to \mathbb{R}^+$ is **supermodular** iff for all $C, D \subseteq P$:

$$v(C \cup D) + v(C \cap D) \geq v(C) + v(D)$$

2. A cooperative game $G = (P, v)$ is **convex** iff v is supermodular.

Observation

Function $v\colon 2^P \to \mathbb{R}^+$ is supermodular iff for all $C \subseteq D \subseteq P$ and all $i \in P \setminus D$:

$$v(C \cup \{i\}) - v(C) \leq v(D \cup \{i\}) - v(D) \tag{1}$$

where $v(C \cup \{i\}) - v(C)$ is player i's **marginal contribution** to coalition C.

- A supermodular function is superadditive (via $v(\emptyset) = 0$),
- but not vice versa.

Cores of Convex Games (1)

Theorem [Shapley, 1971]

Every convex game has a nonempty core.

Proof (1/2).

- Given $G = (P, v)$ with $P = \{1, \ldots, n\}$, we construct $\mathbf{a} = (a_1, \ldots, a_n) \in \mathit{Core}(G)$.
- Define $a_1 := v(\{1\})$, $a_2 := v(\{1, 2\}) - v(\{1\})$, …, $a_n := v(P) - v(P \setminus \{n\})$.
- Payoff vector $\mathbf{a}$ is efficient by construction:
  $$a_1 + a_2 + \ldots + a_n = v(\{1\}) + v(\{1, 2\}) - v(\{1\}) + \ldots + v(P) - v(P \setminus \{n\}) = v(P)$$
- $\mathbf{a}$ is also individually rational: For all $i \in P$, inequality (1) yields
  $$a_i = v(\{1, \ldots, i\}) - v(\{1, \ldots, i - 1\}) \geq v(\{i\}) - v(\emptyset) = v(\{i\})$$
- Thus $\mathbf{a} \in \mathit{Imp}(G)$. It remains to show $\mathbf{a} \in \mathit{Core}(G)$.

Cores of Convex Games (2)

Theorem [Shapley, 1971]

Every convex game has a nonempty core.

Proof (2/2).

- Consider any coalition $C = \{i, j, \ldots, k\}$ with $1 \leq i < j < \ldots < k \leq n$.
- We have $v(C) = v(\{i\}) - v(\emptyset) + v(\{i, j\}) - v(\{i\}) + \ldots + v(C) - v(C \setminus \{k\})$.
- Due to v being supermodular, inequality (1) yields

$$\begin{aligned}
v(\{i\}) - v(\emptyset) &\leq v(\{1, \ldots, i\}) - v(\{1, \ldots, i-1\}) &&= a_i \\
v(\{i, j\}) - v(\{i\}) &\leq v(\{1, \ldots, j\}) - v(\{1, \ldots, j-1\}) &&= a_j \\
&\vdots \\
v(C) - v(C \setminus \{k\}) &\leq v(\{1, \ldots, k\}) - v(\{1, \ldots, k-1\}) &&= a_k
\end{aligned}$$

- Therefore $v(C) \leq a_i + a_j + \ldots + a_k$ and since C was arbitrary, $\mathbf{a} \in Core(G)$. □

Every convex game $G = (P, v)$ also has a unique stable set $S = Core(G)$.

Solution Concept: Shapley Value

Shapley Value: Motivation

- All solution concepts (for cooperative games) we have seen so far yield sets of payoff vectors.
- For stable sets, there might even be different candidates.
- Unless the core contains only one element, there are even infinitely many allocations in it.
- But to actually pay off players, we can only choose one allocation.
- Which one of them should we choose?

Example

In Hospitals and X-Rays,

- the core is empty, and
- no imputation is contained in all stable sets.

Shapley Value

Main Idea: Analyse players' marginal contributions to coalitions.

$\rightsquigarrow$ Issue: Contributions might depend on the order in which players join.

$\rightsquigarrow$ Approach: Look at all possible orders.

Definition

Let $G = (P, v)$ be a cooperative game. For one permutation $\lambda \colon P \to P$ of players, the **marginal contribution of player** $i \in P$ is:

$$\mu_G(\lambda, i) := v(\{ j \in P \mid \lambda(j) \leq \lambda(i)\}) - v(\{ j \in P \mid \lambda(j) < \lambda(i)\})$$

We denote the set of all permutations of players P by L_P; observe $|L_P| = |P|!$.

Intuition

Players who contribute more to more coalitions should get more overall.

Marginal Contributions: Example

Consider $G^4 = (P, v)$ with $P = \{1, 2, 3\}$ and $v(P) = 8$, additionally

$$v(\{1\}) = 1 \qquad v(\{2\}) = 2 \qquad v(\{3\}) = 3$$
$$v(\{1,2\}) = 4 \qquad v(\{1,3\}) = 5 \qquad v(\{2,3\}) = 6$$

The players' marginal contributions are as follows:

λ	$\mu_{G^4}(\lambda, 1)$	$\mu_{G^4}(\lambda, 2)$	$\mu_{G^4}(\lambda, 3)$
$1, 2, 3$	1	3	4
$1, 3, 2$	1	3	4
$2, 1, 3$	2	2	4
$2, 3, 1$	2	2	4
$3, 1, 2$	2	3	3
$3, 2, 1$	2	3	3
$\sum$	10	16	22

Shapley Value (2)

Definition

Let $G = (P, v)$ be a cooperative game and $i \in P$ be a player.

1. Player i's **raw Shapley value** in G is

$$\text{Shapley}^*(G, i) := \sum_{\lambda \in L_P} \mu_G(\lambda, i)$$

2. Player i's **Shapley value** in G is then

$$\text{Shapley}(G, i) := \frac{1}{|P|!} \cdot \text{Shapley}^*(G, i)$$

Taken together, the players' Shapley values constitute an allocation:

$$\big(\text{Shapley}(G, 1), \ldots, \text{Shapley}(G, n)\big)$$

Shapley Value (3)

For computational purposes, it is often better to define the Shapley value in terms of marginal contributions to coalitions instead of permutations.

For $C \subseteq P$ and $i \in P$, we set $\mu_G(C, i) := v(C \cup \{i\}) - v(C)$. Thus: ($2^{n-1}$ vs. $n!$ terms)

$$\text{Shapley}^*(G, i) = \sum_{C \subseteq P \setminus \{i\}} |C|! \cdot (|P| - |C| - 1)! \cdot \mu_G(C, i)$$

Example: Recall G^4 from slide 16:

$C \subseteq P$	$\lvert C\rvert! \cdot (\lvert P\rvert - \lvert C\rvert - 1)! \cdot \mu_{G^4}(C, 1)$	$\lvert C\rvert! \cdot (\lvert P\rvert - \lvert C\rvert - 1)! \cdot \mu_{G^4}(C, 2)$	$\lvert C\rvert! \cdot (\lvert P\rvert - \lvert C\rvert - 1)! \cdot \mu_{G^4}(C, 3)$
$\emptyset$	$1 \cdot 2 \cdot 1 = 2$	$1 \cdot 2 \cdot 2 = 4$	$1 \cdot 2 \cdot 3 = 6$
$\{1\}$		$1 \cdot 1 \cdot 3 = 3$	$1 \cdot 1 \cdot 4 = 4$
$\{2\}$	$1 \cdot 1 \cdot 2 = 2$		$1 \cdot 1 \cdot 4 = 4$
$\{3\}$	$1 \cdot 1 \cdot 2 = 2$	$1 \cdot 1 \cdot 3 = 3$	
$\{1, 2\}$			$2 \cdot 1 \cdot 4 = 8$
$\{1, 3\}$		$2 \cdot 1 \cdot 3 = 6$	
$\{2, 3\}$	$2 \cdot 1 \cdot 2 = 4$		
$\{1, 2, 3\}$			
$\sum$	10	16	22

Towards an Axiomatic Characterisation

Definition

Player $i \in P$ is a **dummy player** in $G = (P, v)$ iff for all $C \subseteq P$:

$$v(C \cup \{i\}) = v(C)$$

Definition

Two players $i, j \in P$ are **symmetric** in $G = (P, v)$ iff for all $C \subseteq P \setminus \{i, j\}$:

$$v(C \cup \{i\}) = v(C \cup \{j\})$$

Definition

Given two games $G_1 = (P, v_1)$ and $G_2 = (P, v_2)$, define the game $G_1 \oplus G_2 := (P, v)$ via $v(C) := v_1(C) + v_2(C)$ for all $C \subseteq P$.

Shapley's Theorem: Axiomatic Characterisation

Theorem (Shapley)

The Shapley value is the only (single-allocation) solution concept for cooperative games that satisfies the following four properties:

1. *Dummy Player:* For every game $G = (P, v)$ it holds that if $i \in P$ is a dummy player, then $\text{Shapley}(G, i) = 0$.
2. *Efficiency:* For every game $G = (P, v)$ it holds that $\sum_{i \in P} \text{Shapley}(G, i) = v(P)$.
3. *Symmetry:* For every game $G = (P, v)$ with symmetric players $i, j \in P$, it holds that $\text{Shapley}(G, i) = \text{Shapley}(G, j)$.
4. *Additivity:* For every pair of games $G_1 = (P, v_1)$ and $G_2 = (P, v_2)$, for every $i \in P$ it holds that $\text{Shapley}(G_1 \oplus G_2, i) = \text{Shapley}(G_1, i) + \text{Shapley}(G_2, i)$.

If G is superadditive, then the Shapley value payoff vector is an imputation.

Shapley Value vs. Core (1)

The Shapley value payoff vector is not necessarily in the core:

Example

Recall the game $G^1 = (P, v)$ with $P = \{1, 2, 3\}$ and

$$v(C) = \begin{cases} 1 & \text{if } 1 \in C \text{ and } |C| \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

The core of G^1 is $\{(1, 0, 0)\}$:

Any imputation (x, y, z) with $x < 1$ has $y + z > 0$ and is thus dominated by (if, say, $y \geq z$) $\left(x + \frac{y}{2}, 0, z + \frac{y}{2}\right)$.

The Shapley value payoff vector is given by $\left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right)$:

λ	$\mu_{G^1}(\lambda, 1)$	$\mu_{G^1}(\lambda, 2)$	$\mu_{G^1}(\lambda, 3)$
1, 2, 3	0	1	0
1, 3, 2	0	0	1
2, 1, 3	1	0	0
2, 3, 1	1	0	0
3, 1, 2	1	0	0
3, 2, 1	1	0	0

Shapley Value vs. Core (2)

Theorem

If $G = (P, v)$ with $P = \{1, \dots, n\}$ is a convex cooperative game, then

$$(\text{Shapley}(G, 1), \dots, \text{Shapley}(G, n)) \in \mathit{Core}(G).$$

Proof.

- For any $\lambda \in L_P$, we obtain an allocation $\mathbf{a}_\lambda \in \mathit{Core}(G)$ (see slide 11):

$$\mathbf{a}_\lambda := \left(\mu_G(\lambda, 1), \dots, \mu_G(\lambda, n)\right)$$

- The core is a convex set, thus for any two $\lambda_1, \lambda_2 \in L_P$ and $\alpha \in [0, 1]$,

$$\alpha \cdot \mathbf{a}_{\lambda_1} + (1 - \alpha) \cdot \mathbf{a}_{\lambda_2} \in \mathit{Core}(G).$$

- Therefore by induction on $|L_P|$, we get $\sum_{\lambda \in L_P} \left(\frac{1}{|L_P|} \cdot \mathbf{a}_\lambda\right) \in \mathit{Core}(G)$. □

Reprise: Solution Concepts

We have seen the following solution concepts for cooperative games:

- stable sets [von Neumann and Morgenstern, 1941] (called "solutions")
  - There can be zero, one, or more stable sets; every stable set is non-empty.
- Shapley value [Shapley, 1953]
  - A unique payoff vector that is efficient, symmetric, and additive.
  - For superadditive games, it is also individually rational (thus an imputation).
- core [Gillies, 1959]
  - A unique set of imputations, but may be empty.
- ε-core [Shapley and Shubik, 1966]
  - A unique set of imputations, (non-)empty depending on $\varepsilon \in \mathbb{R}$.

There are further solution concepts for cooperative games:

- kernel [Davis and Maschler, 1965]
  - A set of imputations stating that no player has "bargaining power" over another.
- nucleolus [Schmeidler, 1969]
  - A unique payoff vector that is contained in both core and kernel.

Conclusion

Summary

- A **stable set** is a set of imputations that do not dominate each other and that dominate every imputation not in the set.
- The core is contained in every (if any) stable set.
- A **convex** game has a non-empty core that equals its unique stable set.
- The **Shapley value** of each player yields a single allocation;
  it is also the only solution concept for cooperative games that:
  - assigns a payoff of zero to dummy players,
  - assigns the same payoff to symmetric players,
  - yields an efficient allocation, and
  - is additive.
- For convex games, the Shapley value payoff vector is in the core.