TECHNISCHE UNIVERSITÄT DRESDEN

COMPLEXITY THEORY

Lecture 2: Turing Machines and Languages

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Knowledge-Based Systems

TU Dresden, 21th Oct 2019

A Model for Computation

Clear

To understand computational problems we need to have a formal understanding of what an **algorithm** is.

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Answer

With Turing machines.

Turing Machines

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Turing Machines

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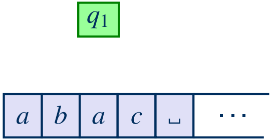
Definition 2.2: A (deterministic) Turing Machine $\mathcal{M} = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}} \rangle$ consists of

- a finite set Q of states,
- an input alphabet Σ not containing $\llcorner\!\lrcorner$,
- a tape alphabet Γ such that $\Gamma \supseteq \Sigma \cup \{\llcorner\!\lrcorner\}$.
- a transition function $\delta \colon Q \times \Gamma \to Q \times \Gamma \times \{\mathsf{L}, \mathsf{R}\}$
- an initial state $q_0 \in Q$,
- an accepting state $q_{\text{accept}} \in Q$, and
- an rejecting state $q_{\text{reject}} \in Q$ such that $q_{\text{accept}} \neq q_{\text{reject}}$.

Example 2.3:



Turing Machines

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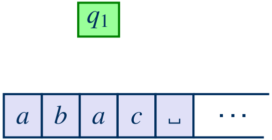

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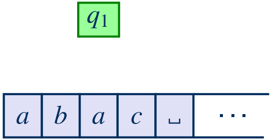

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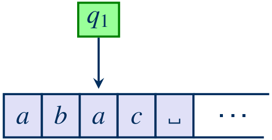

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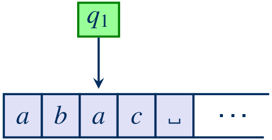

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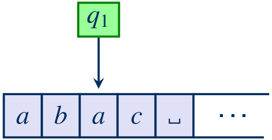

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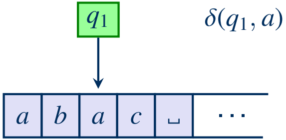

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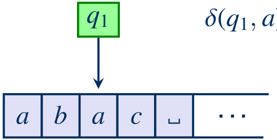

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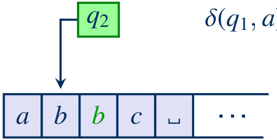

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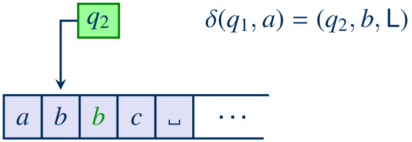

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- The head can read only one symbol at a time
- The head moves and writes according to the transition function δ; the current state also changes accordingly
- The head will stay put when attempting to cross the left tape end

Configurations

Observation: to describe the current step of a computation of a TM it is enough to know

- the content of the tape,
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Some special configurations:

- The **start configuration** for some input word $w \in \Sigma^*$ is the configuration $q_0 w$
- A configuration uqv is **accepting** if $q = q_{\text{accept}}$.
- A configuration uqv is **rejecting** if $q = q_{\text{reject}}$.

Computation

We write

- $\mathcal{C} \vdash_{\mathcal{M}} \mathcal{C}'$ only if $\mathcal{C}'$ can be reached from $\mathcal{C}$ by one computation step of $\mathcal{M}$;
- $\mathcal{C} \vdash^{*}_{\mathcal{M}} \mathcal{C}'$ only if $\mathcal{C}'$ can be reached from $\mathcal{C}$ in a finite number of computation steps of $\mathcal{M}$.

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- $C \vdash_{\mathcal{M}} C'$ only if C' can be reached from C by one computation step of $\mathcal{M}$;
- $C \vdash^*_{\mathcal{M}} C'$ only if C' can be reached from C in a finite number of computation steps of $\mathcal{M}$.

We say that $\mathcal{M}$ **halts** on input w if and only if there is a finite sequence of configurations

$$C_0 \vdash_{\mathcal{M}} C_1 \vdash_{\mathcal{M}} \cdots \vdash_{\mathcal{M}} C_\ell$$

such that C_0 is the start configuration of $\mathcal{M}$ on input w and C_ℓ is an accepting or rejecting configuration. Otherwise $\mathcal{M}$ **loops** on input w.

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We say that $\mathcal{M}$ **accepts** the input w only if $\mathcal{M}$ halts on input w with an accepting configuration.

Recognisability and Decidability

Definition 2.5: Let $\mathcal{M}$ be a Turing machine with input alphabet Σ. The language accepted by $\mathcal{M}$ is the set

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A language $\mathbf{L} \subseteq \Sigma^*$ is called Turing-decidable (decidable, recursive) if and only if there exists a Turing machine $\mathcal{M}$ such that $\mathbf{L} = \mathbf{L}(\mathcal{M})$ and $\mathcal{M}$ halts on every input. In this case we say that $\mathcal{M}$ decides $\mathbf{L}$.

Example

Claim 2.6: The language **L** $:= \{ a^{2^n} \mid n \geq 0 \}$ is decidable.

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Proof: A Turing machine $\mathcal{M}$ that decides **L** is

$\mathcal{M}$:= On input w, where w is a string

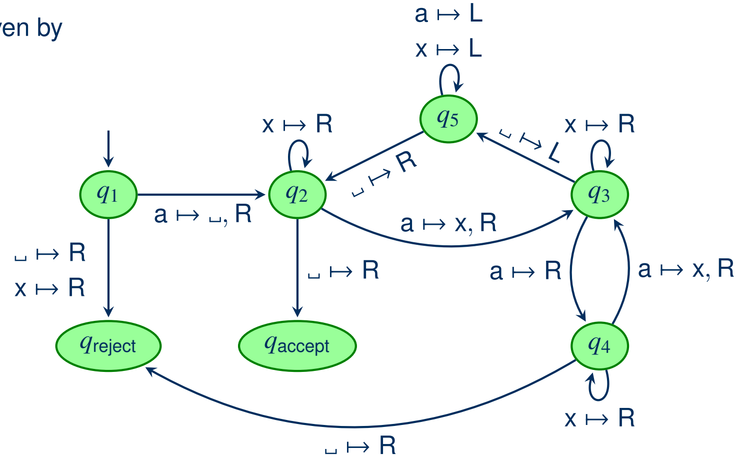
- Go from left to right over the tape and cross off every other a
- If in the first step the tape contained a single a, accept
- If in the first step the number of a's on the tape was odd, reject
- Return the head the beginning of the tape
- Go to the first step

Example (cont'd)

Formally, $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_1, q_{\text{accept}}, q_{\text{reject}})$, where

- $Q = \{ q_1, q_2, q_3, q_4, q_5, q_{\text{accept}}, q_{\text{reject}} \}$
- $\Sigma = \{ \text{a} \}$, $\Gamma = \{ \text{a}, \text{x}, \text{␣} \}$

and δ is given by



Problems as Languages

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Example 2.7 (Graph-Connectedness): The question whether a graph is connected or not can be seen as the **word problem** of the following language

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Notation 2.8: The encoding of objects $O_1, \ldots, O_n$ we denote by $\langle O_1, \ldots, O_n\rangle$.

The Church-Turing Thesis

It turns out that Turing-machines are **equivalent** to a number of formalisations of the intuitive notion of an **algorithm**

- λ-calculus
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Because of this it is believed that Turing-machines completely capture the intuitive notion of an algorithm. $\rightsquigarrow$ **Church-Turing Thesis**:

> "A function on the natural numbers is intuitively computable if and only if it can be computed by a Turing machine."

(→ Wikipedia: Church-Turing Thesis)

Variations of Turing-Machines

It has also been shown that deterministic, single-tape Turing machines are equivalent to a wide range of other forms of Turing machines:

- Multi-tape Turing machines
- Nondeterministic Turing machines
- Turing machines with doubly-infinite tape
- Multi-head Turing machines
- Two-dimensional Turing machines
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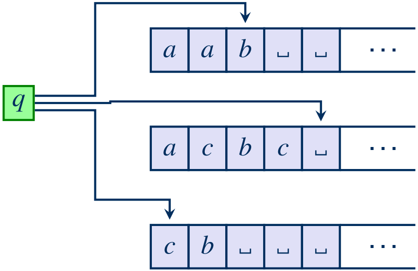

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Definition 2.9: Let $k \in \mathbb{N}$. Then a (deterministic) k**-tape Turing machine** is a tuple $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$, where

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The notions of a **configuration** and of the **language accepted by** $\mathcal{M}$ are defined analogously to the single-tape case.

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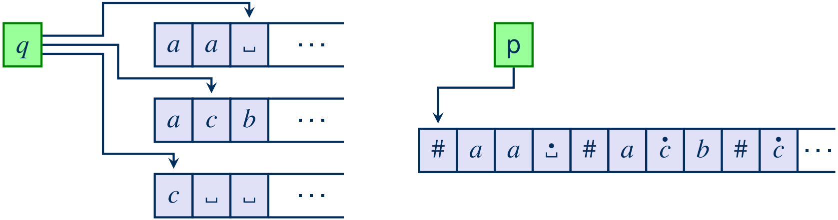

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Note: there may be more than one or no computation of a nondeterministic TM on a given input.
A nondeterministic TM $\mathcal{M}$ **accepts** an input w if and only if **there exists** some accepting computation of $\mathcal{M}$ on input w.

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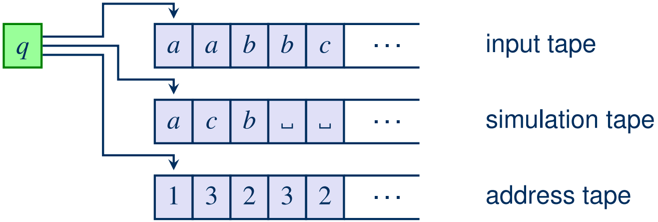
Idea

- D deterministically traverses in breath-first order the tree of configuration of N, where each branch represents a different possibility for N to continue.
- For this, successively try out all possible choices of transitions allowed by N.

Nondeterministic Turing Machines

Sketch of D:



Nondeterministic Turing Machines

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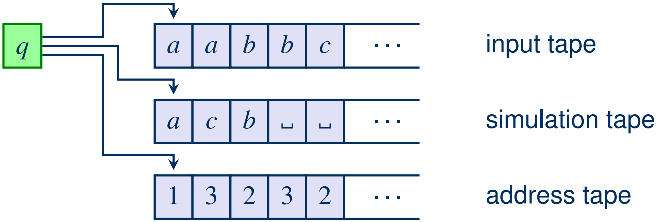

Let b be the maximal number of choices in δ, i.e.,

$$b := \max\{\, |\delta(q, x)| \;\big|\; q \in Q, x \in \Gamma \,\}.$$

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Enumerators

Definition 2.12: A multi-tape Turing machine $\mathcal{M}$ is an enumerator if

- $\mathcal{M}$ has a designated write-only output-tape on which a symbol, once written, can never be changed and where the head can never move left;
- $\mathcal{M}$ has a marker symbol # separating words on the output tape.

We define the language generated by $\mathcal{M}$ to be the set $\mathbf{G}(\mathcal{M})$ of all words that eventually appear between two consecutive # on the output tape of $\mathcal{M}$ when started on the empty word as input.

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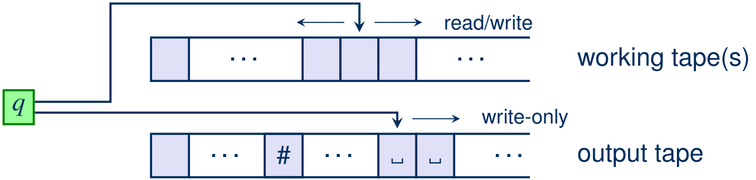

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Proof: Let $\mathcal{E}$ be an enumerator for **L**. Then the following TM accepts **L**:

$\mathcal{M}$:= On input w

- Simulate $\mathcal{E}$ on the empty input. Compare every string output by $\mathcal{E}$ with w
- If w appears in the output of $\mathcal{E}$, accept

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$\mathcal{E} :=$ Ignore the input.

- Print the first # to initialise the output.
- Repeat for $i = 1, 2, 3, \ldots$
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  - If any computation accepts, print the corresponding s_j followed by #

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Theorem 2.14: If $\mathbf{L}$ is Turing-recognisable, then there exists an enumerator for $\mathbf{L}$ that prints each word of $\mathbf{L}$ exactly once.

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Theorem 2.15: A language **L** is decidable if and only if there exists an enumerator for **L** that outputs exactly the words of **L** in some order of non-decreasing length.

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Proof: Suppose **L** to be decidable, and let $\mathcal{M}$ be a TM that decides **L**.

- Define a TM $\mathcal{M}'$ that generates, on some scratch tape, all words over Σ in some order of non-decreasing length. (Exercise!)

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Summary and Outlook

Turing Machines are a simple model of computation

Recognisable (semi-decidable) = recursively enumerable

Decidable = computable = recursive

Many variants of TMs exist – they normally recognise/decide the same languages

What's next?

- A short look into undecidability
- Recursion and self-referentiality
- Actual complexity classes