Lecture 5

Incomplete Constraint Solvers

Outline

Introduce incomplete constraint solvers for

- equality and disequality constraints
- Boolean constraints
- linear constraints over integer intervals and over finite integer domains
- arithmetic constraints over integer intervals
- arithmetic constraints over reals

Equality Rules

Consider equality and disequality constraints over arbitrary domains.

EQUALITY 1

$$\frac{\langle x = x \, ; \, x \in D \rangle}{\langle \, ; \, x \in D \rangle}$$

EQUALITY 2

$$\frac{\langle x = y \, ; \, x \in D_x , y \in D_y \rangle}{\langle x = y \, ; \, x \in D_x \cap D_y , y \in D_x \cap D_y \rangle}$$

Disequality Rules

DISEQUALITY 1

$$\frac{\langle x \neq x \, ; \, x \in D \rangle}{\langle \, ; \, x \in \emptyset \rangle}$$

DISEQUALITY 2

$$\frac{\langle x \neq y \, ; \, x \in D_x, y \in D_y \rangle}{\langle \, ; \, x \in D_x, y \in D_y \rangle}$$ (where $D_x \cap D_y = \emptyset$)

DISEQUALITY 3

$$\frac{\langle x \neq y \, ; \, x \in D, y = a \rangle}{\langle \, ; \, x \in D - \{a\}, y = a \rangle}$$ (where $a \in D$)

Similarly with $x \neq y$ replaced by $y \neq x$.

Characterization Result

Theorem

A CSP with only equality and disequality constraints is hyper-arc consistent iff it is closed under the applications of the EQUALITY 1 – 2 and DISEQUALITY 1 – 3 rules.

Boolean Constaints

Boolean variables: range over {0, 1}

Boolean domain expression: $x \in D$ with $D \subseteq \{0, 1\}$

Boolean expression: built out of Boolean variables using
$\neg$ (negation), $\wedge$ (conjunction), and $\vee$ (disjunction)

Boolean constraints:

$$s = t$$

where s, t Boolean expressions

Simple Boolean Constraints

- $x = y$
- $\neg x = y$
- $x \wedge y = z$
- $x \vee y = z$

Rules

Transformation Rules

Reduce Boolean constraints to simple constraints

Example:

$$\frac{x \wedge s = z}{x \wedge y = z,\, s = y}$$

where *s* is not a variable or is either *x* or *z*

Rules for Simple Constraints (Example)

$$\frac{\langle x \wedge y = z\, ;\, x \in D_x,\, y \in D_y,\, z \in \{1\} \rangle}{\langle\, ;\, x \in D_x \cap \{1\},\, y \in D_y \cap \{1\},\, z \in \{1\} \rangle}$$

Write as $x \wedge y = z$, $z = 1 \rightarrow x = 1$, $y = 1$

Domain Reduction Rules: BOOL

- $x = y, x = 1 \rightarrowtail y = 1$
- $x = y, y = 1 \rightarrowtail x = 1$
- $x = y, x = 0 \rightarrowtail y = 0$
- $x = y, y = 0 \rightarrowtail x = 0$

- $x \wedge y = z, x = 1, y = 1 \rightarrowtail z = 1$
- $x \wedge y = z, x = 1, z = 0 \rightarrowtail y = 0$
- $x \wedge y = z, y = 1, z = 0 \rightarrowtail x = 0$
- $x \wedge y = z, x = 0 \rightarrowtail z = 0$
- $x \wedge y = z, y = 0 \rightarrowtail z = 0$
- $x \wedge y = z, z = 1 \rightarrowtail x = 1, y = 1$

- $\neg x = y, x = 1 \rightarrowtail y = 0$
- $\neg x = y, x = 0 \rightarrowtail y = 1$
- $\neg x = y, y = 1 \rightarrowtail x = 0$
- $\neg x = y, y = 0 \rightarrowtail x = 1$

- $x \vee y = z, x = 1 \rightarrowtail z = 1$
- $x \vee y = z, x = 0, y = 0 \rightarrowtail z = 0$
- $x \vee y = z, x = 0, z = 1 \rightarrowtail y = 1$
- $x \vee y = z, y = 0, z = 1 \rightarrowtail x = 1$
- $x \vee y = z, y = 1 \rightarrowtail z = 1$
- $x \vee y = z, z = 0 \rightarrowtail x = 0, y = 0$

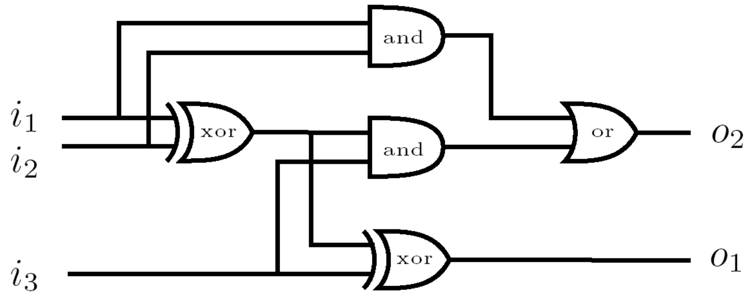
Characterization Result

Theorem

A non-failed Boolean CSP is hyper-arc consistent iff it is closed under the applications of the rules of BOOL.

Constraint Propagation using BOOL: Example

- Full Adder Circuit

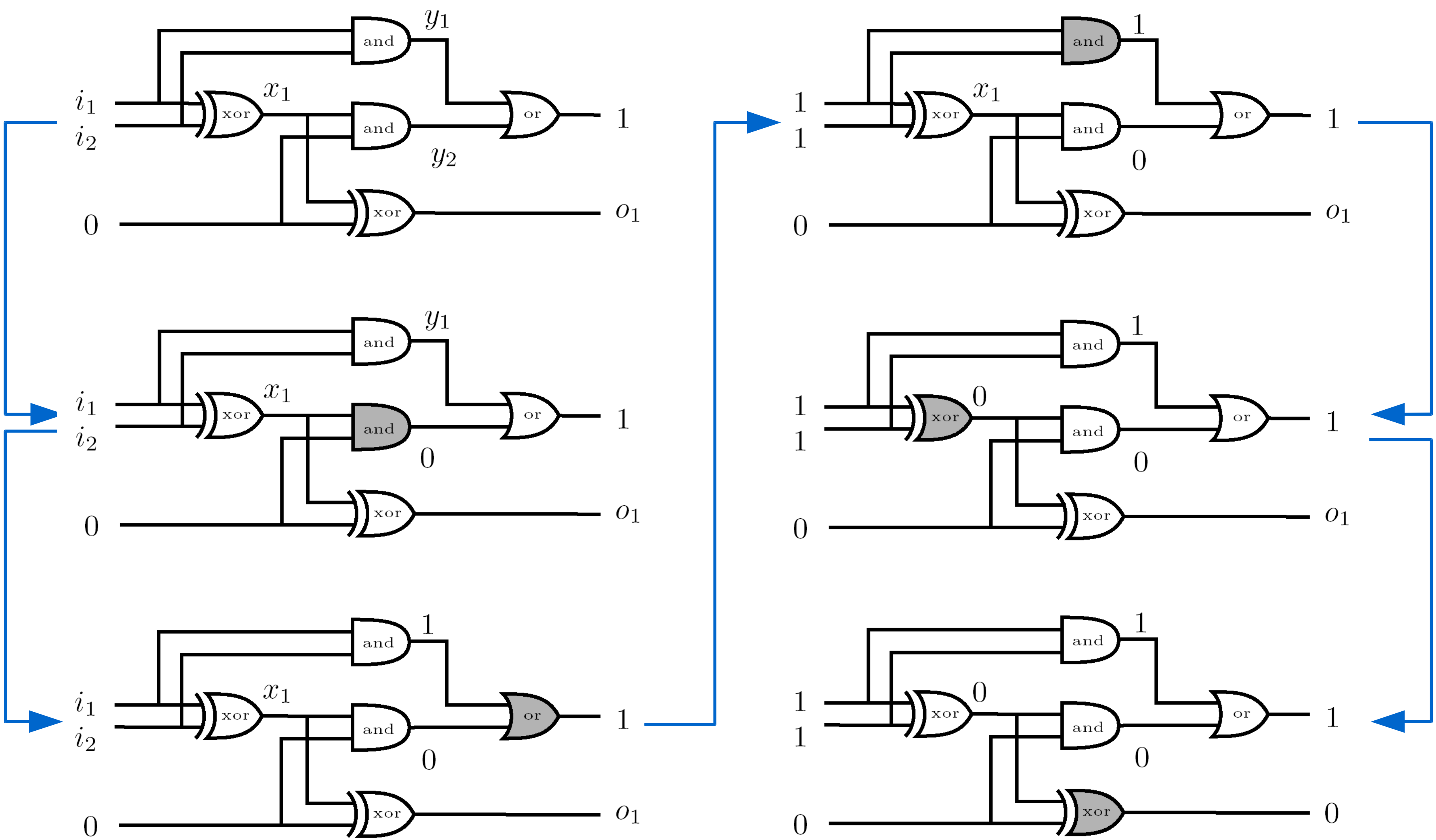


computes the binary sum $i_1 + i_2 + i_3$ in the binary word o_2o_1

- Example: Deduce that $i_1 = 1$, $i_2 = 1$ and $o_1 = 0$ follows from $i_3 = 0$ and $o_2 = 1$
- Two proof rules for *XOR*

XOR 1	$x \oplus y = z, x = 1, y = 1 \rightarrowtail z = 0$
XOR 2	$x \oplus y = z, x = 0, y = 0 \rightarrowtail z = 0$

Constraint Propagation in Full Adder Circuit



Linear Constraints on Integer Intervals

Consider the language with

- two constants 0 and 1
- unary function "–"
- two binary functions "+" and "–"

Linear expression: term in this language

Linear constraint: a formula

$$s \; op \; t$$

where s and t are linear expressions and $op \in \{<, \leq, =, \neq, \geq, >\}$

Abbreviations:

$n \coloneqq 1 + \ldots + 1$ (n times)

$nx \coloneqq x + \ldots + x$ (n times)

Analogously for $-n$ and $-nx$

Simple Disequality Rules

SIMPLE DISEQUALITY 1

$$\frac{\langle x \neq y \,;\, x \in [a..b],\, y \in [c..d] \rangle}{\langle \,;\, x \in [a..b],\, y \in [c..d] \rangle}$$ (where b < c or d < a)

SIMPLE DISEQUALITY 2

$$\frac{\langle x \neq y \,;\, x \in [a..b],\, y = a \rangle}{\langle \,;\, x \in [a+1..\mathrm{b}],\, y = a \rangle}$$

SIMPLE DISEQUALITY 3

$$\frac{\langle x \neq y \,;\, x \in [a..b],\, y = b \rangle}{\langle \,;\, x \in [a..b-1],\, y = b \rangle}$$

Domain Reduction (Example)

Consider

$$3x + 4y - 5z \le 7$$

with $x \in [l_x..h_x]$, $y \in [l_y..h_y]$, $z \in [l_z..h_z]$

Rewrite as

$$x \le \frac{7 - 4y + 5z}{3}$$

Any value of x that satisfies this constraint also satisfies

$$x \le \frac{7 - 4l_y + 5h_z}{3}$$

Hence we can reduce $[l_x..h_x]$ to

$$[l_x..min(\lfloor \frac{7 - 4l_y + 5h_z}{3} \rfloor, h_x)]$$

Domain Reduction for Linear Equality

$$\frac{\left\langle \sum_{i \in POS} a_i x_i - \sum_{i \in NEG} a_i x_i = b \,;\, x_1 \in [l_1..h_1], \ldots, x_n \in [l_n..h_n] \right\rangle}{\left\langle \sum_{i \in POS} a_i x_i - \sum_{i \in NEG} a_i x_i = b \,;\, x_1 \in [l'_1..h'_1], \ldots, x_n \in [l'_n..h'_n] \right\rangle}$$

where $l'_j := max(l_j, \lceil \gamma_j \rceil)$, $h'_j := min(h_j, \lfloor \alpha_j \rfloor)$ for $j \in POS$

and $l'_j := max(l_j, \lceil \beta_j \rceil)$, $h'_j := min(h_j, \lfloor \delta_j \rfloor)$ for $j \in NEG$

and

$$\alpha_j := \frac{b - \sum_{i \in POS - \{j\}} a_i l_i + \sum_{i \in NEG} a_i h_i}{a_j}$$

$$\beta_j := \frac{-b + \sum_{i \in POS} a_i l_i - \sum_{i \in NEG - \{j\}} a_i h_i}{a_j}$$

$$\gamma_j := \frac{b - \sum_{i \in POS - \{j\}} a_i h_i + \sum_{i \in NEG} a_i l_i}{a_j}$$

$$\delta_j := \frac{-b + \sum_{i \in POS} a_i h_i - \sum_{i \in NEG - \{j\}} a_i l_i}{a_j}$$

Example: SEND + MORE = MONEY

```
   SEND
 + MORE
 ------
  MONEY
```

1. Use the transformation rules to transform "SEND + MORE = MONEY" constraint to $9000 \cdot M + 900 \cdot O + 90 \cdot N + Y - (91 \cdot E + D + 1000 \cdot S + 10 \cdot R) = 0$
2. Apply LINEAR EQUALITY reduction rule: $S = 9$, $E \in [0..9]$, $N \in [0..9]$, $D \in [0..9]$, $M = 1$, $O \in [0..1]$, $R \in [0..9]$, $Y \in [0..9]$
3. Apply SIMPLE DISEQUALITY rule to $M \neq O$ to conclude $O = 0$

Example: SEND + MORE = MONEY, ctd

4. Repeatedly use M = 1, O = 0, S = 9 and SIMPLE DISEQUALITY rules. This eventually yields
   $S = 9, E \in [2..8], N \in [2..8], D \in [2..8], M = 1, O = 0, R \in [2..8], Y \in [2..8]$
5. 5 iterations of LINEAR EQUALITY rule yield
   $E \in [2..7], N \in [3..8]$
   $E \in [3..7], N \in [3..8]$
   $E \in [2..7], N \in [4..8]$
   $E \in [4..7], N \in [4..8]$
   $E \in [4..7], N \in [5..8]$
   The other ranges remain unchanged.

Arithmetic Constraints on Integer Intervals

Consider the language with

- two constants 0 and 1
- unary function "–"
- three binary functions "+", "–", and "·" (new)

Arithmetic constraint: a formula

$$s \; op \; t$$

where s and t are terms and $op \in \{<, \leq, =, \neq, \geq, >\}$

Example:

$$x^5 \cdot y^2 \cdot z^4 + 3x \cdot y^3 \cdot z^5 \leq 10 + 4x^4 \cdot y^6 \cdot z^2 - y^2 \cdot x^5 \cdot z^4$$

Approach Based on Atomic Arithmetic Constraints

Atomic arithmetic constraint:

- a linear constraint or
- $x \cdot y = z$

Note: Every arithmetic constraint can be reduced to a sequence of atomic constraints.

Transformation rule (Example):

$$\frac{\sum_{i=1}^{n} m_i \; op \; b}{\sum_{i=1}^{n} v_i \; op \; b, m_1 = v_1, \ldots, m_n = v_n}$$

where $v_1, \ldots, v_n$ are auxiliary variables

Interval Multiplication

X, *Y* sets of integers

- Multiplication:

  $X \cdot Y := \{x \cdot y \mid x \in X, y \in Y\}$

Note: $X \cdot Y$ does not have to be an interval even if *X* and *Y* are

Example: [0..2]·[1..2] = {0, 1, 2, 4}

$$int(A) := \begin{cases} \text{smallest int. interval} \supseteq A & \text{if it exists} \\ \mathbb{Z} & \text{otherwise} \end{cases}$$

(for sets of integers *A*)

Multiplication Rule 1

MULTIPLICATION 1

$$\frac{\langle x \cdot y = z \,;\, x \in D_x,\, y \in D_y,\, z \in D_z \rangle}{\langle x \cdot y = z \,;\, x \in D_x,\, y \in D_y,\, z \in D_z \cap int(D_x \cdot D_y) \rangle}$$

Example:

$\langle x \cdot y = z \,;\, x \in [0..2],\, y \in [1..2],\, z \in [4..6] \rangle$

$int([0..2] \cdot [1..2]) = [0..4]$ and $[4..6] \cap [0..4] = [4..4]$, so we get

$\langle x \cdot y = z \,;\, x \in [0..2],\, y \in [1..2],\, z \in [4..4] \rangle$

Multiplication Rules 2, 3

- Interval division:
  $Z / Y = \{x \in \mathbb{Z} \mid \exists y \in Y \exists z \in Z\ x \cdot y = z\}$

Note: *Z* / *Y* does not have to be an interval even if *Z*, *Y* are

Example: [3..5] / [–1..2] = {–5, –4, –3, 2, 3, 4, 5}

MULTIPLICATION 2

$$\frac{\langle x \cdot y = z \,;\, x \in D_x, y \in D_y, z \in D_z \rangle}{\langle x \cdot y = z \,;\, x \in D_x \cap int(D_z / D_y), y \in D_y, z \in D_z \rangle}$$

MULTIPLICATION 3

$$\frac{\langle x \cdot y = z \,;\, x \in D_x, y \in D_y, z \in D_z \rangle}{\langle x \cdot y = z \,;\, x \in D_x, y \in D_y \cap int(D_z / D_x), z \in D_z \rangle}$$

Example

$\langle x \cdot y = z \,;\, x \in [1..20],\, y \in [9..11],\, z \in [155..161]\rangle$

Applying MULTIPLICATION 2 rule yields

$\langle x \cdot y = z \,;\, x \in [16..16],\, y \in [9..11],\, z \in [155..161]\rangle$

since $[155..161] / [9..11] = [16..16]$ and $[1..20] \cap \mathit{int}([16..16]) = [16..16]$

Applying MULTIPLICATION 3 rule yields

$\langle x \cdot y = z \,;\, x \in [16..16],\, y \in [10..10],\, z \in [155..161]\rangle$

since $[155..161] / [16..16] = [10..10]$ and $[9..11] \cap \mathit{int}([10..10]) = [10..10]$

Applying MULTIPLICATION 1 rule yields

$\langle x \cdot y = z \,;\, x \in [16..16],\, y \in [10..10],\, z \in [160..160]\rangle$

since $[16..16] \cdot [10..10] = [160..160]$ and $[155..161] \cap \mathit{int}([160..160]) = [160..160]$

Arithmetic Constraints on Reals

Consider the language with

- each real number as a constant (new)
- unary function "–"
- three binary functions "+", "–", and "·"

Arithmetic constraint: a formula

$$s \ op \ t$$

where *s* and *t* are terms and *op* ∈ {<, ≤, =, ≠, ≥, >}

Example:

$$2.4 \cdot x^5 \cdot y^2 \cdot z^4 + 3.6 \cdot x \cdot y^3 \cdot z^5 \leq 10.1 + 4.2 \cdot x^4 \cdot y^6 \cdot z^2$$

Domains: Extend Intervals

$$\mathbb{R}^+ := \mathbb{R} \cup \{-\infty, \infty\}$$

Extend < from $\mathbb{R}$ to $\mathbb{R}^+$ as expected

Extend interval: expression

$$\langle a, b \rangle$$

where $a, b \in \mathbb{R}^+$

Meaning: $\langle a, b \rangle = \{r \in \mathbb{R} \mid a \leq r \leq b\}$

Note: For $a, b \in \mathbb{R}$

$$\langle a, a \rangle = \{a\}$$

$$\langle -\infty, b \rangle = \{r \in \mathbb{R} \mid r \leq b\}$$

$$\langle a, \infty \rangle = \{r \in \mathbb{R} \mid a \leq r\}$$

$$\langle -\infty, \infty \rangle = \mathbb{R}$$

Interval Arithmetic

X, Y sets of reals

- $X + Y := \{x + y \mid x \in X, y \in Y\}$
- $X - Y := \{x - y \mid x \in X, y \in Y\}$
- $X \cdot Y := \{x \cdot y \mid x \in X, y \in Y\}$
- $X / Y := \{u \in \mathbb{R} \mid \exists x \in X \exists y \in Y \; u \cdot y = x\}$

For real r and $op \in \{+, -, \cdot, /\}$

$$r \; op \; X := \{r\} \; op \; X$$

$$X \; op \; \mathrm{r} := X \; op \; \{r\}$$

Interval Arithmetic, ctd

X, Y extended intervals, r a real

- $X \cap Y$, $X+Y$, $X-Y$ and $X \cdot Y$ are extended intervals
- $X / \{r\}$ is an extended interval
- X / Y does not have to be an extended interval

Example:

$\langle 2, 16 \rangle / \langle -\infty, -2 \rangle = \{r \in \mathbb{R} \mid -8 \leq r < 0\}$

$int(A) \coloneqq$ smallest extended interval containing A for sets of reals A

Atomic Arithmetic Constraints

- $\sum_{i=1}^{n} a_i x_i = b$

  - n > 0
  - $a_1, \ldots, a_n$ non-zero reals
  - $x_1, \ldots, x_n$ different variables
  - b is a real
- $x \neq y$
- $x \cdot y = z$

Note:

Every arithmetic constraint can be reduced to a sequence of atomic constraints.

Domain Reduction Rules

Intuition: $\sum_{i=1}^{n} a_i x_i = b$ implies that for $j \in [1..n]$: $x_j = \dfrac{b - \sum_{i \in [1..n]-\{j\}} a_i x_i}{a_j}$

ℝ-LINEAR EQUALITY

$$\frac{\left\langle \sum_{i=1}^{n} a_i x_i = b \,;\, x_1 \in D_1, \ldots, x_n \in D_n \right\rangle}{\left\langle \sum_{i=1} a_i x_i = b \,;\, \ldots, x_j \in D'_j, \ldots \right\rangle}$$

where $j \in [1..n]$ and $D'_j := D_j \;\cap\; \dfrac{b - \sum_{i \in [1..n]-\{j\}} a_i \cdot D_i}{a_j}$

DISEQUALITY 2

$$\frac{\langle x \neq y \,;\, x \in D_x, y \in D_y \rangle}{\langle \,;\, x \in D_x, y \in D_y \rangle}$$

where $D_x \cap D_y = \emptyset$

Multiplication Rules

ℝ-MULTIPLICATION 1

$$\frac{\langle x\cdot y=z\,;\,x\in D_x,\,y\in D_y,\,z\in D_z\rangle}{\langle x\cdot y=z\,;\,x\in D_x,\,y\in D_y,\,z\in D_z\cap D_x\cdot D_y\rangle}$$

ℝ-MULTIPLICATION 2

$$\frac{\langle x\cdot y=z\,;\,x\in D_x,\,y\in D_y,\,z\in D_z\rangle}{\langle x\cdot y=z\,;\,x\in D_x\cap int(D_z/D_y),\,y\in D_y,\,z\in D_z\rangle}$$

ℝ-MULTIPLICATION 3

$$\frac{\langle x\cdot y=z\,;\,x\in D_x,\,y\in D_y,\,z\in D_z\rangle}{\langle x\cdot y=z\,;\,x\in D_x,\,y\in D_y\cap int(D_z/D_x),\,z\in D_z\rangle}$$

Example

$\langle x \cdot y = z \,;\, x \in \langle -\infty, -1 \rangle, y \in \langle -\infty, -2 \rangle , z \in \langle -\infty, 161 \rangle \rangle$

By $\mathbb{R}$-MULTIPLICATION 1 rule

$\langle x \cdot y = z \,;\, x \in \langle -\infty, -1 \rangle, y \in \langle -\infty, -2 \rangle , z \in \langle 2, 161 \rangle \rangle$

since $\langle -\infty, -1 \rangle \cdot \langle -\infty, -2 \rangle = \langle 2, \infty \rangle$ and hence $\langle -\infty, 161 \rangle \cap \langle 2, \infty \rangle = \langle 2, 161 \rangle$

By $\mathbb{R}$-MULTIPLICATION 3 rule

$\langle x \cdot y = z \,;\, x \in \langle -\infty, -1 \rangle, y \in \langle -161, -2 \rangle , z \in \langle 2, 161 \rangle \rangle$

since $\langle 2, 161 \rangle / \langle -\infty, -1 \rangle = \{r \in \mathbb{R} \mid -161 \leq r < 0\}$

and hence $int(\langle 2, 161 \rangle / \langle -\infty, -1 \rangle) = \langle -161, 0 \rangle$ and $\langle -\infty, -2 \rangle \cap \langle -161, 0 \rangle = \langle -161, -2 \rangle$

Example, ctd

$\langle x \cdot y = z \, ; \, x \in \langle -\infty, -1\rangle, \, y \in \langle -161, -2\rangle \, , \, z \in \langle 2, 161\rangle\rangle$

By $\mathbb{R}$-MULTIPLICATION 2 rule

$\langle x \cdot y = z \, ; \, x \in \langle -80.5, -1\rangle, \, y \in \langle -161, -2\rangle \, , \, z \in \langle 2, 161\rangle\rangle$

since $\langle 2, 161\rangle \, / \, \langle -161, -2\rangle = \langle -80.5, -2 \, / \, 161\rangle$

and hence $\langle -\infty, -1\rangle \cap \mathit{int}(\langle -80.5, -2 \, / \, 161\rangle) = \langle -80.5, -1\rangle$

Last CSP is closed under the MULTIPLICATION rules

Arithmetic Constraints on Reals: Implementation Issues

Step 1: Extend arithmetic operations from $\mathbb{R}$ to $\mathbb{R}^+$

$\perp$: undefined operation

PR: a positive real

NR: a negative real

x

$x+y$	$-\infty$	NR	0	PR	∞
$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$\perp$
NR		NR	NR	$\mathcal{R}$	∞
0			0	PR	∞
PR				PR	∞
∞					∞

y

x

$x-y$	$-\infty$	NR	0	PR	∞
$-\infty$	$\perp$	∞	∞	∞	∞
NR	$-\infty$	$\mathcal{R}$	PR	PR	∞
0	$-\infty$	NR	0	PR	∞
PR	$-\infty$	NR	NR	$\mathcal{R}$	∞
∞	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$\perp$

y

x

$x \cdot y$	$-\infty$	NR	0	PR	∞
$-\infty$	∞	∞	$\perp$	$-\infty$	$-\infty$
NR		PR	0	NR	$-\infty$
0			0	0	$\perp$
PR				PR	∞
∞					∞

y

x

x/y	$-\infty$	NR	0	PR	∞
$-\infty$	$\perp$	0	0	0	$\perp$
NR	∞	PR	0	NR	$-\infty$
0	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
PR	$-\infty$	NR	0	PR	∞
∞	$\perp$	0	0	0	$\perp$

y

Implementation Issues, ctd

Step 2: Implement intersection, addition, subtraction of extended intervals

Note: For non-empty extended intervals $\langle a, b \rangle$ and $\langle c, d \rangle$

- $\langle a, b \rangle \cap \langle c, d \rangle = \langle max(a, c), min(b, d) \rangle$
- $\langle a, b \rangle + \langle c, d \rangle = \langle a + c, b + d \rangle$
- $\langle a, b \rangle - \langle c, d \rangle = \langle a - d, b - c \rangle$

Classification of Non-Empty Extended Intervals

Depends on the position of 0 w.r.t. such an interval

class of $\langle a, b \rangle$	at least one negative	at least one positive	signs of endpoints
M	yes	yes	$a < 0 \wedge b > 0$
Z	no	no	$a = 0 \wedge b = 0$
P	no	yes	$a \geq 0 \wedge b > 0$
P_0	no	yes	$a = 0 \wedge b > 0$
P_1	no	yes	$a > 0 \wedge b > 0$
N	yes	no	$a < 0 \wedge b \leq 0$
N_0	yes	no	$a < 0 \wedge b = 0$
N_1	yes	no	$a < 0 \wedge b < 0$

Implementation of Multiplication

Step 3: Implement multiplication of extended intervals

$\langle a, b\rangle$ and $\langle c, d\rangle$: non-empty extended intervals

class of $\langle a, b\rangle$	class of $\langle c, d\rangle$	$\langle a, b\rangle \cdot \langle c, d\rangle$
P	P	$\langle a\cdot c, b\cdot d\rangle$
P	M	$\langle b\cdot c, b\cdot d\rangle$
P	N	$\langle b\cdot c, a\cdot d\rangle$
M	P	$\langle a\cdot d, b\cdot d\rangle$
M	M	$\langle min(a\cdot d, b\cdot c), max(a\cdot c, b\cdot d)\rangle$
M	N	$\langle b\cdot c, a\cdot c\rangle$
N	P	$\langle a\cdot d, b\cdot c\rangle$
N	M	$\langle a\cdot d, a\cdot c\rangle$
N	N	$\langle b\cdot d, a\cdot c\rangle$
Z	P, M, N, Z	$\langle 0, 0\rangle$
P, M, N	Z	$\langle 0, 0\rangle$

Example

$\langle -3, 2\rangle \cdot \langle -4, 5\rangle$

Both intervals are of class *M*, so the entry

class of $\langle a, b\rangle$	class of $\langle c, d\rangle$	$\langle a, b\rangle \cdot \langle c, d\rangle$
M	*M*	$\langle min(a \cdot d, b \cdot c), max(a \cdot c, b \cdot d)\rangle$

applies. Thus

$\langle -3, 2\rangle \cdot \langle -4, 5\rangle =$

$\langle min((-3) \cdot 5, 2 \cdot (-4)), max((-3) \cdot (-4), 2 \cdot 5)\rangle =$

$\langle min(-15, -8), max(12, 10)\rangle =$

$\langle -15, 12\rangle$

Implementation of Division

Step 4: Implement division of extended intervals.

$\langle a, b\rangle$ and $\langle c, d\rangle$: non-empty extended intervals

class of $\langle a, b\rangle$	class of $\langle c, d\rangle$	$\langle a, b\rangle / \langle c, d\rangle$
P_1	P_1	$\langle a / d, b / c\rangle \setminus \{0\}$
P_1	P_0	$\langle a / d, \infty\rangle \setminus \{0\}$
P_0	P_1	$\langle 0, b / c\rangle$
M	P_1	$\langle \mathrm{a} / \mathrm{c}, b / c\rangle$
N_0	P_1	$\langle a / c, 0\rangle$
N_1	P_1	$\langle a / c, b / d\rangle \setminus \{0\}$
N_1	P_0	$\langle -\infty, b / d\rangle \setminus \{0\}$
P_1	M	$(\langle -\infty, a / c\rangle \cup \langle a / d, \infty\rangle) \setminus \{0\}$

Implementation of Division, ctd

class of $\langle a, b\rangle$	class of $\langle c, d\rangle$	$\langle a, b\rangle / \langle c, d\rangle$
M, Z, P_0, N_0	M, Z, P_0, N_0	$\langle -\infty, +\infty\rangle$
N_1	M	$(\langle -\infty, b / d\rangle \cup \langle b / c, \infty\rangle) \setminus \{0\}$
P_1	N_1	$\langle b / d, a / c\rangle \setminus \{0\}$
P_1	N_0	$\langle -\infty, a / c\rangle \setminus \{0\}$
P_0	N_1	$\langle b / d, 0\rangle$
M	N_1	$\langle b / d, a / d\rangle$
N_0	N_1	$\langle 0, a / d\rangle$
N_1	N_1	$\langle b / c, a / d\rangle \setminus \{0\}$
N_1	N_0	$\langle b / c, \infty\rangle \setminus \{0\}$
Z	P_1, N_1	$\langle 0, 0\rangle$
P_1, N_1	Z	$\emptyset$

Example

$\langle 2, 16\rangle / \langle -\infty, -2\rangle$

The intervals are of class P_1 and N_1, so the entry

class of $\langle a, b\rangle$	class of $\langle c, d\rangle$	$\langle a, b\rangle / \langle c, d\rangle$
P_1	N_1	$\langle b / d, a / c\rangle \setminus \{0\}$

applies. Thus

$\langle 2, 16\rangle / \langle -\infty, -2\rangle =$

$\langle 16 / (-2), 2 / (-\infty)\rangle \setminus \{0\} =$

$\{r \in \mathbb{R} \mid -8 \le r < 0\}$

Using Floating-point Numbers

Step 5: Introduce Floating-Point Numbers

Motivation: We want to represent solutions to $9 \cdot x^2 = 1$ over $\langle -1, 1\rangle$ as

$$x \in \langle -0.33334, -0.33333\rangle \text{ and } x \in \langle 0.33333, 0.33334\rangle$$

Assume finite subset $\mathcal{F}$ of $\mathbb{R}^+$ containing $-\infty$ and ∞

Elements of $\mathcal{F}$: floating-point numbers

Floating-point interval:

$$\langle a, b\rangle$$

a, b floating-point numbers

$\Gamma(A)$: the least floating-point interval containing A

Amended Multiplication Rules

$\mathcal{F}$-MULTIPLICATION 1

$$\frac{\langle x \cdot y = z\,;\, x \in D_x,\, y \in D_y,\, z \in D_z \rangle}{\langle x \cdot y = z\,;\, x \in D_x,\, y \in D_y,\, z \in D_z \cap \Gamma(D_x \cdot D_y) \rangle}$$

$\mathcal{F}$-MULTIPLICATION 2

$$\frac{\langle x \cdot y = z\,;\, x \in D_x,\, y \in D_y,\, z \in D_z \rangle}{\langle x \cdot y = z\,;\, x \in D_x \cap \Gamma(D_z / D_y),\, y \in D_y,\, z \in D_z \rangle}$$

$\mathcal{F}$-MULTIPLICATION 3

$$\frac{\langle x \cdot y = z\,;\, x \in D_x,\, y \in D_y,\, z \in D_z \rangle}{\langle x \cdot y = z\,;\, x \in D_x,\, y \in D_y \cap \Gamma(D_z / D_x),\, z \in D_z \rangle}$$

- Combined with the implementation $\Gamma(X \cdot Y)$ and $\Gamma(X / Y)$ for the floating-point intervals X , Y
- Similar modification of other domain reduction rules

Objectives

Introduce incomplete constraint solvers for

- equality and disequality constraints
- Boolean constraints
- linear constraints over integer intervals and over finite integer domains
- arithmetic constraints over integer intervals
- arithmetic constraints over reals