Synthesis of Controllable and Normal Sublanguages for Discrete-Event Systems using a Coordinator

Jan Komenda[a], Tomáš Masopust[a,b,*], Jan H. van Schuppen[b]

[a] *Institute of Mathematics, Czech Academy of Sciences, Žižkova 22, 616 62 Brno, Czech Republic*
[b] *CWI, P.O. Box 94079, 1090 GB Amsterdam, The Netherlands*

Abstract

Synthesis of normal or controllable and normal sublanguages of global specification languages without computation of the global modular plant is a difficult problem. In this paper, these sublanguages are computed using a coordinator. We recall the notion of conditional controllability, introduce a notion of conditional normality, and prove necessary and sufficient conditions where such a computation is possible. Specifically, we show that conditionally controllable and conditionally normal languages computed by our method are controllable and normal with respect to the global plant. The optimality (supremality) of the resulting languages is also discussed.

Keywords: Discrete-event system, coordination control, coordinator, supervisory control, conditional controllability, conditional observability, conditional normality, supremal conditionally controllable and conditionally normal sublanguage.
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1. Introduction

In supervisory control with partial observations, specification languages must be observable and controllable for a supervisor to exist so that the closed-loop system equals the specification [1]. This condition is often not satisfied and, moreover, a control requirement is most often a so called safety specification, that is, only the inclusion of the closed-loop language in the specification is required. In the case when the specification language is not controllable or observable, a controllable and observable sublanguage of the specification is considered. The synthesis of observable sublanguages is difficult, especially in the modular setting, where the plant is composed of local subsystems (automata) running in parallel so that the plant is formed as a synchronous product of local components.

Unfortunately, observability is not preserved under union, unlike controllability. Therefore, the supremal observable sublanguage does not always exist, and there are only maximal observable sublanguages, which are not unique in general. A slightly stronger notion, called normality, coincides with observability in the case when all controllable events are observable. Supremal normal sublanguages exist, but they are difficult to compute, especially in the modular framework. We have studied possibilities of local (modular) computations of supremal normal sublanguages in [2] for local specification languages and in [3] for global specification languages. However, the sufficient conditions for their local computation to equal the global computation are too restrictive.

In this paper, another approach is presented for synthesis of controllable and normal sublanguages in modular discrete-event systems. This approach is much less restrictive, but the optimality (supremality of the computed sublanguage) is only guaranteed under some additional conditions. It is based on the coordination control architecture, where a coordinator is added to the plant that takes care of the global part of the specification. Since only prefix-closed specifications are considered in this paper, it is not the coordinator automaton itself, but only its underlying event set that is the basis of our approach. In fact, coordinators themselves are useful for handling the blocking issues, while in

*Corresponding author. CWI, P.O. Box 94079, 1090 GB Amsterdam, The Netherlands, Tel. +31(0)205924122, Fax. +31(0)205924199
Email addresses: `komenda@ipm.cz` (Jan Komenda), `masopust@math.cas.cz` (Tomáš Masopust), `J.H.van.Schuppen@cwi.nl` (Jan H. van Schuppen)

the prefix-closed case it is sufficient to choose a suitable event set containing the intersection of local event sets. The coordinator is then simply the global system projected to this coordinator event set so that the system composed with the coordinator equals the original modular plant. Moreover, the coordinator can be computed modularly because of the conditional independence property (amounting to distributivity of natural projections). The event set of the coordinator is chosen so that the specification language is conditional decomposable as defined in the paper. Our results are based on the notion of conditional controllability and conditional normality, and the computation of supremal conditionally controllable and conditionally normal sublanguages of the specification computed without building the global plant. It is shown that supremal conditionally controllable and conditionally normal sublanguages are controllable and normal with respect to the coordinated plant that equals the original plant as discussed above. Moreover, it is shown that, under some additional conditions, supremal conditionally controllable and conditionally normal, and supremal controllable and normal sublanguages coincide.

The paper is organized as described below. The next section recalls the basic results of supervisory control theory used further in the paper. In Section 3, basic notions that are needed for our approach are introduced. Section 4 contains further concepts on partially observed modular plants, in particular a necessary and sufficient condition for the specification to be exactly achievable using our coordination control scheme, called conditional observability. In Section 5, a stronger notion, called conditional normality, is presented. In Section 6, a procedure for the computation of supremal conditionally controllable and conditional normal sublanguages is proposed. Finally, a conclusion is given in Section 7.

2. Preliminaries

In this section, we briefly recall the elements of supervisory control theory needed in this paper. For more details, the reader is referred to [1, 4].

Discrete-event systems are modeled as deterministic generators that are finite-state machines with partial transition functions, i.e., a *generator* is a quintuple $G = (Q, E, f, q_0, Q_m)$, where Q is a finite set of *states*, E is the finite *event set*, $f : Q \times E \to Q$ is the *partial transition function*, $q_0 \in Q$ is the *initial state*, and $Q_m \subseteq Q$ is the set of *marked states*. As usual, f is extended to $f : Q \times E^* \to Q$. The behaviors of G are defined in terms of languages. The *language* generated by G is defined as $L(G) = \{s \in E^* \mid f(q_0, s) \in Q\}$, and the *language marked* by G is defined as $L_m(G) = \{s \in E^* \mid f(q_0, s) \in Q_m\}$.

The prefix closure $\overline{L}$ of a language $L \subseteq E^*$ is the set of all prefixes of all its words, i.e., $\overline{L} = \{w \in E^* \mid \exists v \in E^* \text{ such that } wv \in L\}$. A language L is prefix-closed if $L = \overline{L}$.

Let L be a prefix-closed language over an event set E with the uncontrollable event set $E_u \subseteq E$. A language $K \subseteq E^*$ is *controllable* with respect to L and E_u if

$$\overline{K}E_u \cap L \subseteq \overline{K}\,.$$

A *natural projection* $P : E^* \to E_o^*$, for some $E_o \subseteq E$, is a homomorphism defined so that $P(a) = \varepsilon$, for $a \in E \setminus E_o$, and $P(a) = a$, for $a \in E_o$. The *inverse image* of P, denoted by $P^{-1} : E_o^* \to 2^{E^*}$, is defined as $P^{-1}(a) = \{s \in E^* \mid P(s) = a\}$. These definitions can naturally be extended to languages.

In what follows, given event sets E_i, E_j, E_k, we denote by P^{i+j}_k the natural projection from $(E_i \cup E_j)^*$ to E_k^*, and by $P^i_{j\cap k}$ the natural projection from E_i^* to $(E_j \cap E_k)^*$. In addition, we define $E_{i,o} = E_i \cap E_o$, $E_{i,u} = E_i \cap E_u$, etc.

Let K and $M = \overline{M}$ be languages over an event set E. Let $E_c \subseteq E$ be the subset of controllable events, and let $E_o \subseteq E$ be the set of *observable* events with P as the corresponding natural projection from E^* to E_o^*. The specification language K is said to be *observable* with respect to M, E_o, and E_c if for all $s \in \overline{K}$ and for all $\sigma \in E_c$,

$$(s\sigma \notin \overline{K}) \text{ and } (s\sigma \in M) \Rightarrow P^{-1}[P(s)]\sigma \cap \overline{K} = \emptyset\,.$$

Consider $M = \overline{M} \subseteq E^*$ and a natural projection $P : E^* \to E_o^*$. A language $K \subseteq M$ is said to be *normal* with respect to M and P if

$$\overline{K} = P^{-1}[P(\overline{K})] \cap M\,.$$

Note that it is known that normality implies observability [1].

A *controlled generator* is a structure (G, E_c, P, Γ), where G is a generator, $E_c \subseteq E$ is the set of *controllable events*, $E_u = E \setminus E_c$ is the set of *uncontrollable events*, $P : E^* \to E_o^*$ is the natural projection, and $\Gamma = \{\gamma \subseteq E \mid E_u \subseteq \gamma\}$ is the *set of control patterns*.

A *supervisor* for the controlled generator (G, E_c, P, Γ) is a map $S : P(L(G)) \to \Gamma$.

A *closed-loop system* associated with the controlled generator (G, E_c, P, Γ) and the supervisor S is defined as the smallest language $L(S/G) \subseteq E^*$ which satisfies

1. $\varepsilon \in L(S/G)$,
2. if $s \in L(S/G)$, $sa \in L(G)$, and $a \in S(P(s))$, then $sa \in L(S/G)$.

In the automata framework where the supervisor is represented as an automaton, one can write $L(S/G)$ as $L(S)\|L(G)$.

Let G be a generator over an event set E. Let $E_u \subseteq E$ be the set of uncontrollable events, $E_c = E \setminus E_u$ be the set of controllable events, and $E_o \subseteq E$ be the set of observable events. Given a prefix-closed specification language $K \subseteq L(G) \subseteq E^*$, the aim of supervisory control theory is to find a supervisor S such that $L(S/G) = K$. It is known that such a supervisor exists if and only if K is controllable with respect to $L(G)$ and E_u and observable with respect to $L(G)$, E_o, and E_c [1]. However, as there are not, in general, supremal observable sublanguages, normality is used instead of observability. Thus, for specifications that are either not controllable or not observable, controllable and normal sublanguages are considered. In what follows, for prefix-closed languages $K \subseteq L \subseteq E^*$, $E_u, E_o \subseteq E$, and $Q : E^* \to E_o^*$, the notation $\sup \mathrm{CN}(K, L, E_u, Q)$ is chosen for the supremal controllable and normal sublanguage of K with respect to L, E_u, and Q. This supremal controllable and normal sublanguage always exists and equals the union of all controllable and normal sublanguages of K, see, e.g, [1]. A formula for calculating supremal controllable and normal sublanguages can be found in [5].

Below, modular discrete-event systems are considered. First, we recall that the synchronous product of languages $L_1 \subseteq E_1^*$ and $L_2 \subseteq E_2^*$ is defined by

$$L_1\|L_2 = P_1^{-1}(L_1) \cap P_2^{-1}(L_2) \subseteq E^*\,,$$

where $P_i : E^* \to E_i^*$, for $i = 1, 2$, are natural projections to local event sets. The synchronous product can also be defined for generators (the reader is referred to [1] for more details). In this case, for two generators G_1 and G_2, it is well known that $L(G_1\|G_2) = L(G_1)\|L(G_2)$ and $L_m(G_1\|G_2) = L_m(G_1)\|L_m(G_2)$.

Decentralized control is control of a monolithic system with two or more controllers each having its own observation channel. The observation event sets of the observation channels are incomparable in general. A modular or a distributed discrete-event system is the synchronous product of two or more modules or subsystems. In control of a distributed system each module or subsystem has its own observation channel and a supervisor or controller either with locally complete or locally partial observations. Control synthesis of a distributed discrete-event system then consists of synthesizing local supervisors, one for each subsystem. This is the main problem of the paper for which the coordinated control synthesis procedure is proposed. The global supervisor then consists of the synchronous product of local supervisors although that product is not computed in practice. In terms of behaviors, the optimal global control synthesis is represented by the closed-loop language $\sup \mathrm{CN}(K, L, E_u, Q) = \sup \mathrm{CN}(\|_{i=1}^n K_i, \|_{i=1}^n L_i, E_u, Q)$.

Given a rational global specification language $K \subseteq E^*$, one can theoretically always compute its supremal controllable and normal sublanguage from which the optimal (least restrictive) supervisor can be built. Such a global control synthesis of a modular discrete-event system consists simply in computing the global plant and then the control synthesis is carried out as described above.

Decentralized control synthesis means that the specification language K is replaced by $K_i = K \cap P_i^{-1}(L_i)$, and the synthesis is done similarly as for local specifications or using the notion of partial controllability [6]. However, the purely decentralized control synthesis is not always possible as the sufficient conditions under which it can be used are quite restrictive. Therefore, we have proposed the coordination control in [7] as a trade-off between the purely decentralized control synthesis, which is in some cases unrealistic, and the global control synthesis, which is naturally prohibitive for complexity reasons.

3. Concepts

Consider three generators G_1, G_2, and G_k. We call G_1 and G_2 *conditionally independent* generators given G_k if there is no simultaneous move in both G_1 and G_2 without the coordinator G_k being also involved, i.e.,

$$E_r(G_1\|G_2) \cap E_r(G_1) \cap E_r(G_2) \subseteq E_r(G_k)\,,$$

where $E_r(G)$ is the set of all reachable symbols in G, i.e., symbols that appear in a word of the language $L(G)$. This concept can easily be extended to the case of three or more generators. The corresponding concept in terms of languages follows. Consider event sets E_1, E_2, E_k, and languages $L_1 \subseteq E_1^*$, $L_2 \subseteq E_2^*$, and $L_k \subseteq E_k^*$. The languages L_1 and L_2 are *conditionally independent* given L_k if

$$E_r(L_1 \| L_2) \cap E_1 \cap E_2 \subseteq E_k\,,$$

where $E_r(L)$ denotes the set of all symbols occurring in words of L.

A language $K \subseteq (E_1 \cup E_2 \cup E_k)^*$ is called *conditionally decomposable* with respect to event sets (E_1, E_2, E_k) if

$$K = P_{1+k}(K) \| P_{2+k}(K) \| P_k(K)\,.$$

Now, the problem studied in this paper is formulated.

Problem 1. Consider generators G_1, G_2, G_k with event sets E_1, E_2, E_k, respectively, and a prefix-closed specification language $K \subseteq L(G_1 \| G_2 \| G_k)$. Assume that the coordinator G_k makes the two generators G_1 and G_2 conditionally independent, and that the specification language K is conditionally decomposable with respect to event sets (E_1, E_2, E_k).

The overall control task is divided into local subtasks and the coordinator subtask [7]. The coordinator takes care of its "part" of the specification, namely $P_k(K)$, i.e., $L(S_k/G_k) \subseteq P_k(K)$. Similarly, supervisors S_1 and S_2 take care of their corresponding "parts" of the specification, namely $P_{i+k}(K)$, i.e., $L(S_i/[G_i \| (S_k/G_k)]) \subseteq P_{i+k}(K)$, for $i = 1, 2$.

Determine supervisors S_1, S_2, and S_k for the respective generators so that the closed-loop system with the coordinator is such that

$$L(S_1/[G_1 \| (S_k/G_k)]) \| L(S_2/[G_2 \| (S_k/G_k)]) \| L(S_k/G_k) = K\,.$$

□

In Section 6.1, we discuss the question of how to find a coordinator as well as we demonstrate the method suggested in this paper. However, note that efficient algorithms that are not discussed in the paper are still a part of our future research.

In what follows, the notion of conditional controllability plays a key role in the theory.

Definition 2. Consider the setting of Problem 1. Call the specification language $K \subseteq E^*$ *conditionally controllable* for generators (G_1, G_2, G_k) and for the (uncontrollable) event subsets $(E_{1+k,u}, E_{2+k,u}, E_{k,u})$ if

(i) The language $P_k(K) \subseteq E_k^*$ is controllable with respect to G_k and $E_{k,u}$; equivalently,

$$P_k(K) E_{k,u} \cap L(G_k) \subseteq P_k(K)\,.$$

(ii.a) The language $P_{1+k}(K) \subseteq (E_1 \cup E_k)^*$ is controllable with respect to $L(G_1) \| P_k(K) \| P_k^{2+k}(L(G_2) \| P_k(K))$ and $E_{1+k,u} = E_u \cap (E_1 \cup E_k)$; equivalently,

$$P_{1+k}(K) E_{1+k,u} \cap L(G_1) \| P_k(K) \| P_k^{2+k}(L(G_2) \| P_k(K)) \subseteq P_{1+k}(K)\,.$$

(ii.b) The language $P_{2+k}(K) \subseteq (E_2 \cup E_k)^*$ is controllable with respect to $L(G_2) \| P_k(K) \| P_k^{1+k}(L(G_1) \| P_k(K))$ and $E_{2+k,u}$; equivalently,

$$P_{2+k}(K) E_{2+k,u} \cap L(G_2) \| P_k(K) \| P_k^{1+k}(L(G_1) \| P_k(K)) \subseteq P_{2+k}(K)\,.$$

4. Control synthesis of conditionally observable and conditionally controllable languages

In this section, we introduce a notion of conditional observability and prove that this condition along with conditional controllability are necessary and sufficient conditions for a specification language to be exactly achieved according to the setting of Problem 1.

Definition 3. Consider the setting of Problem 1. Call the specification language $K \subseteq E^*$ *conditionally observable* for generators (G_1, G_2, G_k), controllable subsets $(E_{1+k,c}, E_{2+k,c}, E_{k,c})$, and natural projections (Q_{1+k}, Q_{2+k}, Q_k), where $Q_i : E_i^* \to E_{i,o}^*$, for $i = 1+k, 2+k, k$, if

(i) The language $P_k(K) \subseteq E_k^*$ is observable with respect to $L(G_k)$, $E_{k,c}$, and Q_k; equivalently, for all $s \in P_k(K)$ and for all $\sigma \in E_{k,c}$,

$$(s\sigma \notin P_k(K)) \text{ and } (s\sigma \in L(G_k)) \Rightarrow Q_k^{-1}[Q_k(s)]\sigma \cap P_k(K) = \emptyset\,.$$

(ii.a) The language $P_{1+k}(K) \subseteq (E_1 \cup E_k)^*$ is observable with respect to $L(G_1)\|P_k(K)\|P_k^{2+k}(L(G_2)\|P_k(K))$, $E_{1+k,c} = E_c \cap (E_1 \cup E_k)$, and Q_{1+k}; equivalently, for all $s \in P_{1+k}(K)$ and for all $\sigma \in E_{1+k,c}$,

$$(s\sigma \notin P_{1+k}(K)) \text{ and } (s\sigma \in L(G_1)\|P_k(K)\|P_k^{2+k}(L(G_2)\|P_k(K))) \Rightarrow Q_{1+k}^{-1}[Q_{1+k}(s)]\sigma \cap P_{1+k}(K) = \emptyset\,.$$

(ii.b) The language $P_{2+k}(K) \subseteq (E_2 \cup E_k)^*$ is observable with respect to $L(G_2)\|P_k(K)\|P_k^{1+k}(L(G_1)\|P_k(K))$, $E_{2+k,c}$, and Q_{2+k}; equivalently, for all $s \in P_{2+k}(K)$ and for all $\sigma \in E_{2+k,c}$,

$$(s\sigma \notin P_{2+k}(K)) \text{ and } (s\sigma \in L(G_2)\|P_k(K)\|P_k^{1+k}(L(G_1)\|P_k(K))) \Rightarrow Q_{2+k}^{-1}[Q_{2+k}(s)]\sigma \cap P_{2+k}(K) = \emptyset\,.$$

A procedure to determine whether a language is conditionally observable is provided in Section 6. The following results are useful.

Lemma 4 ([4]). *Let $P_k : E^* \to E_k^*$ be a natural projection, and let $L_1 \subseteq E_1^*$ and $L_2 \subseteq E_2^*$ be local languages over event sets $E_1 \subseteq E$ and $E_2 \subseteq E$, respectively, such that $E_1 \cap E_2 \subseteq E_k$. Then, $P_k(L_1\|L_2) = P_k(L_1)\|P_k(L_2)$.*

Lemma 5 ([8, 9]). *Let $L \subseteq E^*$ and $P_k : E^* \to E_k^*$ be a natural projection with $E_k \subseteq E$. Then, $L\|P_k(L) = L$.*

Lemma 6 ([10]). *Let $L_i \subseteq E_i^*$ ($i = 1, 2$), and let $E_0 = E_1 \cap E_2$. Define the natural projections $P_i : (E_1 \cup E_2)^* \to E_i^*$ ($i = 0, 1, 2$) and $Q_j : E_j^* \to E_0^*$ ($j = 1, 2$). Then, for $i, j = 1, 2$ and $i \neq j$, $P_i(L_1\|L_2) = L_i \cap Q_i^{-1}Q_j(L_j)$.*

Theorem 7. *Consider the setting of Problem 1. There exists a set of supervisors (S_1, S_2, S_k) such that*

$$L(S_1/[G_1\|(S_k/G_k)]) \,\|\, L(S_2/[G_2\|(S_k/G_k)]) \,\|\, L(S_k/G_k) = K \qquad (1)$$

if and only if (1) the specification language K is conditionally controllable with respect to the set (G_1, G_2, G_k) of generators and $(E_{1+k,u}, E_{2+k,u}, E_{k,u})$ of locally uncontrollable events, and (2) conditionally observable with respect to the set (G_1, G_2, G_k) of generators, $(E_{1+k,c}, E_{2+k,c}, E_{k,c})$ of locally controllable events, and (Q_{1+k}, Q_{2+k}, Q_k) of natural projections from E_i^* to $E_{i,o}^*$, for $i = 1+k, 2+k, k$.

Proof. To prove sufficiency, let K be conditionally controllable with respect to (G_1, G_2, G_k) and $(E_{1+k,u}, E_{2+k,u}, E_{k,u})$, and conditionally observable with respect to (G_1, G_2, G_k), $(E_{1+k,c}, E_{2+k,c}, E_{k,c})$, and (Q_{1+k}, Q_{2+k}, Q_k). Equation (1) must be checked.

Since $K \subseteq L(G_1\|G_2\|G_k) \Rightarrow P_k(K) \subseteq P_k(L(G_1)\|L(G_2)\|L(G_k)) \subseteq L(G_k)$, and $P_k(K)$ is controllable with respect to $L(G_k)$ and $E_{k,u}$, and observable with respect to $L(G_k)$, $E_{k,c}$, and Q_k, it follows from [11] that there exists a supervisor S_k over the event set E_k such that $L(S_k/G_k) = P_k(K)$.

Furthermore, $K \subseteq L(G_1\|G_2\|G_k)$ implies that

$$\begin{aligned}P_{1+k}(K) &\subseteq P_{1+k}(L(G_1\|G_2\|G_k)) = P_{1+k}(L(G_1)\|L(G_k)) \,\|\, P_{k\cap 2}(L(G_2)), \quad \text{by Lemma 4,}\\ &= L(G_1)\|P_{k\cap 2}(L(G_2))\|L(G_k)\,.\end{aligned}$$

Then, $P_{1+k}(K) \subseteq L(G_1)\|P_{k\cap 2}(L(G_2))\|L(G_k)$ and $P_{1+k}(K) \subseteq (P_k^{1+k})^{-1}P_k(K)$ imply that

$$\begin{aligned}P_{1+k}(K) &\subseteq L(G_1)\|P_{k\cap 2}(L(G_2))\|L(G_k)\|P_k(K)\\ &= L(G_1)\|P_{k\cap 2}(L(G_2))\|L(G_k)\|L(S_k/G_k)\\ &= L(G_1)\|P_{k\cap 2}(L(G_2))\|L(S_k/G_k)\,, \qquad \text{by } L(G_k)\|L(S_k/G_k) = L(S_k/G_k)\,,\\ &= L(G_1)\|P_{k\cap 2}(L(G_2))\|L(S_k/G_k)\|P_k(K)\\ &= L(G_1)\|L(S_k/G_k)\|P_k(L(G_2\|(S_k/G_k))), \qquad \text{by } P_{k\cap 2}(L(G_2))\|P_k(K) = P_k(L(G_2\|(S_k/G_k)))\,.\end{aligned}$$

This, the assumption that the specification is conditionally controllable and conditionally observable, and [11] imply that there exists a supervisor S_1 such that $L(S_1/[G_1\|(S_k/G_k)\|P_k(G_2\|(S_k/G_k))]) = P_{1+k}(K)$, where for a generator G

and a natural projection P, $P(G)$ denotes the minimal generator that generates the language $P(L(G))$, i.e., $L(P(G)) = P(L(G))$ (see [1, 4] for details). For (ii.b) of Definition 8, a similar argument shows that there exists a supervisor S_2 such that $L(S_2/[G_2\|(S_k/G_k)\|P_k(G_1\|(S_k/G_k))]) = P_{2+k}(K)$.

In addition,

$$\begin{aligned} L(S_i/[G_i\|(S_k/G_k)\|P_k(G_i\|(S_k/G_k))]) &= L(S_i)\|L(G_i\|(S_k/G_k))\|P_k(L(G_i\|(S_k/G_k))) && (*)\\ &= L(S_i)\|L(G_i\|(S_k/G_k)), && \text{by Lemma 5,}\\ &= L(S_i/[G_i\|(S_k/G_k)]), \end{aligned}$$

which follows from the properties of synchronous product. It is now sufficient to notice that

$$\begin{aligned} &L(S_1/[G_1\|(S_k/G_k)\|P_k(G_2\|(S_k/G_k))])\|L(S_2/[G_2\|(S_k/G_k)\|P_k(G_1\|(S_k/G_k))])\\ &\quad= L(S_1) \parallel L(G_1\|(S_k/G_k)) \parallel P_k(L(G_2\|(S_k/G_k))) \parallel L(S_2) \parallel L(G_2\|(S_k/G_k)) \parallel P_k(L(G_1\|(S_k/G_k)))\\ &\quad= L(S_1) \parallel L(G_1\|(S_k/G_k)) \parallel P_k(L(G_1\|(S_k/G_k))) \parallel L(S_2) \parallel L(G_2\|(S_k/G_k)) \parallel P_k(L(G_2\|(S_k/G_k)))\\ &\quad= L(S_1) \parallel L(G_1\|(S_k/G_k)) \parallel L(S_2) \parallel L(G_2\|(S_k/G_k)), \qquad \text{using } (*),\\ &\quad= L(S_1/[G_1\|(S_k/G_k)]) \parallel L(S_2/[G_2\|(S_k/G_k)]), \end{aligned}$$

where the second equality is by the commutativity of the synchronous product. Summarized, we have shown that

$$\begin{aligned} &L(S_1/[G_1\|(S_k/G_k)\|P_k(G_2\|(S_k/G_k))]) \parallel L(S_2/[G_2\|(S_k/G_k)\|P_k(G_1\|(S_k/G_k))]) \parallel L(S_k/G_k)\\ &\quad= L(S_1/[G_1\|(S_k/G_k)]) \parallel L(S_2/[G_2\|(S_k/G_k)]) \parallel L(S_k/G_k). \end{aligned}$$

Finally, since K is conditionally decomposable and the following equalities are proven above,

$$\begin{aligned} P_{1+k}(K) &= L(S_1/[G_1\|(S_k/G_k)\|P_k(G_2\|(S_k/G_k))])\\ P_{2+k}(K) &= L(S_2/[G_2\|(S_k/G_k)\|P_k(G_1\|(S_k/G_k))])\\ P_k(K) &= L(S_k/G_k), \end{aligned}$$

it follows that $L(S_1/[G_1\|(S_k/G_k)]) \parallel L(S_2/[G_2\|(S_k/G_k)]) \parallel L(S_k/G_k) = P_{1+k}(K)\|P_{2+k}(K)\|P_k(K) = K$. Thus, sufficiency is proven.

To prove necessity, projections P_k, P_{1+k}, and P_{2+k} will be applied to Equation (1). Since all the supervisors cannot disable uncontrollable events, the closed-loop languages can be written as corresponding synchronous products. Thus, (1) can be rewritten as follows.

$$\begin{aligned} K &= L(S_1)\|L(G_1)\|L(S_k)\|L(G_k) \parallel L(S_2)\|L(G_2)\|L(S_k)\|L(G_k) \parallel L(S_k)\|L(G_k)\\ &= L(S_1)\|L(G_1)\|L(S_2)\|L(G_2)\|L(S_k)\|L(G_k), \end{aligned}$$

which yields after projecting by P_k that

$$\begin{aligned} P_k(K) &= P_k(L(S_1)\|L(G_1)\|L(S_2)\|L(G_2)\|L(S_k)\|L(G_k)) = L(S_k)\|L(G_k) \cap P_k(L(S_1)\|L(G_1)\|L(S_2)\|L(G_2))\\ &\subseteq L(S_k)\|L(G_k) = L(S_k/G_k). \end{aligned}$$

On the other hand, we always have $L(S_k/G_k) \subseteq P_k(K)$ because S_k is a supervisor that enforces the coordinator part of the specification $P_k(K)$. Hence, we have that $L(S_k/G_k) = P_k(K)$, which means, according to the basic theorem of supervisory control, that $P_k(K) \subseteq E_k^*$ is controllable with respect to $L(G_k)$ and $E_{k,u}$ and observable with respect to $L(G_k)$, $E_{k,c}$, and Q_k, i.e., (i) of definitions of conditional controllability and conditional observability are satisfied.

Now, (ii.a) of conditional controllability is shown; (ii.b) is a symmetric condition. An application of the projection P_{1+k} to (1) yields $P_{1+k}\big(L(S_k/G_k)\|L(S_1/[G_1\|(S_k/G_k)])\|L(S_2/[G_2\|(S_k/G_k)])\big) = P_{1+k}(K)$. Since $E_{1+k} \cap E_{2+k} = E_k$, and using the fact that $L(S_2)\|L(G_2\|(S_k/G_k)) = L(S_2) \cap L(G_2\|(S_k/G_k))$ because both components are over the same event set E_{2+k}, we obtain that

$$\begin{aligned} P_{1+k}(K) &= L(S_k/G_k) \parallel L(S_1/[G_1\|(S_k/G_k)]) \parallel P_{1+k}(L(S_2/[G_2\|(S_k/G_k)])) \\ &= L(S_k/G_k) \parallel L(S_1/[G_1\|(S_k/G_k)]) \parallel P_{1+k}(L(S_2)\|L(G_2\|(S_k/G_k))) \\ &\subseteq L(S_k/G_k) \parallel L(S_1/[G_1\|(S_k/G_k)]) \parallel P_{1+k}(L(G_2\|(S_k/G_k))) \\ &\subseteq L(S_1/[G_1\|(S_k/G_k)]) \parallel P_{1+k}(L(G_2\|(S_k/G_k))) \\ &\subseteq L(S_1/[G_1\|(S_k/G_k)]) \\ &\subseteq P_{1+k}(K)\,. \end{aligned}$$

Using again the fact that the closed-loop behavior under admissible supervisors can be recast as a synchronous composition of the plant and the supervisor, we get $L(S_1) \parallel L(G_1)\|L(S_k/G_k)\|P_{1+k}(L(G_2\|(S_k/G_k))) = P_{1+k}(K)$. The whole term after $L(S_1)$ can now be taken as a new plant $G_1\|(S_k/G_k)\|P_{1+k}(G_2\|(S_k/G_k))$. According to the basic theorem of supervisory control it implies that $P_{1+k}(K)$ is controllable with respect to $G_1\|(S_k/G_k)\|P_{1+k}(G_2\|(S_k/G_k))$ and $E_{1+k,u}$ and observable with respect to $G_1\|(S_k/G_k)\|P_{1+k}(G_2\|(S_k/G_k))$, $E_{1+k,c}$, and Q_{1+k}, i.e., (ii.a) of definitions of conditional controllability and conditional observability are satisfied, which was to be shown. □

5. Control synthesis of conditionally controllable and conditionally normal languages

As discussed above, it is well known that supremal observable sublanguages do not exist in general and it is not hard to see that this is also the case of conditionally observable sublanguages. Therefore, this section introduces an analogous notion to normality, so called conditional normality, and proves that conditional normality along with conditional controllability are sufficient conditions for the specification language to solve Problem 1.

Definition 8. Consider the setting of Problem 1. Call the specification language $K \subseteq E^*$ *conditionally normal* for generators (G_1, G_2, G_k) and for the natural projections (Q_{1+k}, Q_{2+k}, Q_k), where $Q_i : E_i^* \to E_{i,o}^*$, for $i = 1+k, 2+k, k$, if

(i) The language $P_k(K) \subseteq E_k^*$ is normal with respect to $L(G_k)$ and Q_k; equivalently,

$$Q_k^{-1} Q_k(P_k(K)) \cap L(G_k) = P_k(K)\,.$$

(ii.a) The language $P_{1+k}(K) \subseteq (E_1 \cup E_k)^*$ is normal with respect to $L(G_1)\|P_k(K)\|P_k^{2+k}(L(G_2)\|P_k(K))$ and Q_{1+k}; equivalently,

$$Q_{1+k}^{-1} Q_{1+k}(P_{1+k}(K)) \cap L(G_1)\|P_k(K)\|P_k^{2+k}(L(G_2)\|P_k(K)) = P_{1+k}(K)\,.$$

(ii.b) The language $P_{2+k}(K) \subseteq (E_2 \cup E_k)^*$ is normal with respect to $L(G_2)\|P_k(K)\|P_k^{1+k}(L(G_1)\|P_k(K))$ and Q_{2+k}; equivalently,

$$Q_{2+k}^{-1} Q_{2+k}(P_{2+k}(K)) \cap L(G_2)\|P_k(K)\|P_k^{1+k}(L(G_1)\|P_k(K)) = P_{2+k}(K)\,.$$

A procedure to determine whether a language is conditionally normal is provided in Section 6. To demonstrate that the definition is correct, it remains to show that $P_{i+k}(K) \subseteq L(G_i)\|P_k(K)\|P_k^{j+k}(L(G_j)\|P_k(K))$, for $\{i, j\} = \{1, 2\}$, $i \neq j$. This is shown in the following lemma.

Lemma 9. *For $\{i, j\} = \{1, 2\}$, $i \neq j$, it holds that $P_{i+k}(K) \subseteq L(G_i)\|P_k(K)\|P_k^{j+k}(L(G_j)\|P_k(K))$.*

Proof. By the definition, we need to show that $P_{i+k}(K) \subseteq (P_i^{i+k})^{-1}L(G_i) \cap (P_k^{i+k})^{-1}P_k(K) \cap (P_k^{i+k})^{-1}P_k^{j+k}(L(G_j)\|P_k(K))$. However, $P_{i+k}(K) \subseteq (P_i^{i+k})^{-1}P_i^{i+k}P_{i+k}(K) = (P_i^{i+k})^{-1}P_i(K)$ and $P_{i+k}(K) \subseteq (P_k^{i+k})^{-1}P_k^{i+k}P_{i+k}(K) = (P_k^{i+k})^{-1}P_k(K)$ imply that $P_{i+k}(K) \subseteq P_i(K)\|P_k(K)$, for $i = 1, 2$. In addition, the equality $P_k^{i+k}P_{i+k}(K) = P_k^{j+k}P_{j+k}(K)$ implies that $P_{i+k}(K) \subseteq (P_k^{i+k})^{-1}P_k^{i+k}P_{i+k}(K) = (P_k^{i+k})^{-1}P_k^{j+k}P_{j+k}(K) \subseteq (P_k^{i+k})^{-1}P_k^{j+k}(P_j(K)\|P_k(K))$. Thus, we have shown that $P_{i+k}(K) \subseteq P_i(K)\|P_k(K)\|P_k^{j+k}(P_j(K)\|P_k(K))$. Finally, as $K \subseteq L(G_1\|G_2\|G_k)$, we obtain that $P_i(K) \subseteq P_i(L(G_1\|G_2\|G_k)) \subseteq L(G_i)$. This implies that $P_{i+k}(K) \subseteq P_i(K)\|P_k(K)\|P_k^{j+k}(P_j(K)\|P_k(K)) \subseteq L(G_i)\|P_k(K)\|P_k^{j+k}(L(G_j)\|P_k(K))$, which was to be shown. □

Theorem 10. *Consider the setting of Problem 1. If the specification language K is conditionally controllable with respect to (G_1, G_2, G_k) and $(E_{1+k,u}, E_{2+k,u}, E_{k,u})$ of locally uncontrollable events, and conditionally normal with respect to (G_1, G_2, G_k) and (Q_{1+k}, Q_{2+k}, Q_k) of natural projections from E_i^* to $E_{i,o}^*$, for $i = 1+k, 2+k, k$, then there exist supervisors S_1, S_2, S_k such that*

$$L(S_1/[G_1\|(S_k/G_k)]) \parallel L(S_2/[G_2\|(S_k/G_k)]) \parallel L(S_k/G_k) = K\,.$$

Proof. As normality implies observability, the proof of this theorem follows immediately from Theorem 7. □

6. Computation of supremal conditionally controllable and conditionally normal sublanguages

So far, we have discussed conditions placed on the specification language under which a solution to Problem 1 exists. However, if the specification language does not satisfy these conditions, a supremal sublanguage that satisfies them is to be considered. In what follows, we present a procedure for computation of the supremal conditionally controllable and conditionally normal sublanguage for a given prefix-closed specification.

Theorem 11. *The supremal conditionally controllable sublanguage of a given language K always exists and is equal to the union of all conditionally controllable sublanguages of K.*

Proof. We show that conditional controllability is preserved by union. Let I be an index set, and let K_i, $i \in I$, be conditionally controllable sublanguages of $K \subseteq L(G_1\|G_2\|G_k)$ with respect to generators (G_1, G_2, G_k) and uncontrollable event sets $(E_{1+k,u}, E_{2+k,u}, E_{k,u})$. We prove that $\bigcup_{i\in I} K_i$ is conditionally controllable with respect to those generators and uncontrollable event sets, i.e., the three items of the definition hold.

i) First, we prove that $P_k(\bigcup_{i\in I} K_i)$ is controllable with respect to $L(G_k)$ and $E_{k,u}$. To do this, note that

$$P_k\left(\bigcup_{i\in I} K_i\right) E_{k,u} \cap L(G_k) = \bigcup_{i\in I}\Big(P_k(K_i)E_{k,u} \cap L(G_k)\Big) \subseteq \bigcup_{i\in I} P_k(K_i) = P_k\left(\bigcup_{i\in I} K_i\right),$$

where the inclusion is by controllability of $P_k(K_i)$ with respect to $L(G_k)$ and $E_{k,u}$, $i \in I$.

ii) To prove the other statement, note first that

$$L(G_1)\|P_k\left(\bigcup_{i\in I} K_i\right)\|P_k^{2+k}\left(L(G_2)\|P_k\left(\bigcup_{i\in I} K_i\right)\right) = L(G_1)\|P_k\left(\bigcup_{i\in I} K_i\right)\|P_k^{2+k}(L(G_2)) \tag{1}$$

because $P_k^{2+k}\left(L(G_2)\|P_k\left(\cup_{i\in I} K_i\right)\right) = P_k^{2+k}(L(G_2))\|P_k\left(\cup_{i\in I} K_i\right)$ by Lemma 4, and the second element is already included in the equation. Thus, we need to show that

$$P_{1+k}\left(\bigcup_{i\in I} K_i\right) E_{1+k,u} \cap L(G_1)\|P_k\left(\bigcup_{i\in I} K_i\right)\|P_k^{2+k}(L(G_2)) \subseteq P_{1+k}\left(\bigcup_{i\in I} K_i\right).$$

However, it holds that

$$\begin{aligned}
P_{1+k}\left(\bigcup_{i\in I} K_i\right) E_{1+k,u} \cap L(G_1)\|P_k\left(\bigcup_{i\in I} K_i\right)\|P_k^{2+k}(L(G_2))& \\
= \bigcup_{i\in I}\Big(P_{1+k}(K_i)E_{1+k,u}\Big) \cap \bigcup_{i\in I}\Big(L(G_1)\|P_k(K_i)\|P_k^{2+k}(L(G_2))\Big)& \\
= \bigcup_{i\in I}\bigcup_{j\in I}\Big(P_{1+k}(K_i)E_{1+k,u} \cap L(G_1)\|P_k(K_j)\|P_k^{2+k}(L(G_2))\Big)&.
\end{aligned}$$

For the sake of contradiction, assume that there are two different indexes $i, j \in I$ such that

$$P_{1+k}(K_i)E_{1+k,u} \cap L(G_1)\|P_k(K_j)\|P_k^{2+k}(L(G_2)) \nsubseteq P_{1+k}\left(\bigcup_{i\in I} K_i\right).$$

Then, there exist $x \in P_{1+k}(K_i)$ and $u \in E_{1+k,u}$ such that $xu \in L(G_1)\|P_k(K_j)\|P_k^{2+k}(L(G_2))$, and $xu \notin P_{1+k}\left(\cup_{i\in I} K_i\right)$. It follows that

- $P_k(x) \in P_k P_{1+k}(K_i) = P_k(K_i)$,
- $P_k(xu) \in P_k(K_j)$, and
- $P_k(xu) \notin P_k(K_i)$; otherwise, if $P_k(xu) \in P_k(K_i)$, then $xu \in L(G_1) \| P_k(K_i) \| P_k(L(G_2))$, and controllability of $P_{1+k}(K_i)$ with respect to $L(G_1) \| P_k(K_i) \| P_k(L(G_2))$ implies that $xu \in P_{1+k}(K_i) \subseteq P_{1+k}(\cup_{i \in I} K_i)$, which is not true.

Assume that $u \notin E_{k,u}$. Then, $P_k(xu) = P_k(x) \in P_k(K_i)$, which does not hold. Thus, $u \in E_{k,u}$. As $P_k(K_i) \cup P_k(K_j) \subseteq L(G_k)$, we get that

$$P_k(xu) = P_k(x)u \in L(G_k).$$

However, controllability of $P_k(K_i)$ with respect to $L(G_k)$ and $E_{k,u}$ implies that $P_k(x)u = P_k(xu)$ is in $P_k(K_i)$. This is a contradiction.

iii) As the last item of the definition is proven in the same way, the theorem holds. □

Theorem 12. *The supremal conditionally normal sublanguage of a given language K always exists and is equal to the union of all conditionally normal sublanguages of K.*

Proof. We show that conditional normality is preserved by union. Let I be an index set, and let K_i, $i \in I$, be conditionally normal sublanguages of $K \subseteq L(G_1 \| G_2 \| G_k)$ with respect to generators (G_1, G_2, G_k) and the natural projections (Q_{1+k}, Q_{2+k}, Q_k) to local observable event sets, see Fig. 1. We prove that $\bigcup_{i \in I} K_i$ is conditionally normal with respect to those generators and natural projections, i.e., the three items of the definition hold.

i) First, note that $P_k(\bigcup_{i \in I} K_i)$ is normal with respect to $L(G_k)$ and Q_k because

$$Q_k^{-1} Q_k P_k\left(\bigcup_{i \in I} K_i\right) \cap L(G_k) = \bigcup_{i \in I}\Big(Q_k^{-1} Q_k P_k(K_i) \cap L(G_k)\Big) = \bigcup_{i \in I} P_k(K_i) = P_k\left(\bigcup_{i \in I} K_i\right),$$

where the second equality is by normality of $P_k(K_i)$ with respect to $L(G_k)$ and Q_k, $i \in I$.

ii) To prove the other statement, by (1) we need to show that

$$Q_{1+k}^{-1} Q_{1+k} P_{1+k}\left(\bigcup_{i \in I} K_i\right) \cap L(G_1) \| P_k\left(\bigcup_{i \in I} K_i\right) \| P_k^{2+k}(L(G_2)) = P_{1+k}\left(\bigcup_{i \in I} K_i\right).$$

However, it is true that $P_{1+k}(\cup_{i \in I} K_i) \subseteq Q_{1+k}^{-1} Q_{1+k} P_{1+k}(\cup_{i \in I} K_i) \cap L(G_1) \| P_k(\cup_{i \in I} K_i) \| P_k^{2+k}(L(G_2))$, and that

$$\begin{aligned} &Q_{1+k}^{-1} Q_{1+k} P_{1+k}\left(\bigcup_{i \in I} K_i\right) \cap L(G_1) \| P_k\left(\bigcup_{i \in I} K_i\right) \| P_k^{2+k}(L(G_2)) \\ &\qquad = \bigcup_{i \in I}\Big(Q_{1+k}^{-1} Q_{1+k} P_{1+k}(K_i)\Big) \cap \bigcup_{i \in I}\Big(L(G_1) \| P_k(K_i) \| P_k^{2+k}(L(G_2))\Big) \\ &\qquad\qquad = \bigcup_{i \in I} \bigcup_{j \in I}\Big(Q_{1+k}^{-1} Q_{1+k} P_{1+k}(K_i) \cap L(G_1) \| P_k(K_j) \| P_k^{2+k}(L(G_2))\Big). \end{aligned}$$

For the sake of contradiction, assume that there are two different indexes $i, j \in I$ such that

$$Q_{1+k}^{-1} Q_{1+k} P_{1+k}(K_i) \cap L(G_1) \| P_k(K_j) \| P_k^{2+k}(L(G_2)) \nsubseteq P_{1+k}\left(\bigcup_{i \in I} K_i\right).$$

Then, there is $x \in Q_{1+k}^{-1} Q_{1+k} P_{1+k}(K_i)$ such that $x \in L(G_1) \| P_k(K_j) \| P_k^{2+k}(L(G_2))$, and $x \notin P_{1+k}(\cup_{i \in I} K_i)$. It follows that

$$P_k^{1+k}(x) \in P_k(K_j) \text{ and } P_k^{1+k}(x) \notin P_k(K_i); \tag{2}$$

otherwise, $P_k^{1+k}(x) \in P_k(K_i)$ implies that $x \in L(G_1) \| P_k(K_i) \| P_k(L(G_2))$, and normality of $P_{1+k}(K_i)$ then implies that $x \in P_{1+k}(K_i) \subseteq P_{1+k}(\cup_{i \in I} K_i)$, which does not hold. As $P_k(K_i) \cup P_k(K_j) \subseteq L(G_k)$, we get that

$$P_k^{1+k}(x) \in L(G_k).$$

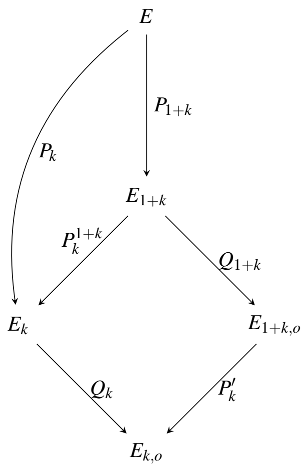


Figure 1: A commutative diagram of the natural projections.

However, as $x \in Q_{1+k}^{-1} Q_{1+k} P_{1+k}(K_i)$, there exists $w \in K_i$ such that $Q_{1+k}(x) = Q_{1+k}P_{1+k}(w)$. Thus, applying the natural projection $P'_k : E_{1+k,o}^* \to E_{k,o}^*$, we get that $P'_k Q_{1+k}(x) = P'_k Q_{1+k} P_{1+k}(w)$. As it holds that $Q_k P_k^{1+k} = P'_k Q_{1+k}$ and $Q_k P_k = P'_k Q_{1+k} P_{1+k}$ (see Fig. 1), we have that

$$Q_k P_k^{1+k}(x) = P'_k Q_{1+k}(x) = P'_k Q_{1+k} P_{1+k}(w) = Q_k P_k(w)\,,$$

i.e., $P_k^{1+k}(x) \in Q_k^{-1} Q_k P_k(K_i)$. By normality of $P_k(K_i)$ with respect to $L(G_k)$ and Q_k, we obtain that $P_k^{1+k}(x) \in P_k(K_i)$, which is a contradiction with (2).

iii) As the last item of the definition is proven in the same way, the theorem holds. □

Given generators G_1, G_2, and G_k. For brevity we denote $L_i = L(G_i)$ in what follows, for $i = 1, 2, k$. In addition, let

$$\sup \mathrm{cCN}(K, L, (E_{1+k,u}, E_{2+k,u}, E_{k,u}), (Q_{1+k}, Q_{2+k}, Q_k))$$

denote the supremal conditionally controllable and conditionally normal sublanguage of the specification language K with respect to the plant language $L = L(G_1 \| G_2 \| G_k)$, the sets of uncontrollable events $(E_{1+k,u}, E_{2+k,u}, E_{k,u})$, and the natural projections (Q_{1+k}, Q_{2+k}, Q_k), where $Q_i : E_i^* \to E_{i,o}^*$, for $i = 1+k, 2+k, k$. The following auxiliary lemmas will be useful.

Lemma 13. *Let $L_1 \subseteq E_1^*$, $L_2 \subseteq E_2^*$, and $E_1 \cap E_2 \subseteq E_k$. Then, $P_k(L_1 \| L_2) = P_k^{1+k}(P_1^{1+k})^{-1}(L_1) \cap P_k^{2+k}(P_2^{2+k})^{-1}(L_2)$.*

Proof. This follows from Lemma 4, the definition of the synchronous product, and Proposition 4.2(6) in [10] showing the commutativity $(P_{i\cap k}^k)^{-1} P_{i\cap k}^i = P_k^{i+k}(P_i^{i+k})^{-1}$, for $i = 1, 2$. Specifically, in turn we have

$$P_k(L_1 \| L_2) = P_k(L_1) \| P_k(L_2) = (P_{1\cap k}^k)^{-1} P_{1\cap k}^1(L_1) \cap (P_{2\cap k}^k)^{-1} P_{2\cap k}^2(L_2) = P_k^{1+k}(P_1^{1+k})^{-1}(L_1) \cap P_k^{2+k}(P_2^{2+k})^{-1}(L_2)\,,$$

which proves the lemma. □

Lemma 14 ([8]). *Let $E = E_1 \cup E_2$ be event sets, and let $L_1 \subseteq E_1^*$ and $L_2 \subseteq E_2^*$ be two languages. Let $P_i : E^* \to E_i^*$ be natural projections, for $i = 1, 2$. Let $A \subseteq E^*$ be a language such that $P_1(A) \subseteq L_1$ and $P_2(A) \subseteq L_2$. Then, $A \subseteq L_1 \| L_2$.*

Lemma 15 ([8]). *Let $K \subseteq L \subseteq M$ be languages over an event set E such that K is controllable with respect to L and E_u, and L is controllable with respect to M and E_u. Then, K is controllable with respect to M and E_u.*

Lemma 16. *Let $K \subseteq L \subseteq M$ be prefix-closed languages such that K is normal with respect to L and Q, and L is normal with respect to M and Q. Then, K is normal with respect to M and Q.*

Proof. We know that $Q^{-1}Q(K) \cap L = K$ and $Q^{-1}Q(L) \cap M = L$. Then, $Q^{-1}Q(K) \cap M \subseteq Q^{-1}Q(L) \cap M = L$. This implies that $Q^{-1}Q(K) \cap M = (Q^{-1}Q(K) \cap M) \cap L = (Q^{-1}Q(K) \cap L) \cap M = K \cap M = K$. □

Lemma 17. *Let $Q : E^* \to E_o^*$, $P_k : E^* \to E_k^*$, and $Q_k : E_k^* \to E_{k,o}^*$ be natural projections. Then, for every language $M \subseteq E_k^*$, the inclusion $Q^{-1}QP_k^{-1}(M) \subseteq P_k^{-1}Q_k^{-1}Q_k(M)$ holds.*

Proof. Let $M \subseteq E_k^*$ be a language, then $QP_k^{-1}(X) \subseteq QP_k^{-1}Q_k^{-1}Q_k(X) = Q(Q_kP_k)^{-1}Q_k(X) = Q(P'_kQ)^{-1}Q_k(X) = QQ^{-1}(P'_k)^{-1}Q_k(X) = (P'_k)^{-1}Q_k(X)$, where $P'_k : E_o^* \to E_{k,o}^*$ is a restriction of the projection P_k to E_o^*. This implies that $Q^{-1}QP_k^{-1}(X) \subseteq Q^{-1}(P'_k)^{-1}Q_k(X) = (P'_kQ)^{-1}Q_k(X) = (Q_kP_k)^{-1}Q_k(X) = P_k^{-1}Q_k^{-1}Q_k(X)$, which was to be shown. □

The following conditions are required in the main result of this section. The reader is referred to [10, 12] for more details.

Definition 18. The natural projection $P_k : E^* \to E_k^*$, where $E_k \subseteq E$, is an *L-observer* for $L \subseteq E^*$ if, for all $t \in P(L)$ and $s \in \overline{L}$, $P(s)$ is a prefix of t implies that there exists $u \in E^*$ such that $su \in L$ and $P(su) = t$.

Definition 19. The natural projection $P_k : E^* \to E_k^*$, where $E_k \subseteq E$, is *output control consistent* (OCC) for $L \subseteq E^*$ if for every $s \in \overline{L}$ of the form

$$s = \sigma_1 \dots \sigma_\ell \quad \text{or} \quad s = s'\sigma_0\sigma_1 \dots \sigma_\ell, \quad \ell \geq 1\,,$$

where $s' \in E^*$, $\sigma_0, \sigma_\ell \in E_k$, and $\sigma_i \in E \setminus E_k$, for $i = 1, 2, \dots, \ell - 1$, if $\sigma_\ell \in E_u$, then $\sigma_i \in E_u$, for all $i = 1, 2, \dots, \ell - 1$.

Note that OCC can be replaced by a similar condition called local control consistency (LCC) discussed in [13, 14].

Theorem 20. *Consider the setting of Problem 1. Define the local languages*

$$\begin{aligned}
\sup \mathrm{CN}_k &= \sup \mathrm{CN}(P_k(K), L(G_k), E_{k,u}, Q_k)\,,\\
\sup \mathrm{CN}_{1+k} &= \sup \mathrm{CN}(P_{1+k}(K), L(G_1) \| \sup \mathrm{CN}_k, E_{1+k,u}, Q_{1+k})\,,\\
\sup \mathrm{CN}_{2+k} &= \sup \mathrm{CN}(P_{2+k}(K), L(G_2) \| \sup \mathrm{CN}_k, E_{2+k,u}, Q_{2+k})\,.
\end{aligned}$$

Let the projection P_k^{i+k} be an $(P_i^{i+k})^{-1}L(G_i)$-observer and OCC for $(P_i^{i+k})^{-1}L(G_i)$, for $i = 1, 2$. Assume that the language $P_k^{1+k}(\sup \mathrm{CN}_{1+k}) \cap P_k^{2+k}(\sup \mathrm{CN}_{2+k})$ is normal with respect to $L(G_k)$ and Q_k. Then,

$$\sup \mathrm{CN}_k \| \sup \mathrm{CN}_{1+k} \| \sup \mathrm{CN}_{2+k} = \sup \mathrm{cCN}(K, L, (E_{1+k,u}, E_{2+k,u}, E_{k,u}), (Q_{1+k}, Q_{2+k}, Q_k))\,.$$

Proof. To prove the theorem, we first denote the left-hand side and the right-hand side as

$$M = \sup \mathrm{CN}_k \| \sup \mathrm{CN}_{1+k} \| \sup \mathrm{CN}_{2+k} \quad \text{and} \quad \sup \mathrm{cCN} = \sup \mathrm{cCN}(K, L, (E_{1+k,u}, E_{2+k,u}, E_{k,u}), (Q_{1+k}, Q_{2+k}, Q_k))\,,$$

respectively, and we denote $L = L(G_1 \| G_2 \| G_k)$ and $L_i = L(G_i)$, $i = 1, 2, k$. To show that the inclusion $M \subseteq \sup \mathrm{cCN}$ holds, we need to show that

1. $M \subseteq K$,
2. M is conditionally controllable with respect to L and $(E_{1+k,u}, E_{2+k,u}, E_{k,u})$, and
3. M is conditionally normal with respect to L and (Q_{1+k}, Q_{2+k}, Q_k).

1. Note that $M = \sup \mathrm{CN}_k \| \sup \mathrm{CN}_{1+k} \| \sup \mathrm{CN}_{2+k} \subseteq P_k(K) \| P_{1+k}(K) \| P_{2+k}(K) = K$ since K is conditionally decomposable. Thus, $M \subseteq K$ holds true.

2. For a proof showing that M is conditionally controllable with respect to L and $(E_{1+k,u}, E_{2+k,u}, E_{k,u})$, the reader is referred to [8].

3. Thus, it remains to prove that M is conditionally normal with respect to L and (Q_{1+k}, Q_{2+k}, Q_k). To do this, we need to show the following three properties of Definition 8:

(I) $Q_k^{-1}Q_k(P_k(M)) \cap L(G_k) = P_k(M)$,

(II) $Q_{1+k}^{-1}Q_{1+k}(P_{1+k}(M)) \cap L(G_1) \| P_k(M) \| P_k^{2+k}(L(G_2) \| P_k(M)) = P_{1+k}(M)$, and

(III) $Q_{2+k}^{-1}Q_{2+k}(P_{2+k}(M)) \cap L(G_2) \| P_k(M) \| P_k^{1+k}(L(G_1) \| P_k(M)) = P_{2+k}(M)$.

As the last two properties are similar, we prove only (II).

(I) To prove $Q_k^{-1}Q_k(P_k(M)) \cap L(G_k) = P_k(M)$, note that

$$P_k(M) = \sup \mathrm{CN}_k \cap P_k^{1+k}(\sup \mathrm{CN}_{1+k}) \cap P_k^{2+k}(\sup \mathrm{CN}_{2+k}) = P_k^{1+k}(\sup \mathrm{CN}_{1+k}) \cap P_k^{2+k}(\sup \mathrm{CN}_{2+k}),$$

where the first equality follows from Lemma 4 by replacing the synchronous product with the intersection, and the other follows from the fact that $\sup \mathrm{CN}_{i+k} \subseteq L(G_i) \| \sup \mathrm{CN}_k \subseteq (P_k^{i+k})^{-1}(\sup \mathrm{CN}_k)$. Thus, by the assumption, $P_k(M)$ is normal with respect to $L(G_k)$ and Q_k, i.e., (I) holds true.

(II) Now, we show the other property, namely

$$Q_{1+k}^{-1}Q_{1+k}(P_{1+k}(M)) \cap L(G_1) \| P_k(M) \| P_k^{2+k}(L(G_2) \| P_k(M)) = P_{1+k}(M).$$

The inclusion $\supseteq$ is proven as in Lemma 9. Thus, it remains to show the other inclusion. First, note that by Lemma 6 and the definition of synchronous product we obtain that

$$P_{1+k}(M) = (P_k^{1+k})^{-1}(\sup \mathrm{CN}_k) \cap \sup \mathrm{CN}_{1+k} \cap (P_k^{1+k})^{-1} P_k^{2+k}(\sup \mathrm{CN}_{2+k}).$$

Assume that $x \in Q_{1+k}^{-1}Q_{1+k}(P_{1+k}(M)) \cap L(G_1) \| P_k(M) \| P_k^{2+k}(L(G_2) \| P_k(M))$. Then, $P_{1+k}(M) \subseteq \sup \mathrm{CN}_{1+k}$ and $P_k(M) \subseteq \sup \mathrm{CN}_k$ imply that

$$\begin{aligned}
&Q_{1+k}^{-1}Q_{1+k}(P_{1+k}(M)) \cap L(G_1) \| P_k(M) \| P_k^{2+k}(L(G_2) \| P_k(M)) \\
&\qquad \subseteq Q_{1+k}^{-1}Q_{1+k}(\sup \mathrm{CN}_{1+k}) \cap L(G_1) \| P_k(M) \| P_k^{2+k}(L(G_2) \| P_k(M)) \\
&\qquad \subseteq Q_{1+k}^{-1}Q_{1+k}(\sup \mathrm{CN}_{1+k}) \cap L(G_1) \| \sup \mathrm{CN}_k = \sup \mathrm{CN}_{1+k}.
\end{aligned}$$

Thus, we have shown that $x \in \sup \mathrm{CN}_{1+k}$. In addition, it is satisfied that $P_k^{1+k}(x) \in P_k^{2+k}(L(G_2) \| P_k(M)) \subseteq P_k(M) \subseteq P_k^{2+k}(\sup \mathrm{CN}_{2+k})$, which implies that $x \in (P_k^{1+k})^{-1} P_k^{2+k}(\sup \mathrm{CN}_{2+k})$. Furthermore, $P_k^{1+k}(x) \in P_k(M) \subseteq \sup \mathrm{CN}_k$ implies that $x \in (P_k^{1+k})^{-1}(\sup \mathrm{CN}_k)$. Hence, $x \in P_{1+k}(M)$. As (III) is proven analogously, we have shown that $M \subseteq \sup \mathrm{cCN}$.

To prove the opposite inclusion, $\sup \mathrm{cCN} \subseteq M$, by Lemma 14 it is sufficient to show that

- $P_k(\sup \mathrm{cCN}) \subseteq \sup \mathrm{CN}_k$ and
- $P_{i+k}(\sup \mathrm{cCN}) \subseteq \sup \mathrm{CN}_{i+k}$, for $i = 1, 2$.

To prove this, note that $P_k(\sup \mathrm{cCN}) \subseteq P_k(K)$ is controllable with respect to $L(G_k)$ and $E_{k,u}$ and normal with respect to $L(G_k)$ and Q_k, which implies that $P_k(\sup \mathrm{cCN}) \subseteq \sup \mathrm{CN}_k$ is satisfied. Furthermore, $P_{1+k}(\sup \mathrm{cCN}) \subseteq P_{1+k}(K)$ is controllable with respect to $L_1 \| P_k(\sup \mathrm{cCN}) \| P_k^{2+k}(L_2 \| P_k(\sup \mathrm{cCN}))$ and $E_{1+k,u}$ and normal with respect to the same language and Q_{1+k}. By Lemma 13, $P_k(\sup \mathrm{cCN}) \subseteq \sup \mathrm{CN}_k \subseteq P_k(L) \subseteq P_k^{2+k}(P_2^{2+k})^{-1}(L_2)$. The following holds:

$$\begin{aligned}
L_1 \| P_k(\sup \mathrm{cCN}) \| P_k^{2+k}(L_2 \| P_k(\sup \mathrm{cCN})) &= L_1 \| P_k(\sup \mathrm{cCN}) \| [P_k(\sup \mathrm{cCN}) \cap P_k^{2+k}(P_2^{2+k})^{-1}(L_2)] \\
&= L_1 \| P_k(\sup \mathrm{cCN}) \| P_k(\sup \mathrm{cCN}) \\
&= L_1 \| P_k(\sup \mathrm{cCN}).
\end{aligned}$$

Since $P_k(\sup \mathrm{cCN})$ is controllable with respect to $L(G_k)$ and $E_{k,u}$, and normal with respect to $L(G_k)$ and Q_k, it is also controllable with respect to $\sup \mathrm{CN}_k \subseteq L(G_k)$ and $E_{k,u}$, and normal with respect to $\sup \mathrm{CN}_k$ and Q_k because $P_k(\sup \mathrm{cCN}) \subseteq \sup \mathrm{CN}_k$. As $P_{1+k}(\sup \mathrm{cCN})$ is controllable with respect to $L_1 \| P_k(\sup \mathrm{cCN})$ and $E_{1+k,u}$, and $L_1 \| P_k(\sup \mathrm{cCN})$ is controllable with respect to $L_1 \| \sup \mathrm{CN}_k$ and $E_{1+k,u}$ by [10, Proposition 4.6] (since all the languages under consideration are prefix-closed), Lemma 15 implies that $P_{1+k}(\sup \mathrm{cCN})$ is controllable with respect to $L_1\|\sup \mathrm{CN}_k$ and $E_{1+k,u}$. Furthermore, as $P_{1+k}(\sup \mathrm{cCN})$ is normal with respect to $L_1\|P_k(\sup \mathrm{cCN})$ and Q_{1+k}, and $L_1\|P_k(\sup \mathrm{cCN})$ is normal with respect to $L_1\|\sup \mathrm{CN}_k$ and Q_{1+k} because (using Lemma 17)

$$
\begin{aligned}
&Q_{1+k}^{-1}Q_{1+k}(L_1\|P_k(\sup \mathrm{cCN})) \cap L_1\|\sup \mathrm{CN}_k \\
&\quad = Q_{1+k}^{-1}Q_{1+k}((P_1^{1+k})^{-1}(L_1) \cap (P_k^{1+k})^{-1}P_k(\sup \mathrm{cCN})) \cap (P_1^{1+k})^{-1}(L_1) \cap (P_k^{1+k})^{-1}(\sup \mathrm{CN}_k) \\
&\quad \subseteq Q_{1+k}^{-1}Q_{1+k}(P_1^{1+k})^{-1}(L_1) \cap Q_{1+k}^{-1}Q_{1+k}(P_k^{1+k})^{-1}P_k(\sup \mathrm{cCN}) \cap (P_1^{1+k})^{-1}(L_1) \cap (P_k^{1+k})^{-1}(\sup \mathrm{CN}_k) \\
&\quad = (P_1^{1+k})^{-1}(L_1) \cap Q_{1+k}^{-1}Q_{1+k}(P_k^{1+k})^{-1}P_k(\sup \mathrm{cCN}) \cap (P_k^{1+k})^{-1}(\sup \mathrm{CN}_k) \\
&\quad \subseteq (P_1^{1+k})^{-1}(L_1) \cap (P_k^{1+k})^{-1}Q_k^{-1}Q_kP_k(\sup \mathrm{cCN}) \cap (P_k^{1+k})^{-1}(\sup \mathrm{CN}_k) \\
&\quad = (P_1^{1+k})^{-1}(L_1) \cap (P_k^{1+k})^{-1}\big(Q_k^{-1}Q_kP_k(\sup \mathrm{cCN}) \cap \sup \mathrm{CN}_k\big) \\
&\quad = (P_1^{1+k})^{-1}(L_1) \cap (P_k^{1+k})^{-1}P_k(\sup \mathrm{cCN}) \\
&\quad = L_1\|P_k(\sup \mathrm{cCN})\,.
\end{aligned}
$$

Then, Lemma 16 implies that $P_{1+k}(\sup \mathrm{cCN})$ is normal with respect to $L_1\|\sup \mathrm{CN}_k$ and Q_{1+k}. Thus, we have shown that $P_{1+k}(\sup \mathrm{cCN}) \subseteq \sup \mathrm{CN}_{1+k}$. The case of the property (ii.b) is proven analogously. Hence, $\sup \mathrm{cCN} \subseteq M$ and the proof is complete. □

Remark 21. The assumption that the language $P_k^{1+k}(\sup \mathrm{CN}_{1+k}) \cap P_k^{2+k}(\sup \mathrm{CN}_{2+k})$ is normal is rather technical. If $E_k = E_{k,o} = E_k \cap E_o$, then the projection Q_k is identity and the condition is trivially satisfied. However, the example below (see Section 6.1) demonstrates that the condition can also be satisfied although E_k and $E_{k,o}$ do not coincide. On the other hand, if $E_o \subseteq E_k$, $P_k^{1+k}(\sup \mathrm{CN}_{1+k}) \cap P_k^{2+k}(\sup \mathrm{CN}_{2+k})$ is normal with respect to $P_k(L_1\|L_2)\|L_k$. Thus, for the coordinator defined as $L_k = P_k(L_1\|L_2)$ discussed in the example below, the technical assumption is satisfied. The proof is as follows: $E_o \subseteq E_k$ implies $Q_kP_k = Q$. Then, normality of K with respect to L and Q implies normality of $P_k(K)$ with respect to $P_k(L)$ and Q_k. To see this, let $t \in P_k(L)$, $t' \in P_k(K)$, and $Q_k(t) = Q_k(t')$. There exist $s \in L$ and $s' \in K$ such that $P_k(s) = t$ and $P_k(s') = t'$. Thus, $Q(s) = Q_kP_k(s) = Q_k(t) = Q_k(t') = Q_kP_k(s') = Q(s')$. Now, normality of K with respect to L and Q implies $s \in K$, i.e., $t = P_k(s) \in P_k(K)$. Note that $\sup \mathrm{CN}_{1+k}$ is normal with respect to $L_1\|\sup \mathrm{CN}_k$ and Q_{1+k}, $\sup \mathrm{CN}_k$ is normal with respect to L_k and Q_k, and L_1 is normal with respect to L_1 and Q_1. This implies that $L_1\|\sup \mathrm{CN}_k$ is normal with respect to $L_1\|L_k$ and Q_{1+k}. Lemma 16 then implies that $\sup \mathrm{CN}_{1+k}$ is normal with respect to $L_1\|L_k$ and Q_{1+k}. The same arguments show that $\sup \mathrm{CN}_{2+k}$ is normal with respect to $L_2\|L_k$ and Q_{2+k}. Now, two applications of the preservation of normality under projection yield normality of $P_k^{i+k}(\sup \mathrm{CN}_{i+k})$ with respect to $P_k(L_i\|L_k) = L_k\|P_k(L_i)$ and Q_k, for $i = 1, 2$. Then, $P_k^{1+k}(\sup \mathrm{CN}_{1+k}) \cap P_k^{2+k}(\sup \mathrm{CN}_{2+k})$ is normal with respect to $L_k\|P_k(L_1)\|P_k(L_2) = L_k$ and Q_k as required in the technical condition.

Let us mention that even if $P_k(\sup \mathrm{CN}_{1+k}) \cap P_k(\sup \mathrm{CN}_{2+k})$ fails to be normal with respect to L_k and Q_k, and the equality in Theorem 20 does not need to hold, it does not mean that our approach is useless. In fact, our procedure to compute the supremal normal sublanguage as the composition of corresponding supremal normal sublanguages over the alphabets E_k, $E_1 \cup E_k$, and $E_2 \cup E_k$ is natural and we always compute a normal sublanguage of $L_1\|L_2\|L_k$ using the computation scheme. Indeed, it follows by the same argument as the proof of normality of $P_k(\sup \mathrm{CN}_{1+k}) \cap P_k(\sup \mathrm{CN}_{2+k})$ with respect to L_k and Q_k in the special case $E_o \subseteq E_k$ discussed above. Namely, $\sup \mathrm{CN}_k$ is by definition normal with respect to L_k and Q_k, and for $i = 1, 2$ we have that $\sup \mathrm{CN}_{i+k}$ is normal with respect to $L_i\|L_k$ and Q_{i+k}. Since we deal with prefix-closed (hence nonconflicting) languages, this gives that $M = \sup \mathrm{CN}_k\|\sup \mathrm{CN}_{1+k}\|\sup \mathrm{CN}_{2+k}$ is normal with respect to $L_k\|(L_1\|L_k)\|(L_2\|L_k) = L_1\|L_2\|L_k$. This is because normality (as well as controllability) of prefix-closed languages is preserved by the parallel composition under a very mild assumption that all shared events have the same observability status in all subsystems that share them. Formally, $E_{o,2} \cap E_1 = E_2 \cap E_{o,1}$, which is automatically satisfied because we defined locally observable events as $E_{o,i} = E_o \cap E_i$.

As the assumption that the projection P_k^{i+k} is an $(P_i^{i+k})^{-1}L(G_i)$-observer and OCC for $(P_i^{i+k})^{-1}L(G_i)$, for $i = 1, 2$, is required only for controllability, we have the following corollary. Let $\sup \mathrm{N}(K, L, Q)$ denote the supremal normal sublanguage of K with respect to L and Q.

Corollary 22. *Consider the setting of Problem 1. Define the local languages*

$$\begin{aligned}\sup \mathrm{N}_k &= \sup \mathrm{N}(P_k(K), L(G_k), Q_k)\,,\\ \sup \mathrm{N}_{1+k} &= \sup \mathrm{N}(P_{1+k}(K), L(G_1)\|\sup \mathrm{N}_k, Q_{1+k})\,,\\ \sup \mathrm{N}_{2+k} &= \sup \mathrm{N}(P_{2+k}(K), L(G_2)\|\sup \mathrm{N}_k, Q_{2+k})\,.\end{aligned}$$

Assume that the language $P_k^{1+k}(\sup \mathrm{N}_{1+k}) \cap P_k^{2+k}(\sup \mathrm{N}_{2+k})$ is normal with respect to $L(G_k)$ and Q_k. Then,

$$\sup \mathrm{N}_k\|\sup \mathrm{N}_{1+k}\|\sup \mathrm{N}_{2+k} = \sup \mathrm{cN}(K, L, (Q_{1+k}, Q_{2+k}, Q_k))\,.$$

The minimal cardinality of the coordinator event set depends on the specification language. If we assume that the projections are observers, the state size of the computed models is guaranteed to be no larger than that of the original models. In a typical situation, the projected models are significantly smaller than the original models. The reader is referred to [15, 16] for more details.

In [17], the computational complexity of the supremal controllable sublanguage of a specification language K with respect to the plant language L with n and m states in their minimal generator representations, respectively, is shown (for prefix-closed languages) to be $O(mn)$. The computational complexity of the supremal controllable and normal sublanguage is $O(2^{mn})$, see [5]. We denote the number of states of minimal generators for $L(G_1)$, $L(G_2)$, and $L(G_k)$ by m_1, m_2, and m_k, respectively. As the specification language K is conditionally decomposable, i.e., $K = P_{1+k}(K)\|P_{2+k}(K)\|P_k(K)$, we denote the number of states of minimal generators for $P_{1+k}(K)$, $P_{2+k}(K)$, and $P_k(K)$ by n_1, n_2, and n_k, respectively. Then, in the worst case, $m = O(m_1m_2m_k)$ and $n = O(n_1n_2n_k)$. The computational complexity of $\sup \mathrm{C}_k$, $\sup \mathrm{C}_{1+k}$, and $\sup \mathrm{C}_{2+k}$ (see [8]) gives the formula $O(m_kn_k + m_1n_1m_kn_k + m_2n_2m_kn_k)$, which is better than $O(mn) = O(m_1m_2m_kn_1n_2n_k)$ of the monolithic case. The situation is more complicated for supremal controllable and normal sublanguages. In this case, the computational complexity of $\sup \mathrm{CN}_k$, $\sup \mathrm{CN}_{1+k}$, and $\sup \mathrm{CN}_{2+k}$ gives the formula $O(2^{m_kn_k} + 2^{m_1n_12^{m_kn_k}} + 2^{m_2n_22^{m_kn_k}})$, which is better than $O(2^{m_1m_2m_kn_1n_2n_k})$ of the monolithic case if $m_in_i > \frac{2^{m_kn_k}}{m_kn_k}$, for $i = 1, 2$, i.e., if the coordinator is significantly smaller than the subsystems. As the coordinator (and its event set) can be chosen to be minimal, there is a possibility to choose the coordinator so that it, in addition, satisfies the condition that the number of states of the minimal generator of $\sup \mathrm{CN}_k$ is in $O(m_kn_k)$ or even in $O(\min\{m_k, n_k\})$. However, this question requires further investigation.

In addition to the procedure for computation of sup cCN in a distributed way, another consequence is of interest. Namely, under the conditions of Theorem 20, sup cCN is conditionally decomposable, cf. Lemma 23.

Lemma 23 ([8, 9]). *A language $M \subseteq E^*$ is conditionally decomposable with respect to event sets $E_1 \cup E_2 \cup E_k = E$ if and only if there exist languages $M_i \subseteq E_i^*$, for $i = 1, 2, k$, such that $M = M_1\|M_2\|M_k$.*

Even more, this implies that the supremal conditionally controllable and conditionally normal sublanguage is controllable and normal with respect to the global plant as we show below and, consequently, the supremal conditionally controllable and conditionally normal sublanguage is included in the global supremal controllable and normal sublanguage. This is because the language synthesized using our coordination architecture is more restrictive than the language synthesized using the supervisory control for the global plant.

Theorem 24. *In the setting of Theorem 20,* $\sup \mathrm{cCN}(K, L, (E_{k,u}, E_{1+k,u}, E_{2+k,u}), (Q_k, Q_{1+k}, Q_{2+k}))$ *is controllable with respect to $L = L_1\|L_2\|L_k$ and E_u, and normal with respect to L and $Q : E^* \to E_o^*$, where $E = E_{1+k} \cup E_{2+k}$.*

Proof. Let $\sup \mathrm{cCN} = \sup \mathrm{cCN}(K, L, (E_{k,u}, E_{1+k,u}, E_{2+k,u}), (Q_k, Q_{1+k}, Q_{2+k}))$. Controllability of sup cCN is shown in [8], thus we only prove normality here and refer the reader to [8]. Note that according to Theorem 20, there exist $\sup \mathrm{CN}_k \subseteq E_k^*$, $\sup \mathrm{CN}_{1+k} \subseteq E_{1+k}^*$, and $\sup \mathrm{CN}_{2+k} \subseteq E_{2+k}^*$ so that $\sup \mathrm{cCN} = \sup \mathrm{CN}_k\|\sup \mathrm{CN}_{1+k}\|\sup \mathrm{CN}_{2+k}$. In addition, the following three properties hold: (1) $\sup \mathrm{CN}_k$ is normal with respect to L_k and Q_k, (2) $\sup \mathrm{CN}_{1+k}$ is normal with respect to $L_1\|\sup \mathrm{CN}_k$ and Q_{1+k}, and (3) $\sup \mathrm{CN}_{2+k}$ is normal with respect to $L_2\|\sup \mathrm{CN}_k$ and Q_{2+k}. Then, using Lemma 17,

$$
\begin{aligned}
&Q^{-1}Q(\sup \mathrm{cCN}) \cap L \| \sup \mathrm{CN}_k \\
&\qquad = Q^{-1}Q(\sup \mathrm{CN}_k \| \sup \mathrm{CN}_{1+k} \| \sup \mathrm{CN}_{2+k}) \cap L \| \sup \mathrm{CN}_k \\
&\qquad = Q^{-1}Q\left(P_k^{-1}(\sup \mathrm{CN}_k) \cap P_{1+k}^{-1}(\sup \mathrm{CN}_{1+k}) \cap P_{2+k}^{-1}(\sup \mathrm{CN}_{2+k})\right) \cap L \| \sup \mathrm{CN}_k \\
&\qquad \subseteq Q^{-1}QP_k^{-1}(\sup \mathrm{CN}_k) \cap Q^{-1}QP_{1+k}^{-1}(\sup \mathrm{CN}_{1+k}) \cap Q^{-1}QP_{2+k}^{-1}(\sup \mathrm{CN}_{2+k}) \cap L \| \sup \mathrm{CN}_k \\
&\qquad \subseteq P_k^{-1}Q_k^{-1}Q_k(\sup \mathrm{CN}_k) \cap P_{1+k}^{-1}Q_{1+k}^{-1}Q_{1+k}(\sup \mathrm{CN}_{1+k}) \cap P_{2+k}^{-1}Q_{2+k}^{-1}Q_{2+k}(\sup \mathrm{CN}_{2+k}) \cap L_1 \| L_2 \| L_k \| \sup \mathrm{CN}_k \\
&\qquad = P_k^{-1}\left(Q_k^{-1}Q_k(\sup \mathrm{CN}_k) \cap L_k\right) \cap P_{1+k}^{-1}\left(Q_{1+k}^{-1}Q_{1+k}(\sup \mathrm{CN}_{1+k}) \cap L_1 \| \sup \mathrm{CN}_k\right) \\
&\qquad\quad \cap P_{2+k}^{-1}\left(Q_{2+k}^{-1}Q_{2+k}(\sup \mathrm{CN}_{2+k}) \cap L_2 \| \sup \mathrm{CN}_k\right) \\
&\qquad = \sup \mathrm{cCN}\,.
\end{aligned}
$$

Thus, $\sup \mathrm{cCN}$ is normal with respect to $L \| \sup \mathrm{CN}_k$ and Q. Analogously, we can prove that $L \| \sup \mathrm{CN}_k$ is normal with respect to $L \| L_k = L$ and Q. Finally, by the transitivity of normality for prefix-closed languages (Lemma 16) we obtain that $\sup \mathrm{cCN}$ is normal with respect to L and Q, which was to be shown. □

Theorem 24 demonstrates that the result of our approach is controllable and normal with respect to L, E_u, and Q. To complete this study, we show that if some additional conditions are also satisfied, then the constructed supremal conditionally controllable and conditionally normal sublanguage is optimal.

Theorem 25. *Consider the setting of Theorem 20. If, in addition, $L_k \subseteq P_k(L)$ and P_{i+k} is OCC for the language $P_{i+k}^{-1}(L_i \| L_k)$, for $i = 1, 2$, then*

$$\sup \mathrm{cCN}(K, L, (E_{k,u}, E_{1+k,u}, E_{2+k,u}), (Q_k, Q_{1+k}, Q_{2+k})) = \sup \mathrm{CN}(K, L, E_u, Q)$$

if and only if

- $P_k(Q^{-1}Q(\sup \mathrm{CN}) \cap L) = P_kQ^{-1}Q(\sup \mathrm{CN}) \cap L_k$ *and*
- $P_{i+k}(Q^{-1}Q(\sup \mathrm{CN}) \cap L_1 \| L_2 \| P_k(\sup \mathrm{CN})) = P_{i+k}Q^{-1}Q(\sup \mathrm{CN}) \cap P_{i+k}(L_1 \| L_2 \| P_k(\sup \mathrm{CN}))$*, for $i = 1, 2$,*

where $\sup \mathrm{CN} = \sup \mathrm{CN}(K, L, E_u, Q)$.

Proof. First, note that the case considering controllability is proven in [8], i.e., $\sup \mathrm{cC}(K, L, (E_{k,u}, E_{1+k,u}, E_{2+k,u})) = \sup \mathrm{C}(K, L, E_u)$. Moreover, $\sup \mathrm{cCN}(K, L, (E_{k,u}, E_{1+k,u}, E_{2+k,u}), (Q_k, Q_{1+k}, Q_{2+k})) \subseteq \sup \mathrm{CN}(K, L, E_u, Q)$ is proven in Theorem 24. Thus, it remains to show that $\sup \mathrm{CN}$ is conditionally normal if and only if $P_k(Q^{-1}Q(\sup \mathrm{CN}) \cap L) = P_kQ^{-1}Q(\sup \mathrm{CN}) \cap L_k$ and $P_{i+k}(Q^{-1}Q(\sup \mathrm{CN}) \cap L_1 \| L_2 \| P_k(\sup \mathrm{CN})) = P_{i+k}Q^{-1}Q(\sup \mathrm{CN}) \cap P_{1+k}(L_1 \| L_2 \| P_k(\sup \mathrm{CN}))$, for $i = 1, 2$.

The assumption $L_k \subseteq P_k(L)$ implies that $L_k = P_k(L)$ because $P_k(L) \subseteq L_k$ always holds. Moreover, for any $A \subseteq E^*$, $Q_k^{-1}Q_kP_k(A) = P_kQ^{-1}Q(A)$. This can be proven as follows. Let $x \in Q_k^{-1}Q_kP_k(A)$, then there exists $z \in A$ such that $Q_k(x) = Q_kP_k(z)$, $P_k(z) = y$, and $Q(z) = w$, for some $y \in E_k^*$ and $w \in E_o^*$. Assume $Q_k(x) = Q_k(y) = v = \sigma_0\sigma_1 \ldots \sigma_n$, for $\sigma_i \in E_{k,o}$, or $v = \varepsilon$. Then, $x = u_0\sigma_0u_1\sigma_1 \ldots u_n\sigma_nu_{n+1}$, for some $u_i \in (E_k \setminus E_o)^*$. As $Q_kP_k(z) = P'_kQ(z) = v$, where P'_k denotes the restriction of P_k to E_o^*, $w = w_0\sigma_0w_1\sigma_1 \ldots w_n\sigma_nw_{n+1}$, for $w_i \in (E_o \setminus E_k)^*$. Set $z' = u_0w_0\sigma_0u_1w_1\sigma_1 \ldots u_nw_n\sigma_nu_{n+1}w_{n+1}$. Then, $P_k(z') = x$ and $Q(z') = w$, which implies that $z' \in Q^{-1}Q(z)$. Thus, $x \in P_kQ^{-1}Q(A)$. On the other hand, $x \in P_kQ^{-1}Q(A)$ implies that there exists $y \in Q^{-1}Q(A)$ such that $P_k(y) = x$, and that there is $z \in A$ such that $Q(y) = Q(z)$. Thus, considering the image of z under $Q_k \circ P_k$, we obtain that $Q_kP_k(z) = P'_kQ(z) = P'_kQ(y) = Q_kP_k(y) = Q_k(x)$, which implies that $x \in Q_k^{-1}Q_kP_k(A)$. Thus, $Q_k^{-1}Q_kP_k(A) = P_kQ^{-1}Q(A)$ is shown.

Based on these observations and the normality of $\sup \mathrm{CN}$, we obtain that the natural projection $P_k(\sup \mathrm{CN}) = P_k(Q^{-1}Q(\sup \mathrm{CN}) \cap L) \subseteq P_kQ^{-1}Q(\sup \mathrm{CN}) \cap P_k(L) = P_kQ^{-1}Q(\sup \mathrm{CN}) \cap L_k = Q_k^{-1}Q_kP_k(\sup \mathrm{CN}) \cap L_k$. It follows that $P_k(\sup \mathrm{CN})$ is normal if and only if $P_k(Q^{-1}Q(\sup \mathrm{CN}) \cap L) = P_kQ^{-1}Q(\sup \mathrm{CN}) \cap L_k$. Now, we show that the language $P_{1+k}(\sup \mathrm{CN})$ is normal with respect to $L_1 \| P_k(\sup \mathrm{CN}) \| P_k^{2+k}(L_2 \| P_k(\sup \mathrm{CN}))$ and Q_{1+k} if and only if the assumption is satisfied. The case of the language $P_{2+k}(\sup \mathrm{CN})$ is proven analogously. First, as $P_k(\sup \mathrm{CN}) \subseteq L_k$, we obtain that $Q^{-1}Q(\sup \mathrm{CN}) \cap L_1 \| L_2 \| P_k(\sup \mathrm{CN}) = Q^{-1}Q(\sup \mathrm{CN}) \cap P_1^{-1}(L_1) \cap P_2^{-1}(L_2) \cap P_k^{-1}(L_k) \cap P_k^{-1}P_k(\sup \mathrm{CN}) = Q^{-1}Q(\sup \mathrm{CN}) \cap L_1 \| L_2 \| L_k \cap P_k^{-1}P_k(\sup \mathrm{CN}) = \sup \mathrm{CN} \cap P_k^{-1}P_k(\sup \mathrm{CN}) = \sup \mathrm{CN}$, i.e., $\sup \mathrm{CN}$ is normal with respect to $L_1 \| L_2 \| P_k(\sup \mathrm{CN})$ and Q. Thus,

$$
\begin{aligned}
P_{1+k}(\sup \mathrm{CN}) &= P_{1+k}(Q^{-1}Q(\sup \mathrm{CN}) \cap L_1 \| L_2 \| P_k(\sup \mathrm{CN})) \\
&\subseteq P_{1+k}Q^{-1}Q(\sup \mathrm{CN}) \cap P_{1+k}(L_1 \| L_2 \| P_k(\sup \mathrm{CN})) \\
&= Q_{1+k}^{-1}Q_{1+k}P_{1+k}(\sup \mathrm{CN}) \cap P_{1+k}(L_1 \| L_2 \| P_k(\sup \mathrm{CN}))\,, \qquad \text{by an analogous argument as above,} \\
&= Q_{1+k}^{-1}Q_{1+k}P_{1+k}(\sup \mathrm{CN}) \cap L_1 \| P_k(\sup \mathrm{CN}) \| P_{2\cap k}^{2}(L_2)\,, \qquad \text{by Lemma 4,} \\
&= Q_{1+k}^{-1}Q_{1+k}P_{1+k}(\sup \mathrm{CN}) \cap L_1 \| P_k(\sup \mathrm{CN}) \| P_{2\cap k}^{2}(L_2) \| P_k(\sup \mathrm{CN}) \\
&= Q_{1+k}^{-1}Q_{1+k}P_{1+k}(\sup \mathrm{CN}) \cap L_1 \| P_k(\sup \mathrm{CN}) \| P_k^{2+k}(L_2 \| P_k(\sup \mathrm{CN}))\,,
\end{aligned}
$$

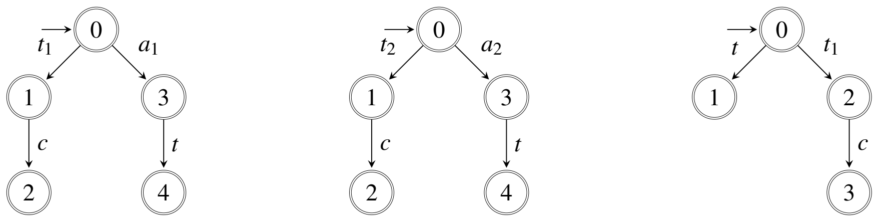
which implies that the language $P_{1+k}(\sup \mathrm{CN})$ is normal with respect to $L_1 \| P_k(\sup \mathrm{CN}) \| P_k^{2+k}(L_2 \| P_k(\sup \mathrm{CN}))$ and Q_{1+k} if and only if the projection P_{1+k} distributes over the languages $Q^{-1}Q(\sup \mathrm{CN})$ and $L_1 \| L_2 \| P_k(\sup \mathrm{CN})$. Hence, the proof is complete. $\square$

Note that to verify this condition, we need to compute the plant language L. However, this language is not computed during the computational process because of the complexity reasons. Thus, it is an open problem how to verify the optimality of the computed result based only on the local languages L_1, L_2, and L_k.

6.1. An example

In this section, we demonstrate our approach on an example. As controllability is demonstrated on an example presented in [8], we consider only the case of conditional normality here.

Let $G = G_1 \| G_2$ be a plant defined over an event set $E = E_1 \cup E_2 = \{a_1, c, t, t_1\} \cup \{a_2, c, t, t_2\} = \{a_1, a_2, c, t, t_1, t_2\}$ as a synchronous composition of two systems G_1 and G_2 defined as shown in Figure 2, where the set of unobservable events is $E_{uo} = \{t, t_1, t_2\}$. The behaviors of these systems are $L(G_1) = \overline{\{t_1c, a_1t\}}$, $L(G_2) = \overline{\{t_2c, a_2t\}}$, and $L(G) = \overline{\{a_1a_2t, a_2a_1t, a_1t_2, a_2t_1, t_1t_2c, t_2t_1c, t_1a_2, t_2a_1\}}$. The specification $K = \overline{\{t_1t_2c, t_1a_2, a_2t_1, a_2a_1t, a_1a_2t, a_1t_2, t_2a_1, t_2t_1\}}$ is define by the generator shown in Figure 3.



(a) Generator G_1. (b) Generator G_2. (c) Coordinator.

Figure 2: Generators G_1, G_2, and the coordinator.

Now, we need to find a coordinator G_k; specifically, its event set E_k. Note that, by the definition, E_k has to contain both shared events c and t. In addition, to ensure that the specification language K is conditionally decomposable, at least one of t_1 and t_2 has to be added to E_k. Assume t_1 is added, i.e., $E_k = \{c, t, t_1\}$. Thus, K is conditionally decomposable.

Moreover, as we consider only prefix-closed languages in this paper, and the choice of a coordinator plays a role in solving blocking issues, we choose the coordinator so that its behavior $L_k = L(G_k)$ does not change the original system when composed together, i.e., $L(G_1 \| G_2) \| L_k = L(G_1 \| G_2)$ is satisfied, see Figure 2(c). In fact, due to the absence of blocking issues, the important aspect in the choice of the coordinator is the choice of its alphabet, E_k, so that OCC and observer properties are satisfied together with the technical condition on normality.

Our choice is thus $L_k = L(P_k^1(G_1) \| P_k^2(G_2))$, which means that $L_k = \overline{\{t, t_1c\}}$. The projections of K are then the following languages $P_k(K) = \overline{\{t, t_1c\}}$, $P_{1+k}(K) = \overline{\{a_1t, t_1c\}}$, and $P_{2+k}(K) = \overline{\{t_2t_1, a_2t, a_2t_1, t_1a_2, t_1t_2c\}}$. We can compute

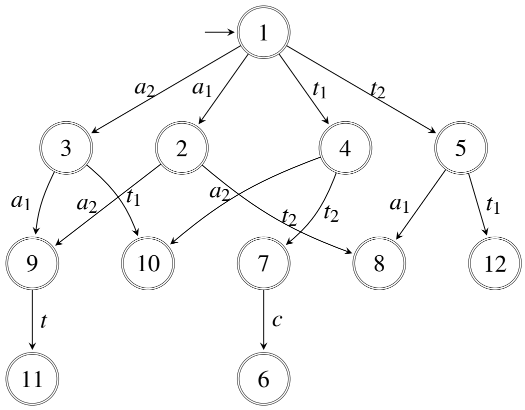


Figure 3: Generator for the specification language K.

the languages $\sup \mathrm{N}_k = \overline{\{t, t_1 c\}}$, $\sup \mathrm{N}_{1+k} = \overline{\{t_1 c, a_1 t\}}$, $\sup \mathrm{N}_{2+k} = \overline{\{t_2 t_1, t_1 t_2, t_1 a_2, a_2 t_1, a_2 t\}}$, as defined in Corollary 22, where their composition

$$\sup \mathrm{N}_k \| \sup \mathrm{N}_{1+k} \| \sup \mathrm{N}_{2+k} = \overline{\{t_2 t_1, t_2 a_1, a_1 t_2, a_1 a_2 t, t_1 t_2, t_1 a_2, a_2 a_1 t, a_2 t_1\}}$$

is the supremal conditionally normal sublanguage of K, which is also normal by Theorem 24. In addition, it can be verified that the resulting language coincides with the supremal normal sublanguage of K with respect to $L(G)$ and Q.

7. Conclusion

In this paper, we have investigated coordination control of modular discrete-event systems with partial observations. We have established that conditional observability together with conditional controllability form a necessary and sufficient condition for a global specification language to be exactly achievable within our coordination control that consists of local supervisors and a supervisor for the coordinator.

Similarly as observability in monolithic supervisory control with partial observations, conditional observability is not preserved by language union, hence we have studied conditional normality and supremal conditional normal sublanguages that always exist. We have shown that under quite weak assumptions supremal conditionally controllable sublanguages are conditionally decomposable and can be fairly easily computed. Sufficient conditions have been established, where a distributed computation is possible, which consists of computing the supremal conditionally controllable and conditionally normal sublanguage of a global specification language as the synchronous composition of the supremal controllable and normal sublanguages for the coordinator and those for the coordinator combined with local subsystems.

Moreover, conditions have been found under which our supremal conditionally controllable and conditionally normal sublanguage coincides with the globally optimal solution, i.e. the supremal controllable and normal sublanguage. Hence, as a consequence we have proposed an efficient computation of supremal controllable and normal sublanguage of a global specification, a very difficult problem.

For a future consideration, several extensions of our approach are left open. Let us note that the approach can be fairly easily extended to the general case of n subsystems running in parallel. In fact, it is possible to introduce one central coordinator which should dispose of all events shared by at least two subsystems. Another extension to non-prefix-closed global specification languages is currently investigated. It should be noted that there is an implicit and simple form of communication between coordinator and local controllers, namely , there is a two way communication channel between the coordinator and the local supervisors such that all coordinator events are communicated between local supervisors via the coordinator. In this respect, generalizations to coordination control with more general forms of communication between coordinator and local supervisors should be investigated. Finally, it would be nice to extend the coordination control to classes of timed automata.

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