4. Bisimilarity for Processes with Internal Activities

Lecture on Models of Concurrent Systems

(Summer 2022)

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May 3-10, 2022

Weak Transition Relation

Let us denote by Act an action alphabet, excluding the internal action τ.

By Act_τ we denote the set $Act \cup \{\tau\}$.

Definition 4.1: We call an LTS $(Pr, Act_\tau, \rightarrow)$ a **weak LTS** as it may use internal actions between processes. Likewise, an LTS $(Pr, Act, \rightarrow)$ is a **strong LTS**.

We may turn every weak LTS into a strong LTS
by abstracting from internal transitions. Thereby, so-called **weak transition relations are used**.

Definition 4.2: For LTS $(Pr, Act_\tau, \rightarrow)$ define its strong version by

$$(Pr, Act, \Rightarrow \cup \bigcup_{\alpha \in Act} \overset{\alpha}{\Longrightarrow}),$$

where $\Rightarrow := \xrightarrow{\tau}^{*}$ and $\overset{\alpha}{\Longrightarrow} := \Rightarrow \xrightarrow{\alpha} \Rightarrow$.

Weak Bisimilarity

Definition 4.3: A process relation $\mathcal{W}$ is a **weak bisimulation** if, for all $(P, Q) \in \mathcal{W}$ and $\alpha \in Act$,

1. for all P' with $P \stackrel{\alpha}{\Longrightarrow} P'$, there is a Q', such that $Q \stackrel{\alpha}{\Longrightarrow} Q'$ and $(P', Q') \in \mathcal{W}$;
2. for all P' with $P \Rightarrow P'$, there is a Q', such that $Q \Rightarrow Q'$ and $(P', Q') \in \mathcal{W}$;
3. for all Q' with $Q \stackrel{\alpha}{\Longrightarrow} Q'$, there is a P', such that $P \stackrel{\alpha}{\Longrightarrow} P'$ and $(P', Q') \in \mathcal{W}$;
4. for all Q' with $Q \Rightarrow Q'$, there is a P', such that $P \Rightarrow P'$ and $(P', Q') \in \mathcal{W}$.

Processes P and Q are **weakly bisimilar**, denoted $P \leftrightsquigarrow Q$, if, and only if, there is a weak bisimulation $\mathcal{W}$, such that $(P, Q) \in \mathcal{W}$.

Properties of Weak Bisimilarity

Theorem 4.4: Weak bisimilarity is an equivalence relation.

However, weak bisimilarity is not a congruence for CCS. More specifically, weak bisimilarity may fail in choice contexts: Consider weakly bisimilar processes $P = a.\mathbf{0}$ and $Q = \tau.a.\mathbf{0}$ (proof: $\mathcal{W} = \{(a.\mathbf{0}, \tau.a.\mathbf{0}), (a.\mathbf{0}, a.\mathbf{0}), (\mathbf{0}, \mathbf{0})\}$ is a weak bisimulation). However, $C[P] \not\leftrightsquigarrow C[Q]$ for $C = \bullet + b.\mathbf{0}$.

Weak bisimilarity does not recognize the first change of states of $C[Q]$, disabling b.

Observational Congruence (aka. Rooted Weak Bisimilarity)

Fix: Make the very first move observable! Here, we allow for $\overset{\tau}{\Longrightarrow}$-transitions, being $\Rightarrow\overset{\tau}{\rightarrow}\Rightarrow$.

Definition 4.5: Observational congruence is the largest relation, such that $P \leftrightsquigarrow^c Q$ if, and only if, for all $\alpha \in Act_\tau$

1. $P \overset{\alpha}{\rightarrow} P'$ implies there is a Q', such that $Q \overset{\alpha}{\Longrightarrow} Q'$ and $P' \leftrightsquigarrow Q'$, and
2. $Q \overset{\alpha}{\rightarrow} Q'$ implies there is a P', such that $P \overset{\alpha}{\Longrightarrow} P'$ and $P' \leftrightsquigarrow Q'$.

Now, $P = a.\mathbf{0}$ and $Q = \tau.a.\mathbf{0}$ are not observationally congruent anymore, as $Q \overset{\tau}{\rightarrow} a.\mathbf{0}$, but P has only a enabled.

Theorem 4.6: Observational congruence is a congruence for CCS.

τ-Laws

The following equalities hold for $= \in \{\leftrightsquigarrow, \leftrightsquigarrow^c\}$:

$$\begin{aligned} \alpha.\mathbf{0} &= \alpha.\tau.\mathbf{0} && (1)\\ P + \tau.P &= \tau.P && (2)\\ \alpha.(P + \tau.Q) &= \alpha.(P + \tau.Q) + \alpha.Q && (3) \end{aligned}$$

Note, for weak bisimilarity, law (2) can be adapted to $P + \tau.P = P$, while this is impossible for observational congruence.

One Last Thing: Expansion Lemma

Definition 4.7: Let I be a finite index set. A process of the form $\sum_{i \in I} \alpha_i.P_i$ is in **head standard form**.

The following result is known as the **Expansion Lemma**:

Theorem 4.8: For processes $P = \sum_{i \in I} \alpha_i.P_i$ and $Q = \sum_{j \in J} \beta_j.Q_j$,

$$P \parallel Q \leftrightarrows \sum_{i \in I} \alpha_i.(P_i \parallel Q') + \sum_{j \in J} \beta_j.(P \parallel Q_j) + \sum_{\alpha_i = \overline{\beta_j}} \tau.(P_i \parallel Q_j).$$

Consequence: Parallel composition can be implemented by the choice operator.