Exercise Sheet 5: Treewidth and Hypertreewidth

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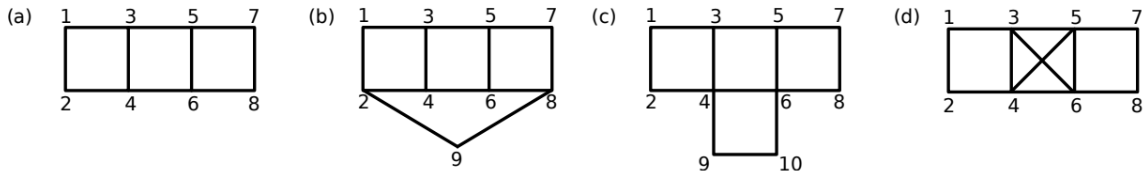
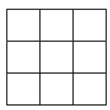
Exercise 5.1. Construct the query hypergraph and the primal graph for the following queries:

1. $\exists x, y, z, u, v.\ r(x, y, z, u) \wedge s(z, u, v)$
2. $\exists x, y, z, u, v.\ a(x, y) \wedge b(y, z) \wedge c(z, u) \wedge d(u, v) \wedge e(v, z) \wedge f(z, x) \wedge d(x, u) \wedge d(u, y)$

Exercise 5.2. Determine the treewidth of each of the following graphs and provide a suitable tree decomposition. Argue why there cannot be a tree decomposition of smaller width.



Exercise 5.3. Show that the $n \times n$ grid has a treewidth $\leq n$ by finding a suitable tree decomposition of width n. For example, the following 4×4 grid has treewidth 4:



Hint: Develop a winning strategy for $n + 1$ cops in the cops & robbers game.

Exercise 5.4. Show that a clique (fully connected graph) of size n has treewidth $n - 1$.

Exercise 5.5. Recall that a graph is 3-colourable if one can assign three colours to its vertices in such a way that neighbouring vertices never share the same colour.Let C_3 be the set of all 3-colourable graphs. Are the graphs in C_3 of bounded or unbounded treewidth? Explain your answer.

Exercise 5.6. Decide whether the following claims are true or false. Explain your answer.

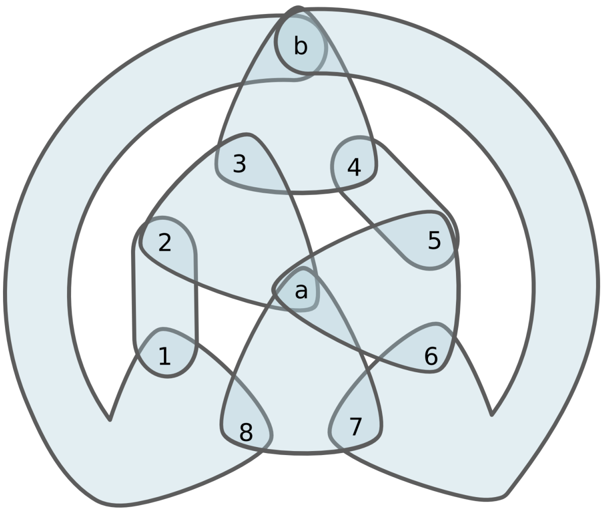
1. Deleting an edge from a graph may make the treewidth smaller but never larger.
2. Deleting a vertex from a graph (and removing all of its edges) may make the treewidth smaller but never larger.
3. Deleting a hyperedge from a hypergraph may make the hypertree width smaller but never larger.
4. Deleting a vertex from a hypergraph (and contracting all of its edges) may make the hypertree width smaller but never larger.

Exercise 5.7. The following BCQ corresponds to graph (a) in Exercise 5.2:

$$\exists x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8.\ r(x_1, x_2) \wedge r(x_1, x_3) \wedge r(x_2, x_4) \wedge r(x_3, x_4) \wedge r(x_3, x_5) \wedge \\ r(x_4, x_6) \wedge r(x_5, x_6) \wedge r(x_5, x_7) \wedge r(x_6, x_8) \wedge r(x_7, x_8)$$

According to the logical characterisation from the lecture, this query can be expressed in the $\exists$-$\wedge$-fragment of FO using only treewidth$+1$ variables. Find such a formula.

Exercise 5.8. Consider *Adler's Hypergraph*:



Play the marshals & robber game on this graph. It might be convenient to use small objects that can be moved around on the printed exercise sheet.

1. Can one marshal catch the robber?
2. Can two marshals catch the robber?
3. Can three marshals catch the robber?

Adler [Journal of Graph Theory, 2001] proposed this graph as an example where fewer marshals can win if they are allowed to play non-monotonically, that is, if they are not required to shrink the remaining space in each turn. Can you confirm her findings?

(∗) Can you explain why non-monotone play it is unavoidable in one of the above cases if the marshals want to win?

Hint: Controlling the nodes a and b is of special importance to win the game.