Lecture 7

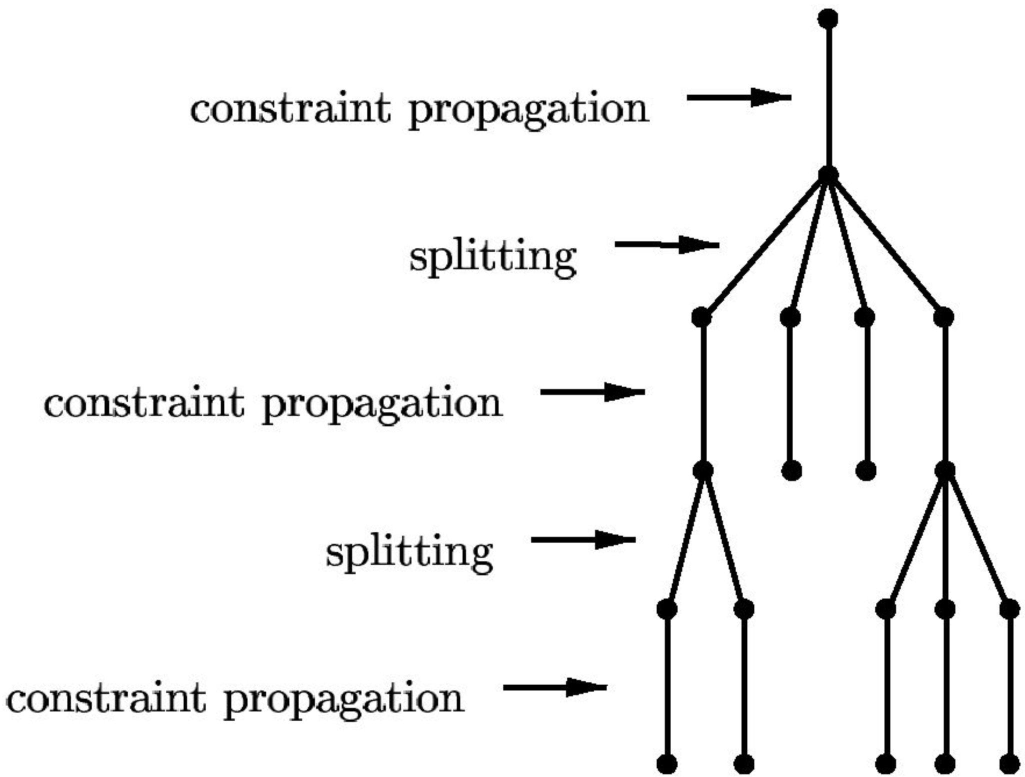
Search

Outline

- Introduce search trees
- Discuss various types of labeling trees, in particular trees for
  - forward checking
  - partial look ahead
  - maintaining arc consistency (MAC)
- Discuss various search algorithms for labeling trees
- Discuss search algorithms for constrained optimization problems
- Introduce various heuristics for search algorithms

Useful Slogan

Search Algorithm = Search Tree + Traversal Algorithm



Search Trees

Consider a CSP $\mathcal{P}$ with a sequence of variables X

Search tree for $\mathcal{P}$: a finite tree such that

- its nodes are CSP's
- its root is $\mathcal{P}$
- the nodes at an even level have exactly one direct descendant
- if $\mathcal{P}_1, ..., \mathcal{P}_m$ are direct descendants of $\mathcal{P}_0$, then the union of $\mathcal{P}_1, ..., \mathcal{P}_m$ is equivalent w.r.t. X to $\mathcal{P}_0$

Labeling Trees

Specific search trees for finite CSP's

- Splitting consists of labeling of the domain of a variable
- Constraint propagation consists of a domain reduction method

Complete Labeling Trees

Constraint propagation absent

Given:

- a CSP $\mathcal{P}$ with non-empty domains

- $x_1, ..., x_n$ the sequence of its variables linearly ordered by $\prec$

Complete labeling tree associated with $\mathcal{P}$ and $\prec$:

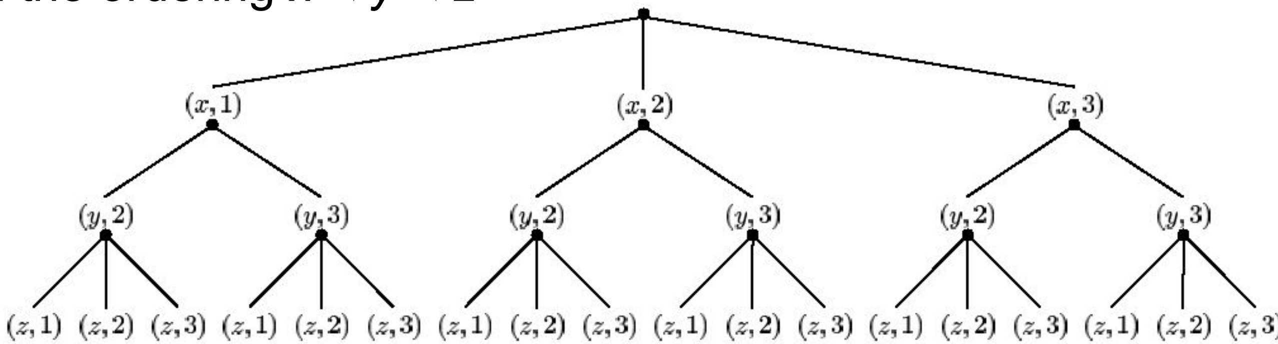
- the direct descendants of the root are of the form (x_1, d)
- the direct descendants of a node (x_j, d), where $j \in [1..n - 1]$, are of the form (x_{j+1}, e)
- its branches determine all the instantiations with the domain $\{x_1, ..., x_n\}$

Examples

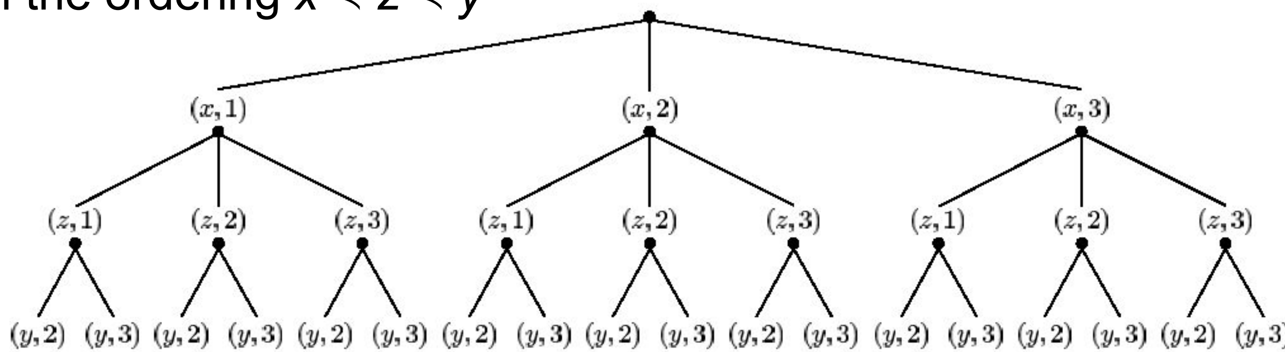
Consider

$\langle x < y,\ y < z\ ;\ x \in \{1, 2, 3\},\ y \in \{2, 3\},\ z \in \{1, 2, 3\}\rangle$

1. with the ordering $x \prec y \prec z$



2. with the ordering $x \prec z \prec y$



Sizes of Complete Labeling Trees

Given:

- a CSP with non-empty domains
- $x_1, ..., x_n$ the sequence of its variables linearly ordered by $\prec$
- $D_1, ..., D_n$ the corresponding variable domains

- The number of nodes in the complete labeling tree associated with $\prec$ is

$$1+\sum_{i=1}^{n}\left(\prod_{j=1}^{i}|D_j|\right)$$

  $|A|$: the cardinality of set A

- The complete labeling tree has the least number of nodes if the variables are ordered by their domain sizes in increasing order

Examples

Tree in 1. (cf. Slide 7):

The cardinalities of the domains: 3, 2, 3

The tree has $1 + 3 + 3 \cdot 2 + 3 \cdot 2 \cdot 3$, i.e., 28 nodes

Tree in 2. (cf. Slide 7):

The cardinalities of the domains: 3, 3, 2

The tree has $1 + 3 + 3 \cdot 3 + 3 \cdot 3 \cdot 2$, i.e., 31 nodes

Both trees have the same number of leaves: 18

Reduced Labeling Trees

An instantiation I is along the ordering $x_1, ..., x_n$ if its domain is $\{x_1, ..., x_j\}$ for some $j \in [1..n]$.

Given:

- a CSP $\mathcal{P}$ with non-empty domains
- $x_1, ..., x_n$ the sequence of its variables linearly ordered by $\prec$

Reduced labeling tree associated with $\mathcal{P}$ and $\prec$:

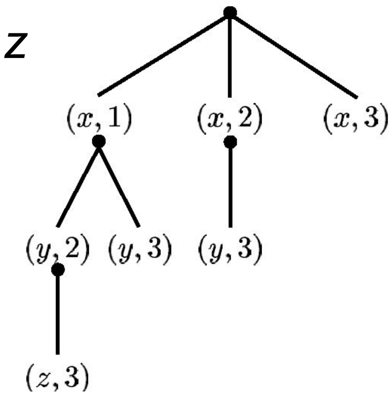
- the direct descendants of the root are of the form (x_1, d)
- the direct descendants of a node (x_j, d), where $j \in [1..n-1]$, are of the form (x_{j+1}, e)
- its branches determine all consistent instantiations along the ordering $x_1, ..., x_n$

Examples

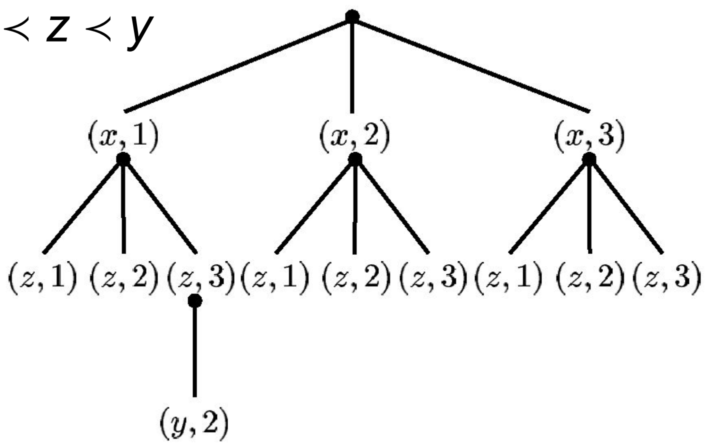
Consider

$\langle x < y, y < z \,;\, x \in \{1, 2, 3\}, y \in \{2, 3\}, z \in \{1, 2, 3\}\rangle$

1. with the ordering $x \prec y \prec z$



2. with the ordering $x \prec z \prec y$



Reduced labeling trees can have different number of nodes and different number of leaves.

Labeling Trees with Constraint Propagation

Given: $\mathcal{P} \coloneqq \langle \mathcal{C} \,;\, x_1 \in D_1, \ldots, x_n \in D_n \rangle$

- Assume fixed form of constraint propagation *prop(i)* in the form of a domain reduction, where $i \in [0..n-1]$
- i determines the sequence $x_{i+1}, \ldots, x_n$ of the variables to whose domains *prop(i)* is applied
- Given current variable domains $E_1, \ldots, E_n$, constraint propagation *prop(i)* transforms only $E_{i+1}, \ldots, E_n$
- *prop(i)* depends on the original constraints $\mathcal{C}$ of $\mathcal{P}$ and on the domains $E_1, \ldots, E_i$

prop Labeling Trees

prop labeling tree associated with $\mathcal{P}$:

- its nodes are sequences of the domain expressions $x_1 \in E_1, \ldots, x_n \in E_n$
- its root is $x_1 \in D_1, x_2 \in D_2, \ldots, x_n \in D_n$
- each node at an even level $2i$ with $i \in [0..n]$ is of the form

  $$x_1 \in \{d_1\}, \ldots, x_i \in \{d_i\}, x_{i+1} \in E_{i+1}, \ldots, x_n \in E_n$$

  If $i = n$, this node is a leaf. Otherwise, it has exactly one direct descendant, obtained using *prop(i)*:

  $$x_1 \in \{d_1\}, \ldots, x_i \in \{d_i\}, x_{i+1} \in E'_{i+1}, \ldots, x_n \in E'_n$$

  where $E'_j \subseteq E_j$ for $j \in [i + 1..n]$

prop Labeling Trees, ctd

- each node at an odd level $2i + 1$ with $i \in [0..n-1]$ is of the form

  $$x_1 \in \{d_1\}, ..., x_i \in \{d_i\}, x_{i+1} \in E_{i+1}, ..., x_n \in E_n$$

  If $E_j = \emptyset$ for some $j \in [i+1..n]$, this node is a leaf. Otherwise, it has direct descendants of the form

  $$x_1 \in \{d_1\}, ..., x_i \in \{d_i\}, x_{i+1} \in \{d\}, x_{i+2} \in E_{i+2}, ..., x_n \in E_n$$

  for all $d \in E_{i+1}$ such that the instantiation $\{(x_1, d_1), ..., (x_i, d_i), (x_{i+1}, d)\}$ is consistent

Intuition

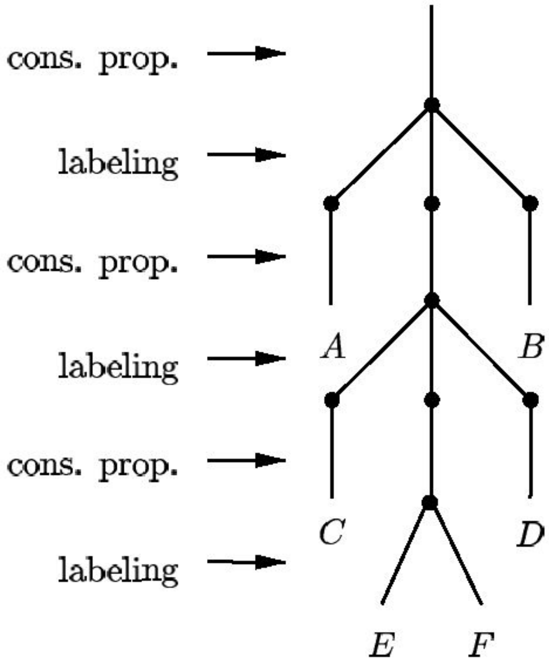
Given: node $x_1 \in E_1, ..., x_n \in E_n$ at level $2i - 1$ or $2i$

- if $i \in [2..n - 1]$, we call $x_1, ..., x_{i-1}$ its past variables
- if $i \in [1..n]$, we call x_i its current variable
- if $i \in [0..n - 1]$, we call $x_{i+1}, ..., x_n$ its future variables

prop(i) affects only the domains of the future variables.

Example of a *prop* Labeling Tree

Consider a CSP with three variables, x_1, x_2, x_3

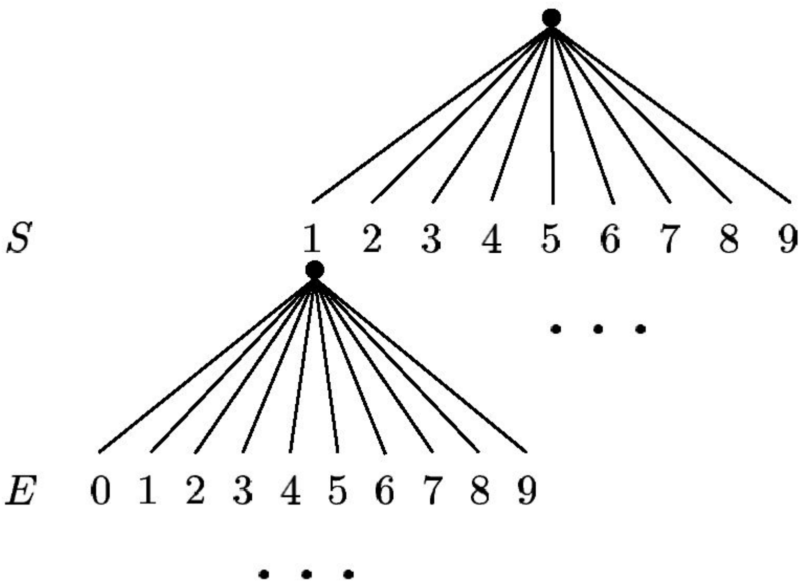


levels	past variables	current variable	future variables
0			
1		x_1	x_2, x_3
2		x_1	x_2, x_3
3	x_1	x_2	x_3
4	x_1	x_2	x_3
5	x_1, x_2	x_3	
6	x_1, x_2	x_3	

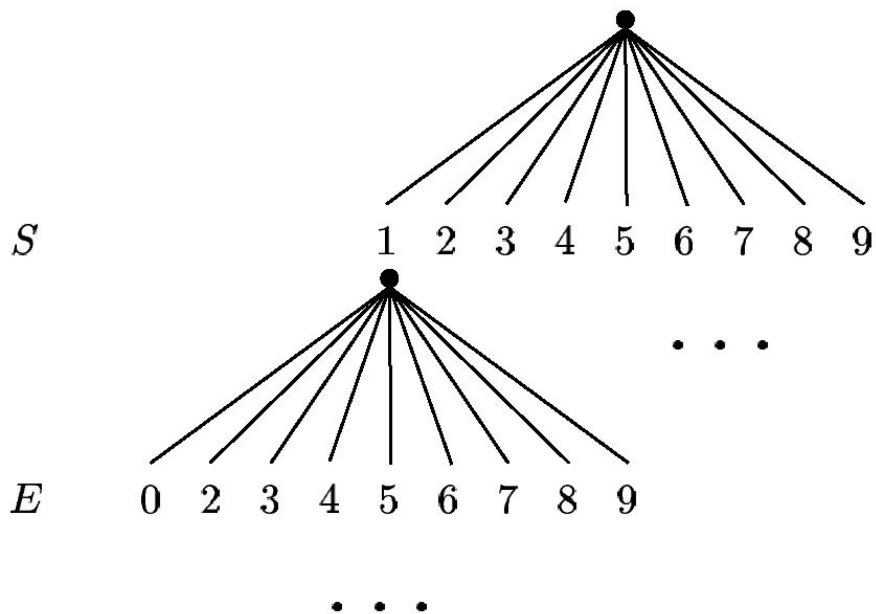
A, *B*, *C*, and *D* are failed nodes. *E* and *F* are success nodes.

Example: SEND + MORE = MONEY

Complete Labeling Tree:



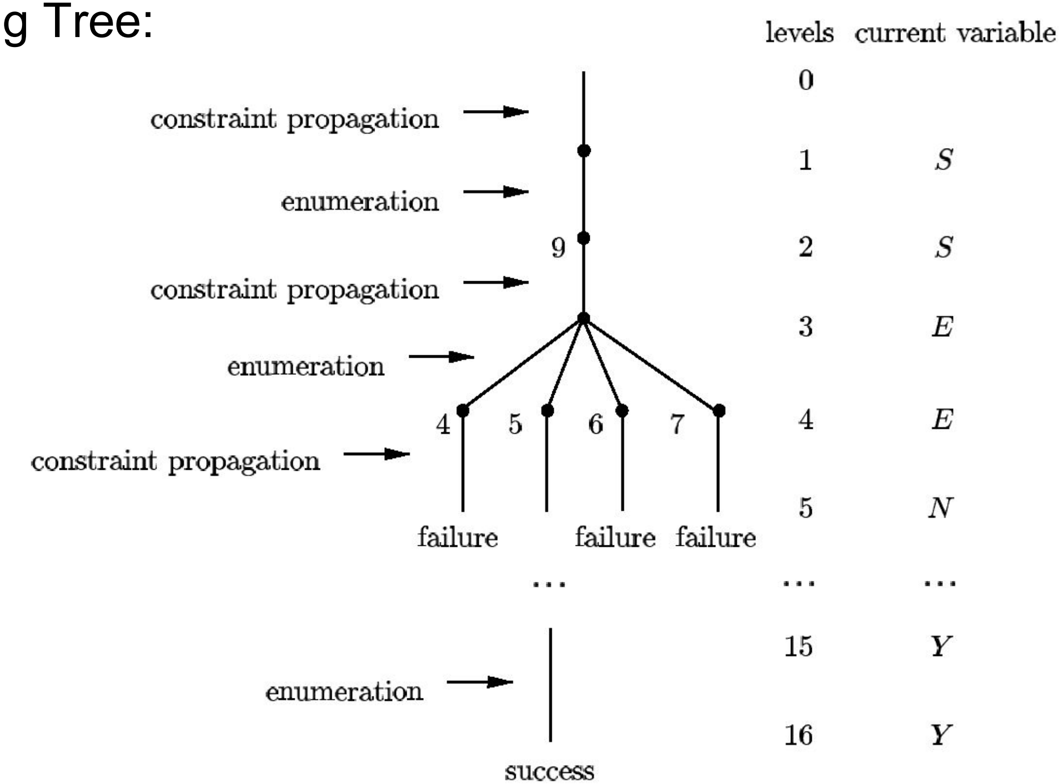
Reduced Labeling Tree:



SEND + MORE = MONEY, ctd

Use as *prop* the domain reduction rules for linear constraints over integer intervals from Chapter 5.

prop Labeling Tree:



Sizes of Generated Trees

For SEND + MORE = MONEY:

- Complete labeling tree
  Total number of leaves: $9^2 \cdot 10^6 = 81000000$
- Reduced labeling tree
  Total number of leaves: $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 - 2 \cdot (9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4) = 483840$
  Gain: 99.4% with respect to the complete labeling tree
- *prop* labeling tree
  Total number of leaves: 4

Instances of *prop* Labeling Trees

- forward checking
- partial look ahead
- maintainting arc consistency (MAC)
  (aka full look ahead)

Forward Checking Search Tree

Recall from the definition of *prop* labeling trees:

- Each node at an even level $2i$ with $i \in [0..n]$ is of the form

  $$x_1 \in \{d_1\}, ..., x_i \in \{d_i\}, x_{i+1} \in E_{i+1}, ..., x_n \in E_n$$

  If $i = n$, this node is a leaf. Otherwise, it has exactly one direct descendant, obtained using *prop(i)*:

  $$x_1 \in \{d_1\}, ..., x_i \in \{d_i\}, x_{i+1} \in E'_{i+1}, ..., x_n \in E'_n$$

  where $E'_j \subseteq E_j$ for $j \in [i + 1..n]$

Define

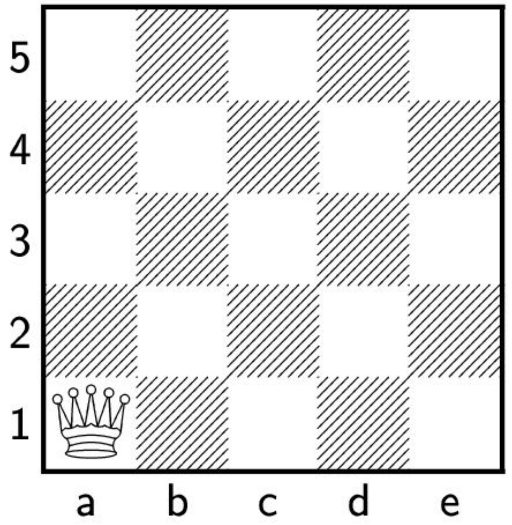
$$E'_j := \{e \in E_j \mid \{(x_1, d_1), ..., (x_i, d_i), (x_j, e)\} \text{ is consistent}\}$$

Example: 5 Queens Problem

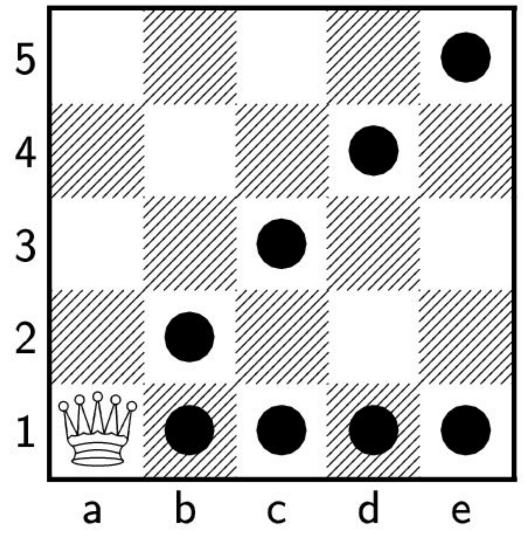
Take the standardized CSP corresponding to 5 Queens Problem.

Interpretation: the variables x_1, x_2, x_3, x_4, x_5 correspond to the columns a, b, c, d, e

First queen placed at `a1`:



Effect of forward checking:

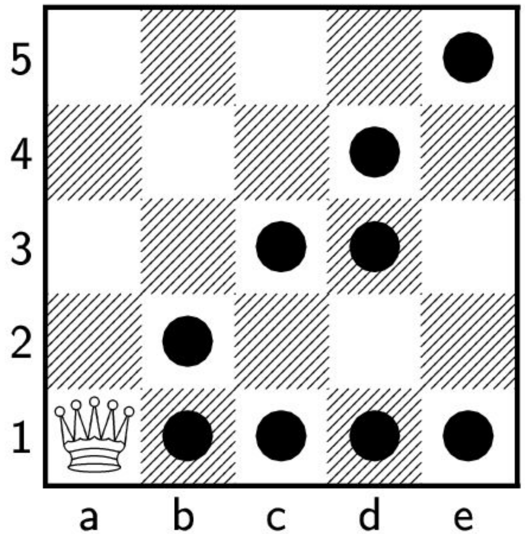


Partial Look Ahead Search Tree

- Impose forward checking
- Impose directional arc consistency, e.g. using the DARC algorithm

Example: 5 Queens Problem

Effect of partial look ahead in the example:

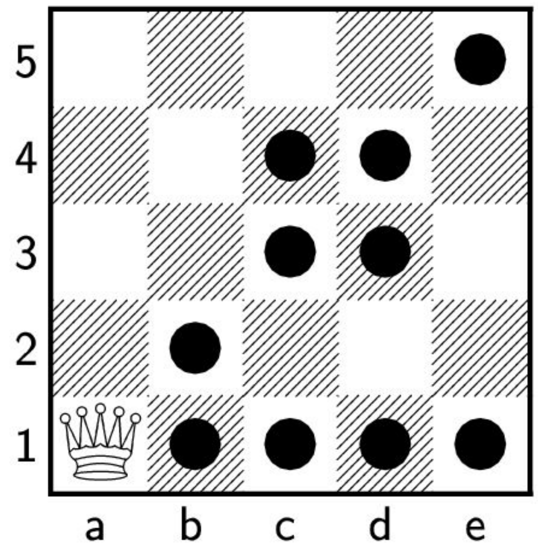


MAC Search Tree

- Impose forward checking
- Impose arc consistency, e.g. using the ARC algorithm

Example: 5 Queens Problem

Effect of MAC in the example:



Search Algorithms for Labeling Trees

- Backtrack-free search
- Backtrack-free search with constraint propagation
- Backtrack search
- Backtrack search with constraint propagation
  - forward checking
  - partial look ahaed
  - MAC

Search algorithms for constrained optimization problems:

- Branch and bound search
- Branch and bound with constraint propagation search

Declarations

cons(inst, j, d) ≡ "the instantiation
$\{(x_1, \text{inst}[1]), \ldots, (x_{j-1}, \text{inst}[j - 1]), (x_j, d)\}$ is consistent"

type domains = **array** [1..n] **of** domain;
instantiation = **array** [1..n] **of** elements;

var inst: instantiation;
failure: **boolean**

Backtracking

```
procedure backtrack(j: integer; D: domains; var success: boolean);
begin
    while D[j] ≠ ∅ and not success do
        choose d from D[j];
        D[j] ≔ D[j] – {d};
        if cons(inst, j, d) then
            inst[j] ≔ d;
            success ≔ (j = n);
            if not success then backtrack(j + 1, D, success)
        end-if
    end-while
end

begin
    success ≔ false;
    backtrack(1, D, success)
end
```

Backtracking with Constraint Propagation

```
procedure backtrack_prop(j: integer; D: domains; var success: boolean);
begin
    while D[j] ≠ ∅ and not success do
        choose d from D[j];
        D[j] ≔ D[j] – {d};
        if cons(inst, j, d) then
            inst[j] ≔ d;
            success ≔ (j = n);
            if not success then
                prop(j, D, failure);
                if not failure then backtrack_prop(j + 1, D, success)
            end-if
        end-if
    end-while
end

begin
    success ≔ false;
    prop(0, D, failure);
    if not failure then backtrack_prop(1, D, success)
end
```

Forward Checking

```
procedure revise(j, k: integer; var D: domains);
begin
    D[k] ≔ {d ∈ D[k] | { (x_1, inst[1]), ..., (x_j, inst[j]), (x_k, d)} is a consistent instantiation}
end

procedure prop(j: integer; var D: domains; var failure: boolean);
var k: integer;
begin
    failure ≔ false;
    k ≔ j + 1;
    while k < n + 1 and not failure do
        revise(j ,k, D);
        failure ≔ (D[k] = ∅);
        k ≔ k + 1
    end-while
end
```

Partial Look Ahead

procedure prop(*j*: **integer**; **var** D: domains; **var** failure: **boolean**);
var *k*: **integer**;
begin
    failure ≔ **false**;
    k ≔ *j* + 1;
    while *k* < *n* + 1 **and not** failure **do**
        revise(*j* ,*k*, *D*);
        failure ≔ ($D[k] = \emptyset$);
        k ≔ *k* + 1
    end-while
    if not failure **then** darc(*j* + 1, *D*, failure)
end

MAC (Full Look Ahead)

```
procedure prop(j: integer; var D: domains; var failure: boolean);
...
    if not failure then arc(j + 1, D, failure)
end
```

Finite Constrained Optimization Problems

- $\mathcal{P} := \langle \mathcal{C} \,;\, x_1 \in D_1, \ldots, x_n \in D_n \rangle$
- $obj : Sol \to \mathbb{R}$ from the set Sol of all solutions to $\mathcal{P}$ to $\mathbb{R}$
- Heuristic function $h : \mathcal{P}(D_1) \times \ldots \times \mathcal{P}(D_n) \to \mathbb{R} \cup \{\infty\}$

Monotonicity: If $\bar{E}_1 \subseteq \bar{E}_2$, then $h(\bar{E}_1) \leq h(\bar{E}_2)$

Bound: $obj(d_1, \ldots, d_n) \leq h(\{d_1\}, \ldots, \{d_n\})$

procedure *obj*(inst: instantiation): **real**;

procedure *h*(inst: instantiation; *j*: **integer**; *D*: domains): **real**;

h(inst, *j*, *D*) returns the value of *h* on ({inst[1]}, ..., {inst[*j*]}, *D*[*j* + 1], ..., *D*[*n*])

Branch and Bound with Constraint Propagation

```
procedure branch_and_bound_prop(j: integer; D: domains; var solution: instantiation; var bound: real);
begin
    while D[j] ≠ ∅ do
        choose d from D[j];
        D[j] ≔ D[j] – {d};
        if cons(inst, j, d) then
            inst[j] ≔ d;
            if j = n then
                if obj(inst) > bound then
                    bound ≔ obj(inst); solution ≔ inst
                end-if
            else
                prop(j, D, failure);
                if not failure and h(inst, j, D) > bound then
                    branch_and_bound_prop(j + 1, D, solution, bound)
            end-if
        end-if
    end-while
end
```

Branch and Bound with Constraint Propagation, ctd

```
begin
    solution ≔ nil;
    bound ≔ -∞;
    prop(0, D, failure);
    if not failure then
        branch_and_bound_prop(1, D, solution, bound)
end
```

Heuristics for Search Algorithms

Variable Selection

- Select a variable with the smallest domain
- Select a most constrained variable
- (For numeric domains)
  Select a variable with the smallest difference between its domain bounds

Value Selection

- Select a value for the heuristic function that yields the highest outcome
- Select the smallest value
- Select the largest value
- Select the middle value

Objectives

- Introduce search trees
- Discuss various types of labeling trees, in particular trees for
  - forward checking
  - partial look ahead
  - maintaining arc consistency (MAC)
- Discuss various search algorithms for labeling trees
- Discuss search algorithms for constrained optimization problems
- Introduce various heuristics for search algorithms