

FOUNDATIONS OF COMPLEXITY THEORY

Lecture 11: Space Hierarchy and Gaps

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Review

Review: Time Hierarchy Theorems

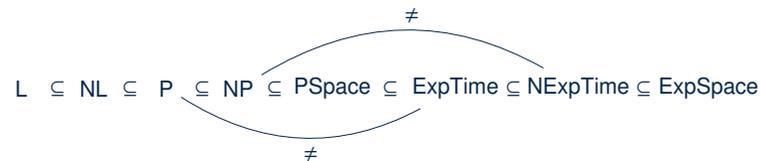
Time Hierarchy Theorem 12.12 If $f, g : \mathbb{N} \rightarrow \mathbb{N}$ are such that f is time-constructible, and $g \cdot \log g \in o(f)$, then

$$DTime_*(g) \subsetneq DTime_*(f)$$

Nondeterministic Time Hierarchy Theorem 12.14 If $f, g : \mathbb{N} \rightarrow \mathbb{N}$ are such that f is time-constructible, and $g(n+1) \in o(f(n))$, then

$$NTime_*(g) \subsetneq NTime_*(f)$$

In particular, we find that $P \neq ExpTime$ and $NP \neq NExpTime$:



A Hierarchy for Space

Space Hierarchy

For space, we can always assume a single working tape:

- Tape reduction leads to a constant-factor increase in space
- Constant factors can be eliminated by space compression

Therefore, $\text{DSpace}_k(f) = \text{DSpace}_1(f)$.

Space turns out to be easier to separate – we get:

Space Hierarchy Theorem 11.1: If $f, g : \mathbb{N} \rightarrow \mathbb{N}$ are such that f is space-constructible, and $g \in o(f)$, then

$$\text{DSpace}(g) \subsetneq \text{DSpace}(f)$$

Challenge: TMs can run forever even within bounded space.

Proving the Space Hierarchy Theorem (1)

Proof (continued): It remains to show that \mathcal{D} implements diagonalisation:

$\mathbf{L}(\mathcal{D}) \in \text{DSpace}(f)$:

- f is space-constructible, so both the marking of tape symbols and the initialisation of the counter are possible in $\text{DSpace}(f)$
- The simulation is performed so that the marked $O(f)$ -space is not left

There is some w such that $\langle \mathcal{M}, w \rangle \in \mathbf{L}(\mathcal{D})$ iff $\langle \mathcal{M}, w \rangle \notin \mathbf{L}(\mathcal{M})$:

- As for time, we argue that some w is long enough to ensure that f is sufficiently larger than g , so \mathcal{D} 's simulation can finish.
- The countdown measures $2^{f(n)}$ steps. The number of possible distinct configurations of \mathcal{M} on w is $|Q| \cdot n \cdot g(n) \cdot |\Gamma|^{g(n)} \in 2^{O(g(n)+\log n)}$, and due to $f(n) \geq \log n$ and $g \in o(f)$, this number is smaller than $2^{f(n)}$ for large enough n .
- If \mathcal{M} has d tape symbols, then \mathcal{D} can encode each in $\log d$ space, and due to \mathcal{M} 's space bound \mathcal{D} 's simulation needs at most $\log d \cdot g(n) \in o(f(n))$ cells.

Therefore, there is w for which \mathcal{D} simulates \mathcal{M} long enough to obtain (and flip) its output, or to detect that it is not terminating (and to accept, flipping again). \square

Proving the Space Hierarchy Theorem (1)

Space Hierarchy Theorem 11.1: If $f, g : \mathbb{N} \rightarrow \mathbb{N}$ are such that f is space-constructible, and $g \in o(f)$, then

$$\text{DSpace}(g) \subsetneq \text{DSpace}(f)$$

Proof: Again, we construct a diagonalisation machine \mathcal{D} . We define a multi-tape TM \mathcal{D} for inputs of the form $\langle \mathcal{M}, w \rangle$ (other cases do not matter), assuming that $|\langle \mathcal{M}, w \rangle| = n$

- Compute $f(n)$ in unary to mark the available space on the working tape
- Initialise a separate countdown tape with the largest binary number that can be written in $f(n)$ space
- Simulate \mathcal{M} on $\langle \mathcal{M}, w \rangle$, making sure that only previously marked tape cells are used
- Time-bound the simulation using the content of the countdown tape by decrementing the counter in each simulated step
- If \mathcal{M} rejects (in this space bound) or if the time bound is reached without \mathcal{M} halting, then accept; otherwise, if \mathcal{M} accepts or uses unmarked space, reject

Space Hierarchies

Like for time, we get some useful corollaries:

Corollary 11.2: $\text{PSPACE} \subsetneq \text{ExpSpace}$

Proof: As for time, but easier. \square

Corollary 11.3: $\text{NL} \subsetneq \text{PSPACE}$

Proof: Savitch tells us that $\text{NL} \subseteq \text{DSpace}(\log^2 n)$. We can apply the Space Hierarchy Theorem since $\log^2 n \in o(n)$. \square

Corollary 11.4: For all real numbers $0 < a < b$, we have $\text{DSpace}(n^a) \subsetneq \text{DSpace}(n^b)$.

In other words: The hierarchy of distinct space classes is very fine-grained.

The Gap Theorem

Proving the Gap Theorem

Special Gap Theorem 11.5: There is a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $\text{DTime}(f(n)) = \text{DTime}(2^{f(n)})$.

Proof idea: We divide time into exponentially long intervals of the form:

$$[0, n], [n + 1, 2^n], [2^n + 1, 2^{2^n}], [2^{2^n} + 1, 2^{2^{2^n}}], \dots$$

(for some appropriate starting value n)

We are looking for **gaps of time** where no TM halts, since:

- for every finite set of TMs,
- and every finite set of inputs to these TMs,
- there is some interval of the above form $[m + 1, 2^m]$

such none of the TMs halts in between $m + 1$ and 2^m steps on any of the inputs.

The task of f is to find the start m of such a gap for a suitable set of TMs and words

Why Constructibility?

The hierarchy theorems require that resource limits are given by constructible functions

Do we really need this?

Yes. The following theorem shows why (for time):

Special Gap Theorem 11.5: There is a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $\text{DTime}(f(n)) = \text{DTime}(2^{f(n)})$.

This has been shown independently by Boris Trakhtenbrot (1964) and Allan Borodin (1972).

Reminder: For this we continue to use the strict definition of $\text{DTime}(f)$ where no constant factors are included (no hidden $O(f)$). This simplifies proofs; the factors are easy to add back.

Gaps in Time

We consider an (effectively computable) enumeration of all Turing machines:

$$\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2, \dots$$

Definition 11.6: For arbitrary numbers $i, a, b \in \mathbb{N}$ with $a \leq b$, we say that $\text{Gap}_i(a, b)$ is true if:

- Given any TM \mathcal{M}_j with $0 \leq j \leq i$,
- and any input string w for \mathcal{M}_j of length $|w| = i$,

\mathcal{M}_j on input w will halt in less than a steps, in more than b steps, or not at all.

Lemma 11.7: Given $i, a, b \geq 0$ with $a \leq b$, it is decidable if $\text{Gap}_i(a, b)$ holds.

Proof: We just need to ensure that none of the finitely many TMs $\mathcal{M}_0, \dots, \mathcal{M}_i$ will halt after a to b steps on any of the finitely many inputs of length i . This can be checked by simulating TM runs for at most b steps. \square

Find the Gap

We can now define the value $f(n)$ of f for some $n \geq 0$:

Let $\text{in}(n)$ denote the number of runs of TMs $\mathcal{M}_0, \dots, \mathcal{M}_n$ on words of length n , i.e.,

$$\text{in}(n) = |\Sigma_0|^n + \dots + |\Sigma_n|^n \quad \text{where } \Sigma_i \text{ is the input alphabet of } \mathcal{M}_i$$

We recursively define a **series of numbers** k_0, k_1, k_2, \dots by setting $k_0 = 2n$ and $k_{i+1} = 2^{k_i}$ for $i \geq 0$, and we consider the following **list of intervals**:

$$\begin{array}{ccccccc} [k_0 + 1, k_1], & [k_1 + 1, k_2], & \dots, & [k_{\text{in}(n)} + 1, k_{\text{in}(n)+1}] \\ \parallel & \parallel & & \parallel \\ [2n + 1, 2^{2n}], & [2^{2n} + 1, 2^{2^{2n}}], & \dots, & [2^{2^{2^{2n}}} + 1, 2^{2^{2^{2^{2n}}}}] \end{array}$$

Let $f(n)$ be the least number k_i with $0 \leq i \leq \text{in}(n)$ such that $\text{Gap}_n(k_i + 1, k_{i+1})$ is true.

Finishing the Proof

We can now complete the proof of the theorem:

Claim: $\text{DTime}(f(n)) = \text{DTime}(2^{f(n)})$.

Consider any $\mathbf{L} \in \text{DTime}(2^{f(n)})$.

Then there is an $2^{f(n)}$ -time bounded TM \mathcal{M}_j with $\mathbf{L} = \mathbf{L}(\mathcal{M}_j)$.

For any input w with $|w| \geq j$:

- The definition of $f(|w|)$ took the run of \mathcal{M}_j on w into account
- \mathcal{M}_j on w halts after less than $f(|w|)$ steps, or not until after $2^{f(|w|)}$ steps (maybe never)
- Since \mathcal{M}_j runs in time $\text{DTime}(2^{f(n)})$, it must halt in $\text{DTime}(f(n))$ on w

For the finitely many inputs w with $|w| < j$:

- We can augment the state space of \mathcal{M}_j to run a finite automaton to decide these cases
- This will work in $\text{DTime}(f(n))$

Therefore we have $\mathbf{L} \in \text{DTime}(f(n))$. □

Properties of f

We first establish some basic properties of our definition of f :

Claim: The function f is well-defined.

Proof: For finding $f(n)$, we consider $\text{in}(n) + 1$ intervals. Since there are only $\text{in}(n)$ runs of TMs $\mathcal{M}_0, \dots, \mathcal{M}_n$, at least one interval remains a “gap” where no TM run halts. □

Claim: The function f is computable.

Proof: We can compute $\text{in}(n)$ and k_i for any i , and we can decide $\text{Gap}_n(k_i + 1, k_{i+1})$. □

Papadimitriou: “notice the fantastically fast growth, as well as the decidedly unnatural definition of this function.”

Discussion: The case $|w| < j$

Borodin says: It is meaningful to state complexity results if they hold for “almost every” input (i.e., for all but a finite number)

Papadimitriou says: These words can be handled since we can check the length and then recognise the word in less than $2j$ steps

Really?

- If we do these $< 2j$ steps before running \mathcal{M}_j , the modified TM runs in $\text{DTime}(f(n) + 2j)$
- This does not show $\mathbf{L} \in \text{DTime}(f(n))$

A more detailed argument:

- Make the intervals larger: $[k_i + 1, 2^{k_i+2n} + 2n]$, that is $k_{i+1} = 2^{k_i+2n} + 2n$.
- Select $f(n)$ to be $k_i + 2n + 1$ if the least gap starts at $k_i + 1$.

The same pigeon hole argument as before ensures that an empty interval is found.

But now the $f(n)$ time bounded machine \mathcal{M}_j from the proof will be sure to stop after $f(n) - 2n - 1$ steps, so a shift of $2j \leq 2n$ to account for the finitely many cases will not make it use more than $f(n)$ steps either

Discussion: Generalising the Gap Theorem

- Our proof uses the function $n \mapsto 2^n$ to define intervals
- Any other computable function could be used without affecting the argument

This leads to a generalised Gap Theorem:

Gap Theorem 11.8: For every computable function $g : \mathbb{N} \rightarrow \mathbb{N}$ with $g(n) \geq n$, there is a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $\text{DTime}(f(n)) = \text{DTime}(g(f(n)))$.

Example 11.9: There is a function f such that

$$\text{DTime}(f(n)) = \text{DTime}\left(\underbrace{2^{2^{\cdot^{\cdot^{\cdot}}}}}_{f(n) \text{ times}}\right)$$

Moreover, the Gap Theorem can also be shown for space (and for other resources) in a similar fashion (space is a bit easier since the case of short words $|w| < j$ is easy to handle in very little space)

Discussion: Significance of the Gap Theorem

What have we learned?

- More time (or space) does not always increase computational power
- However, this only works for extremely fast-growing, very unnatural functions

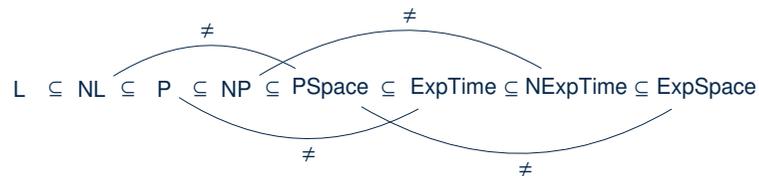
“Fortunately, the gap phenomenon cannot happen for time bounds t that anyone would ever be interested in”¹

Main insight: better stick to constructible functions

¹Allender, Loui, Reagan: Complexity Theory. In Computing Handbook, 3rd ed., CRC Press, 2014
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Summary and Outlook

Hierarchy theorems tell us that more time/space leads to more power:



However, they don't help us in comparing different resources and machine types (P vs. NP, or PSpace vs. ExpTime)

With non-constructible functions as time/space bounds, arbitrary (constructible or not) boosts in resources do not lead to more power

What's next?

- The inner structure of NP revisited
- Computing with oracles (reprise)
- The limits of diagonalisation, proved by diagonalisation