The Expressive Power of Description Logics with Numerical Constraints over Restricted Classes of Models

Franz Baader[1,2][0000−0002−4049−221X] and Filippo De Bortoli[1,2][0000−0002−8623−6465]

[1] TU Dresden, Institute of Theoretical Computer Science, Dresden, Germany
[2] Center for Scalable Data Analytics and AI (ScaDS.AI) Dresden/Leipzig, Germany

Abstract. For Description Logics (DLs), different approaches for extending the expressive power using numerical constraints have been introduced. Here, we consider the logic $\mathcal{ALCSCC}$, which can state powerful numerical constraints on the number of role successors satisfying certain properties, and logics of the form $\mathcal{ALC}(\mathfrak{D})$, in which individuals can be assigned numerical or other concrete values, which can be compared using predefined predicates of $\mathfrak{D}$. Instead of investigating the complexity of reasoning in these logics, we are interested in characterizing their expressive power. We improve on our previous work in this direction in several respects. For $\mathcal{ALCSCC}$, we develop a method that can deal with the finitely branching interpretations considered in the original paper on this logic, rather than moving to the variant $\mathcal{ALCSCC}^\infty$, where arbitrary interpretations are allowed. The main idea is to employ, in the proof of the characterization, locality properties of first-order logic over certain restricted classes of models (such as finite and finitely branching models) rather than compactness, which does not hold in the finitely branching case. For logics of the form $\mathcal{ALC}(\mathfrak{D})$, we consider a notion of expressive power that takes the concrete values assigned to individuals into account, rather than the abstract expressive power investigated in our previous work. The characterization of the expressive power of $\mathcal{ALC}(\mathfrak{D})$ obtained this way works not only for arbitrary interpretations, but also for finite and finitely branching ones.

1 Introduction

Description logics (DLs) [6,13] are a prominent family of logic-based knowledge representation languages, which can be used to formalize the terminological knowledge of an application domain in a machine-processable way. For instance, the standard Web Ontology Language OWL[3] is based on an expressive DL and the large medical ontology SNOMED CT[4] has been developed using a rather inexpressive DL. The *expressive power* of a DL is determined by the constructors

[3] https://www.w3.org/TR/owl2-overview/
[4] https://www.snomed.org/

that are available for building complex concept descriptions out of concept names (unary predicates) and role names (binary predicates). For example, the concept description $\mathsf{Person} \sqcap \exists \mathsf{pet}.\mathsf{Dog}$, describing persons that have a dog as a pet, uses conjunction ($\sqcap$) and existential restriction ($\exists r.C$) as constructors, where Person and Dog are concept names and pet is a role name. To show that a given DL $\mathcal{L}_1$ can be expressed by another DL $\mathcal{L}_2$ using the same concept and role names, we can provide a semantic-preserving translation of $\mathcal{L}_1$ concept descriptions into $\mathcal{L}_2$ concept descriptions. Proving inexpressivity is more challenging. The first formal investigation of the expressive power of DLs was performed in [1,2], but in a rather ad hoc manner. More fundamental characterizations of the expressive power of various concept description languages up to the DL $\mathcal{ALC}$ based on the model-theoretic notion of *bisimulation* are given in [20]. Basically, this approach (pioneered by van Benthem [28] for the modal logic K, which is a syntactic variant of $\mathcal{ALC}$) characterizes a given DL as the fragment of first-order logic (FOL) that is invariant under an appropriate notion of bisimulation.

The expressive power of $\mathcal{ALC}$ can, for instance, be extended by enabling the use of numerical constraints within concept descriptions. In the extension $\mathcal{ALCQ}$ of $\mathcal{ALC}$, qualified number restrictions [18] can be employed to constrain the number of role successors belonging to a certain concept; e.g., $\mathsf{Person} \sqcap (\geq 3\, \mathsf{child}.\mathsf{Female}) \sqcap (\leq 2\, \mathsf{pet}.\mathsf{Dog})$ describes persons that have at least 3 daughters and at most 2 dogs as pets. The DL $\mathcal{ALCSCC}$ [3] extends $\mathcal{ALCQ}$ with very expressive counting constraints on role successors expressed in the logic QFBAPA [19]. Since QFBAPA only considers finite sets and their cardinalities, the semantics of $\mathcal{ALCSCC}$ is restricted to finitely branching interpretations, where each element can have only finitely many role successors. In $\mathcal{ALCSCC}$ one can, e.g., describe persons that have more daughters than they have dogs as pets, without using specific numbers as upper/lower bounds for the numbers of pet dogs and daughters. Bisimulation-based characterizations of $\mathcal{ALCQ}$ (or its modal logic variant of K extended with graded modalities) can be found in [26,22,25]. In [7,8], we have investigated the expressivity of DLs with expressive counting constraints. However, to dispense with the requirement that interpretations be finitely branching, we used an infinite variant QFBAPA$^\infty$ of QFBAPA to formulate these constraints, which yields the variant $\mathcal{ALCSCC}^\infty$ of $\mathcal{ALCSCC}$. We were able to show that $\mathcal{ALCSCC}^\infty$ is not a fragment of FOL and characterized the first-order fragment of this logic ($\mathcal{ALCCQU}$ or equivalently $\mathcal{ALCQ}t$) using a form of counting bisimulation [22]. The *first major contribution* of the present paper is to prove the same results for $\mathcal{ALCSCC}$, where only finitely branching interpretations are available. The proof techniques used in [7,8], which were inspired by the ones in [22], cannot be employed in this setting since they depend on compactness of FOL, which does not hold for the restriction of FOL to finitely branching interpretations. Instead, we employ a proof technique inspired by [27,25], which utilizes locality properties of FOL rather than compactness. Interestingly, this approach can deal with arbitrary interpretations, finitely branching interpretations, and finite interpretations in a uniform way.

An orthogonal approach for employing numerical constraints within concept descriptions is the use of numerical concrete domains [21,14]. In a DL with a concrete domain, concrete objects such as numbers or strings can be assigned to individuals using partial functions called *features*. For example, the concept description $\mathsf{Person} \sqcap \exists \mathsf{child\ age}, \mathsf{pet\ age}.<$ describes persons that have a child that is younger than one of their pets. Here, age is a feature that assigns a rational number, their age, to some of the elements of the interpretation domain, and $<$ is the usual smaller relation between rational numbers. In [9,10], we have investigated the abstract expressive power of DLs with concrete domains, which only considers the abstract part of interpretations, i.e., ignores the values assigned to features. We have shown that the abstract expressive power of $\mathcal{ALC}(\mathfrak{D})$, i.e., $\mathcal{ALC}$ extended with the concrete domains $\mathfrak{D}$, is contained in FOL for certain concrete domains, but have also exhibited a large class of concrete domains for which this is not the case. The *second major contribution* of the present paper is to introduce a notion of concrete expressive power for DLs with concrete domains that also takes the feature values into account. For example, if we take two concrete domains over the rational numbers, where one has as only predicate $+_1$ (relating $q \in \mathbb{Q}$ with $q + 1$) and the other $+_2$ (relating $q \in \mathbb{Q}$ with $q + 2$), then the extensions of $\mathcal{ALC}$ with these concrete domains have the same abstract expressive power, but their concrete expressive power is incomparable. Using proof techniques similar to the ones employed for $\mathcal{ALCSCC}$ we can characterize $\mathcal{ALC}(\mathfrak{D})$ as the fragment of $\mathsf{FOL}(\mathfrak{D})$ (i.e., FOL extended with the concrete domain $\mathfrak{D}$) that is invariant under an appropriate notion of bisimulation.

A technical report containing detailed proofs of all the results introduced in this paper is available online [11].

2 Preliminaries

We start by introducing the base logic $\mathcal{ALC}$ before defining its two orthogonal extensions with numerical constraints. Since here we focus on the expressivity of concept description languages, we do not introduce TBoxes, ABoxes, or reasoning problems (see [13] for more details on $\mathcal{ALC}$ and other classical DLs).

The classical DL $\mathcal{ALC}$ Given disjoint, at most countable sets $\mathsf{N_C}$ and $\mathsf{N_R}$ of *concept* and *role names*, $\mathcal{ALC}$ *concept descriptions* (*concepts* for short) are built from concept names using negation ($\neg C$), conjunction ($C \sqcap D$), and *existential restrictions* ($\exists r.C$), where $r \in \mathsf{N_R}$ and C, D are $\mathcal{ALC}$ concept descriptions. As usual, we define $C \sqcup D := \neg(\neg C \sqcap \neg D)$ (disjunction), $\forall r.C := \neg\exists r.\neg C$ (value restriction) and $\top := A \sqcup \neg A$ (top concept). An *interpretation* $\mathcal{I}$ consists of a non-empty *domain* $\Delta^{\mathcal{I}}$ and a mapping $\cdot^{\mathcal{I}}$ assigning a set $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ to $A \in \mathsf{N_C}$ and a binary relation $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ to $r \in \mathsf{N_R}$. For $d \in \Delta^{\mathcal{I}}$, we define $r^{\mathcal{I}}(d) := \{e \in \Delta^{\mathcal{I}} \mid (d, e) \in r^{\mathcal{I}}\}$. We extend $\cdot^{\mathcal{I}}$ to concepts by $(\neg C)^{\mathcal{I}} := \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$, $(C \sqcap D)^{\mathcal{I}} := C^{\mathcal{I}} \cap D^{\mathcal{I}}$ and $(\exists r.C)^{\mathcal{I}} := \{d \in \Delta^{\mathcal{I}} \mid r^{\mathcal{I}}(d) \cap C^{\mathcal{I}} \neq \emptyset\}$. In this DL, the concept of a person not having a dog as a pet can be written as $\mathsf{Person} \sqcap \forall \mathsf{pet}.\neg\mathsf{Dog}$.

The DL $\mathcal{ALCSCC}$ This DL employs the logic *QFBAPA* [19] to state cardinality constraints on role successors that are more expressive than existential and value restrictions. In QFBAPA, *set terms* are built from *set variables* and the constants $\emptyset$ and $\mathcal{U}$ using intersection $\cap$, union $\cup$ and complement ^{c}. A *QFBAPA formula* is a Boolean combination of *atomic formulae* of the form

$$m_0 + m_1|s_1| + \cdots + m_k|s_k| \leqslant n_0 + n_1|t_1| + \cdots + n_\ell|t_\ell| \tag{1}$$

where each s_i, t_j is a set term and each m_i, n_j is a natural number.[5] A *solution* σ of a QFBAPA formula ϕ assigns a *finite* set $\sigma(\mathcal{U})$ to $\mathcal{U}$, the empty set to $\emptyset$ and subsets of $\sigma(\mathcal{U})$ to set variables such that ϕ is satisfied by σ, in the standard way. Checking if a QFBAPA formula has a solution is an NP-complete problem [19]. The logic QFBAPA$^\infty$ [7] has the same syntax as QFBAPA, but solutions may assign infinite sets to $\mathcal{U}$. Its satisfiability problem is also NP-complete [7].

$\mathcal{ALCSCC}$ extends the syntax of $\mathcal{ALC}$ with the new constructor *role successor restriction* (or succ-*restriction*) $\mathsf{succ}(\mathfrak{con})$, where $\mathfrak{con}$ is an atomic QFBAPA formula with role names and $\mathcal{ALCSCC}$ concept descriptions as set variables [3]. For instance, the concept of all persons that have more daughters than they have dogs as pets can be expressed in $\mathcal{ALCSCC}$ as $\mathsf{succ}(|\mathsf{pet} \cap \mathsf{Dog}| < |\mathsf{child} \cap \mathsf{Female}|)$. Note that existential restrictions $\exists r.C$ are not needed as explicit constructors in this DL since they can be expressed as $\mathsf{succ}(|r \cap C| \geqslant 1)$.

When defining the semantics of $\mathcal{ALCSCC}$, interpretations $\mathcal{I}$ are required in [3] to be *finitely branching*, i.e. such that the set of all role successors $\mathsf{ars}^{\mathcal{I}}(d) := \bigcup_{r \in \mathsf{N_R}} r^{\mathcal{I}}(d)$ is finite, for all $d \in \Delta^{\mathcal{I}}$. Then, each $d \in \Delta^{\mathcal{I}}$ induces a QFBAPA assignment σ_d, where $\sigma_d(\mathcal{U}) := \mathsf{ars}^{\mathcal{I}}(d)$, $\sigma_d(r) := r^{\mathcal{I}}(d)$ for $r \in \mathsf{N_R}$ and $\sigma_d(C) := C^{\mathcal{I}} \cap \mathsf{ars}^{\mathcal{I}}(d)$ for concepts C. The mapping $\cdot^{\mathcal{I}}$ is extended to succ-restrictions by defining $d \in \mathsf{succ}(\mathfrak{con})^{\mathcal{I}}$ iff σ_d is a solution of $\mathfrak{con}$.

The DL $\mathcal{ALCSCC}^\infty$ is defined in [7] with the same syntax as $\mathcal{ALCSCC}$, but in the semantics arbitrary interpretations are allowed. Consequently, the assignment σ_d may be such that $\sigma_d(\mathcal{U})$ is infinite, and thus satisfaction of the constraint $\mathfrak{con}$ by σ_d is evaluated in QFBAPA$^\infty$ rather than QFBAPA.

In the definitions of $\mathcal{ALCSCC}^\infty$ and $\mathcal{ALCSCC}$, we considered two classes of first-order interpretations: the class $\mathbb{C}_{\mathsf{all}}$ of all interpretations and the class $\mathbb{C}_{\mathsf{fb}}$ of finitely branching interpretations. Later on, we will also consider the class $\mathbb{C}_{\mathsf{fin}}$ of all finite interpretations, which is also of interest in DL research [17,23]. Our results on the expressive power will be parameterized with a class $\mathbb{C}$ of interpretations satisfying certain restrictions. Since the syntax of $\mathcal{ALCSCC}^\infty$ and $\mathcal{ALCSCC}$ coincide, we will in the following always talk about $\mathcal{ALCSCC}$ concepts. However, if $\mathbb{C}$ contains interpretations that are not finitely branching, then the semantics uses QFBAPA$^\infty$ rather than QFBAPA.

DLs with concrete domains Following [12,21,14], we use the term *concrete domain* to refer to a relational structure $\mathfrak{D} = (D, \ldots, P^D, \ldots)$ over a non-empty, at most countable relational signature, where D is a non-empty set,

[5] Following [8], we use a streamlined definition of QFBAPA that does not explicitly introduce set constraints and divisibility constraints.

and each predicate P has an associated arity $k_P \in \mathbb{N}$ and is interpreted by a relation $P^D \subseteq D^{k_P}$. An example is the structure $\mathfrak{Q} := (\mathbb{Q}, <, =, >)$ over the rational numbers $\mathbb{Q}$ with standard binary ordering and equality relations. Given a countable set V of variables, a *constraint system* over V is a set $\mathfrak{C}$ of *constraints* $P(v_1, \dots, v_k)$, where $v_1, \dots, v_k \in V$ and P is a k-ary predicate of $\mathfrak{D}$. We denote by $V(\mathfrak{C})$ the set of variables that occur in $\mathfrak{C}$. The constraint system $\mathfrak{C}$ is *satisfiable* if there is a mapping $h\colon V(\mathfrak{C}) \to D$ such that $P(v_1, \dots, v_k) \in \mathfrak{C}$ implies $(h(v_1), \dots, h(v_k)) \in P^D$. The *constraint satisfaction problem* for $\mathfrak{D}$, denoted CSP($\mathfrak{D}$), asks if a given finite constraint system $\mathfrak{C}$ over $\mathfrak{D}$ is satisfiable. The CSP of $\mathfrak{Q}$ is decidable in polynomial time, by reduction to $<$-cycle detection: for example, the system $\{x_1 < x_2,\ x_2 < x_3,\ x_3 < x_1\}$ is unsatisfiable over $\mathfrak{Q}$.

When integrating such a concrete domain into the DL $\mathcal{ALC}$, it needs to satisfy certain restrictions to obtain a decidable DL. Without a TBox, admissibility is required in [12] whereas in the presence of a TBox the stronger ω-admissibility is required in [21,14]. In the context of our investigation of the expressive power of DLs with concrete domains, it is sufficient to assume that negated constraints can be expressed using one or more non-negated ones.

Definition 1. *A structure $\mathfrak{D}$ is* weakly closed under negation (WCUN) *if for all $k \geqslant 1$ and all k-ary relations P of $\mathfrak{D}$ there are k-ary relations $P_1, \dots, P_{n_P}$ such that $(d_1, \dots, d_k) \notin P^D$ iff $(d_1, \dots, d_k) \in \bigcup_{i=1}^{n_P} P_i^D$ for all $d_1, \dots, d_k \in D^k$.*

It is easy to see that both admissible and ω-admissible concrete domains satisfy this property. Examples of ω-admissible, and thus WCUN, concrete domains are Allen's interval algebra, RCC8 and $\mathfrak{Q}$ [21,14]. For example the negated predicate $\neq$ in $\mathfrak{Q}$ is obtained as the union of $<$ and $>$.

To integrate a given concrete domain $\mathfrak{D}$ into $\mathcal{ALC}$, we complement $\mathsf{N_C}$ and $\mathsf{N_R}$ with a finite set $\mathsf{N_F}$ of *feature names* that connect individuals with values in D [12]. A *feature path* p is of the form f or rf with $r \in \mathsf{N_R}$ and $f \in \mathsf{N_F}$. For instance, age is a feature name as well as a feature path, while child age is a feature path including the role name child. The DL $\mathcal{ALC}(\mathfrak{D})$ extends $\mathcal{ALC}$ with *concrete domain restrictions* (or *CD-restrictions*) of the form $\exists p_1, \dots, p_k.P$ and $\forall p_1, \dots, p_k.P$, where p_i are feature paths and P is a k-ary predicate of $\mathfrak{D}$. An interpretation $\mathcal{I}$ assigns to $f \in \mathsf{N_F}$ a *partial* function $f^\mathcal{I}\colon \Delta^\mathcal{I} \rightharpoonup D$. A feature path p is mapped to $p^\mathcal{I} \subseteq \Delta^\mathcal{I} \times D$ by defining[6] $p^\mathcal{I}(d) := \{f^\mathcal{I}(d)\}$ if $p = f$ and $p^\mathcal{I}(d) := \{f^\mathcal{I}(e) \mid e \in r^\mathcal{I}(d)\}$ if $p = rf$. Then we can define

$$(\exists p_1, \dots, p_k.P)^\mathcal{I} := \big\{d \in \Delta^\mathcal{I} \mid \textit{some} \text{ tuple in } p_1^\mathcal{I}(d) \times \dots \times p_k^\mathcal{I}(d) \text{ is in } P^D\big\}$$
$$(\forall p_1, \dots, p_k.P)^\mathcal{I} := \big\{d \in \Delta^\mathcal{I} \mid \textit{every} \text{ tuple in } p_1^\mathcal{I}(d) \times \dots \times p_k^\mathcal{I}(d) \text{ is in } P^D\big\}.$$

For example, one can describe individuals having a child that is younger than one of their pets using $\exists$child age, pet age.$<$.

[6] In a slight abuse of notation, we view $f^\mathcal{I}(d)$ both as a value and as a singleton set.

3 The Expressive Power of $\mathcal{ALCSCC}$

In this section, we first introduce a notion of bisimulation, called Presburger bisimulation, such that $\mathcal{ALCSCC}$ concept descriptions are invariant under such bisimulations, i.e., bisimilar elements belong to the same $\mathcal{ALCSCC}$ concept descriptions. Next, we consider an approximate variant of Presburger bisimulation and show that, while not all $\mathcal{ALCSCC}$ concept descriptions are invariant under this notion, the ones that are expressible in first-order logic are. This shows that there are $\mathcal{ALCSCC}$ concept descriptions that are not expressible in FOL. Finally, we characterize the fragment of $\mathcal{ALCSCC}$ that is first-order definable as the logic $\mathcal{ALCQ}t$, for which successor constraints have a restricted form.

Presburger bisimulation Assume that $\mathsf{N_C}$ and $\mathsf{N_R}$ are finite. We base our definition of Presburger bisimulations on the notion of *safe role types*, which are non-empty subsets of $\mathsf{N_R}$. Intuitively, such a role type stands for the intersection of its elements intersected with the complements of the non-elements. For example, if $\mathsf{N_R} = \{r, s, t\}$, then the safe role type $\{r, s\}$ corresponds to the set term $r \cap s \cap t^c$. More formally, safe role types τ are interpreted in an interpretation $\mathcal{I}$ as the binary relation

$$\tau^{\mathcal{I}} := (\bigcap_{r\in\tau} r^{\mathcal{I}} \setminus (\bigcup_{r\in\mathsf{N_R}\setminus\tau} r^{\mathcal{I}})) \subseteq \bigcup_{r\in\mathsf{N_R}} r^{\mathcal{I}}.$$

The fact that safe role types are non-empty sets of role names ensures the inclusion stated above, i.e., any $\tau^{\mathcal{I}}$ is an $r^{\mathcal{I}}$ successor for at least one role name r, which justifies the name *safe*. Consequently, for all $d \in \Delta^{\mathcal{I}}$, the set $\tau^{\mathcal{I}}(d) := \{e \in \Delta^{\mathcal{I}} \mid (d, e) \in \tau^{\mathcal{I}}\}$ is a subset of $\mathsf{ars}^{\mathcal{I}}(d)$, and every $e \in \mathsf{ars}^{\mathcal{I}}(d)$ belongs to $\tau^{\mathcal{I}}(d)$ for exactly one safe role type τ. The set $\mathsf{N_R}$ must be finite, in order to encode safe role types as well-defined set terms. For $\mathcal{ALCSCC}^{\infty}$ it was shown in [7] that each set term s occurring within a succ-restriction can be rewritten as the disjoint union of terms of the form $\tau \cap C$ where τ is a safe role type and C an $\mathcal{ALCSCC}^{\infty}$ concept [7]. The same also holds for $\mathcal{ALCSCC}$. Following [7], we modify the notion of *counting bisimulation* from [22] by using safe role types in place of role names to obtain Presburger bisimulations (called $\mathcal{ALCQ}t$ bisimulations in [7]).

Definition 2. *Let $\mathsf{N_C}$ and $\mathsf{N_R}$ be finite and $\mathbb{C}$ a class of interpretations. The binary relation $\rho \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{J}}$ is a* Presburger (Pr) bisimulation *between the interpretations $\mathcal{I}$ and $\mathcal{J}$ if for all $A \in \mathsf{N_C}$ and all safe role types τ over $\mathsf{N_R}$ the following properties are satisfied:*

Atomic *$(d, e) \in \rho$ implies $d \in A^{\mathcal{I}}$ iff $e \in A^{\mathcal{J}}$;*
Forth *if $(d, e) \in \rho$ and $D \subseteq \tau^{\mathcal{I}}(d)$ is finite, then there is a set $E \subseteq \tau^{\mathcal{J}}(e)$ such that ρ contains a bijection between D and E;*
Back *if $(d, e) \in \rho$ and $E \subseteq \tau^{\mathcal{J}}(e)$ is finite, then there is a set $D \subseteq \tau^{\mathcal{I}}(d)$ such that ρ contains a bijection between D and E.*

We call $d \in \Delta^{\mathcal{I}}$ *and* $e \in \Delta^{\mathcal{J}}$ Pr bisimilar *if* $(d, e) \in \rho$ *for some* Pr *bisimulation* ρ *between* $\mathcal{I}$ *and* $\mathcal{J}$*. A concept* C *is* $\mathbb{C}$-invariant *under* Pr *bisimulation if* $d \in C^{\mathcal{I}}$ *iff* $e \in C^{\mathcal{J}}$ *holds for all* Pr *bisimilar individuals* $d \in \Delta^{\mathcal{I}}$*,* $e \in \Delta^{\mathcal{J}}$ *with* $\mathcal{I}, \mathcal{J} \in \mathbb{C}$*.*

In [7] we proved that $\mathcal{ALCSCC}^{\infty}$ concepts are $\mathbb{C}_{\mathsf{all}}$-invariant under Pr bisimulation. A very similar proof (by induction on the structure of concept descriptions) can be used to show the corresponding result for $\mathcal{ALCSCC}$, where only finitely branching interpretations are considered.

Theorem 1. *Every* $\mathcal{ALCSCC}$ *concept is* $\mathbb{C}_{\mathsf{fb}}$*-invariant under* Pr *bisimulation.*

Proof. Let $\mathcal{I}, \mathcal{J} \in \mathbb{C}_{\mathsf{fb}}$ and ρ a Pr bisimulation relating $d \in \Delta^{\mathcal{I}}$ and $e \in \Delta^{\mathcal{J}}$. We show by induction on the structure of an $\mathcal{ALCSCC}$ concept C that $d \in C^{\mathcal{I}}$ iff $e \in C^{\mathcal{J}}$ holds. The cases where C is a concept name, a conjunction of concepts or the negation of a concept are similar to the analogous cases in the proof of a corresponding result for $\mathcal{ALC}$ [13], and are omitted.

Thus, we focus on the case $C = \mathsf{succ}(\mathsf{con})$, where we inductively assume that every subconcept of C is $\mathbb{C}_{\mathsf{fb}}$-invariant under Pr bisimulation. Recall that con is of the form (1). By applying distributivity of set intersection over set union, it is easy to show that any set term occurring in con can be written as the disjoint union of set terms of the form $\tau \cap F$ where τ is a safe role type and F is a Boolean combination of concepts to which the induction assumption applies. The reason we can restrict the attention to safe role types here lies in the semantics of $\mathcal{ALCSCC}$, which considers only role successors when evaluating set terms. We provide for every $\mathcal{ALCSCC}$ concept F and safe role type τ over $\mathsf{N_R}$ an injective mapping from $D := \tau^{\mathcal{I}}(d) \cap F^{\mathcal{I}}$ to $E := \tau^{\mathcal{J}}(e) \cap F^{\mathcal{J}}$ and vice versa. This proves that these sets have the same size, and thus that con is evaluated equally w.r.t. d and e. Note that, since $\mathcal{I}$ and $\mathcal{J}$ are finitely branching, the sets D and E are both finite. Overall, this implies that $d \in C^{\mathcal{I}}$ iff $e \in C^{\mathcal{J}}$.

The required injections are obtained as follows. Thanks to the forth property, we find a set $E' \subseteq \tau^{\mathcal{J}}(e)$ such that ρ contains a bijection between D and E'. By our induction hypothesis, the concept F is $\mathbb{C}_{\mathsf{fb}}$-invariant under Pr bisimulation, so we obtain that $E' \subseteq C^{\mathcal{J}}$. Then, $E' \subseteq E$ holds, and the bijection between D and E' is the sought injective mapping from D to E. Using the back property, we similarly prove that there is an injective mapping from E to D.

Together with the other cases, this concludes our proof, and thus we conclude that every $\mathcal{ALCSCC}$ concept is $\mathbb{C}_{\mathsf{fb}}$-invariant under Pr bisimulation. □

Since finite interpretations are finitely branching, this also implies $\mathbb{C}_{\mathsf{fin}}$-invariance of $\mathcal{ALCSCC}$ concepts under Pr bisimulation.

As usual, such invariance results can be employed to prove that a certain DL $\mathcal{L}$ cannot be expressed in $\mathcal{ALCSCC}$. For this, it is sufficient to find an example of an $\mathcal{L}$ concept that is not invariant under Pr bisimulation. In [11], we apply this approach to show that the abstract expressive power [10] of $\mathcal{ALC}(\mathfrak{Q})$ is not contained in that of $\mathcal{ALCSCC}$ on finitely branching interpretations, and that $\mathcal{ALCSCC}^{++}$ [4] is a strict extension of $\mathcal{ALCSCC}$ on finite interpretations.

***$\mathcal{ALCSCC}$* goes beyond FOL** $\mathcal{ALC}$ and many other DLs are fragments of first-order logic (FOL) [15], in the sense that for every concept description C of the given DL there is a FOL formula $\phi(x)$ such that $\phi^{\mathcal{I}} = C^{\mathcal{I}}$ for all interpretations $\mathcal{I}$, where $\phi^{\mathcal{I}} := \{d \in \Delta^{\mathcal{I}} \mid \mathcal{I} \models \phi(d)\}$. This notion of definability of a concept description by an FOL formula in one free variable can be relativized to a class of models $\mathbb{C}$ in an obvious way. $\mathbb{C}$-invariance of an FOL formula in one free variable under a given notion of bisimulation is also defined in an obvious way.

In [7], we have shown that there are $\mathcal{ALCSCC}^\infty$ concepts that are not FOL-definable in this sense w.r.t. $\mathbb{C}_{\mathsf{all}}$. However, since the semantics of $\mathcal{ALCSCC}$ is defined w.r.t. a restricted class of interpretations, this result does not directly transfer to $\mathcal{ALCSCC}$. Our tool for showing non-FOL-definability for $\mathcal{ALCSCC}$ (and incidentally also for $\mathcal{ALCSCC}^\infty$ w.r.t. other classes of interpretations) is a bounded version of Pr bisimulation where one makes only a bounded number ℓ of steps into the interpretation and bounds the cardinalities of the sets considered in the back and forth conditions by a number q. This notion of bisimulation is obtained by adapting the bisimulation-based characterization of modal logic with graded modalities w.r.t. finite models in [25] to our more expressive logic.

Definition 3. *Let $\mathsf{N_C}$ and $\mathsf{N_R}$ be finite and $q, \ell \in \mathbb{N}$. The relation $\rho \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{J}}$ is a* Pr $(q$,0)-bisimulation *between the interpretations $\mathcal{I}$ and $\mathcal{J}$ if it satisfies the (atomic) condition of Definition 2, and it is a* Pr $(q,\ell+1)$-bisimulation *if it is a Pr (q,ℓ)-bisimulation that satisfies the following for all safe role types τ:*

(q,ℓ)-forth *if $(d, e) \in \rho$ and $D \subseteq \tau^{\mathcal{I}}(d)$ with $|D| \leqslant q$, then there are $E \subseteq \tau^{\mathcal{J}}(e)$ and a Pr (q,ℓ)-bisimulation ρ' that contains a bijection between D and E;*
(q,ℓ)-back *if $(d, e) \in \rho$ and $E \subseteq \tau^{\mathcal{J}}(e)$ with $|E| \leqslant q$, then there are $D \subseteq \tau^{\mathcal{I}}(d)$ and a Pr (q,ℓ)-bisimulation ρ' that contains a bijection between D and E.*

The notions of Pr (q,ℓ)-bisimilarity and $\mathbb{C}$-invariance w.r.t. Pr (q,ℓ)-bisimulation are defined similarly to how it was done in Definition 2.

Theorem 1 states that all $\mathcal{ALCSCC}$ concepts are invariant under Pr bisimulation. For Pr (q,ℓ)-bisimulation, this need not hold, as stated in the next theorem.

Theorem 2. *There is an $\mathcal{ALCSCC}$ concept C such that, for all values of q and ℓ, the concept C is not $\mathbb{C}_{\mathsf{fb}}$-invariant under Pr (q,ℓ)-bisimulation.*

Proof. Consider the $\mathcal{ALCSCC}$ concept $C := \mathsf{succ}(|r \cap A| = |r \cap \neg A|)$, which has been used in [7] to show that $\mathcal{ALCSCC}^\infty$ is not a fragment of FOL. For $n, m \in \mathbb{N}$, let $\mathcal{I}_{m,n}$ be the finitely branching interpretation containing individuals d and d_i for $i = 1, \ldots, m + n$, where r is interpreted as the set of tuples (d, d_i) for $i = 1, \ldots, m+n$, every d_i with $i = 1, \ldots, m$ is in A and every other individual is not in A. Given $q \in \mathbb{N}$ we consider $\mathcal{I}_{q,q}$ and $\mathcal{I}_{q,q+1}$, and notice that $d \in \Delta^{\mathcal{I}_{q,q}}$ and $d \in \Delta^{\mathcal{I}_{q,q+1}}$ are Pr (q,ℓ)-bisimilar: the relation mapping $d \in \Delta^{\mathcal{I}_{q,q}}$ to $d \in \Delta^{\mathcal{I}_{q,q+1}}$ and $d_i \in \Delta^{\mathcal{I}_{q,q}}$ to $d_i \in \Delta^{\mathcal{I}_{q,q+1}}$ is a Pr (q,ℓ)-bisimulation for all $\ell \in \mathbb{N}$. However, $d \in C^{\mathcal{I}_{q,q}}$ holds, whereas $d \notin C^{\mathcal{I}_{q,q+1}}$. □

Our goal is now to show that this cannot happen for $\mathcal{ALCSCC}$ concepts that are FOL-definable w.r.t. $\mathbb{C}_{\mathsf{fb}}$ or $\mathbb{C}_{\mathsf{fin}}$ (or more generally a class $\mathbb{C}$ of interpretations satisfying certain closure properties). The proof of this result uses certain locality properties of FOL formulae that are invariant under Pr bisimulation.

Definition 4. *Let $\mathcal{I}$ be an interpretation. The* distance *of d and d' in $\mathcal{I}$ is the smallest value $\ell \in \mathbb{N}$ for which there is a sequence of elements $d_1, \ldots, d_{\ell+1} \in \Delta^{\mathcal{I}}$ where $d_1 = d$, $d_{\ell+1} = d'$ and d_i is a role successor or predecessor of d_{i+1} for $i = 1, \ldots, \ell$, or ∞ if such a number does not exist. The* ℓ-neighborhood $\mathcal{N}_\ell^{\mathcal{I}}[\![d]\!]$ *of d is derived from $\mathcal{I}$ by taking the substructure consisting of all individuals with distance at most ℓ from d.*

The class $\mathbb{C}$ of interpretations is closed under neighborhoods *if $\mathcal{N}_\ell^{\mathcal{I}}[\![d]\!] \in \mathbb{C}$ for all $\mathcal{I} \in \mathbb{C}$, $d \in \Delta^{\mathcal{I}}$ and $\ell \in \mathbb{N}$. The FOL formula $\phi(x)$ is* ℓ-local *w.r.t. $\mathbb{C}$ if for all $\mathcal{I} \in \mathbb{C}$ and all $d \in \Delta^{\mathcal{I}}$ we have that $\mathcal{I} \models \phi(d)$ iff $\mathcal{N}_\ell^{\mathcal{I}}[\![d]\!] \models \phi(d)$.*

Interestingly, there is a close relationship between ℓ-locality of FOL formulae and invariance under finite disjoint union.

Definition 5 (Disjoint union). *Given a finite index set $\mathbb{I}$ and a family of interpretations $(\mathcal{I}_\nu)_{\nu \in \mathbb{I}} \subseteq \mathbb{C}$, their* finite disjoint union *$\mathcal{I}$ is defined by:*

$$\Delta^{\mathcal{I}} := \{(d, \nu) \mid \nu \in \mathbb{I} \text{ and } d \in \Delta^{\mathcal{I}_\nu}\},$$
$$A^{\mathcal{I}} := \{(d, \nu) \mid \nu \in \mathbb{I} \text{ and } d \in A^{\mathcal{I}_\nu}\} \quad \text{for all } A \in \mathsf{N_C},$$
$$r^{\mathcal{I}} := \{((d, \nu), (e, \nu)) \mid \nu \in \mathbb{I} \text{ and } (d, e) \in r^{\mathcal{I}_\nu}\} \quad \text{for all } r \in \mathsf{N_R}.$$

The FOL formula $\phi(x)$ is $\mathbb{C}$-invariant under finite disjoint unions *if, for any finite disjoint union constructed as above, $\mathcal{I}_\nu \models \phi(d)$ iff $\mathcal{I} \models \phi((d, \nu))$ holds for every $\nu \in \mathbb{I}$ and $d \in \Delta^{\mathcal{I}_\nu}$. We say that $\mathbb{C}$ is* closed under finite disjoint unions *if $\mathcal{I}_\nu \in \mathbb{C}$ for all $\nu \in \mathbb{I}$ implies that the disjoint union of $(\mathcal{I}_\nu)_{\nu \in \mathbb{I}}$ also belongs to $\mathbb{C}$ whenever the index set $\mathbb{I}$ is finite.*

By proving that $\rho := \{(d, (d, \nu)) \mid d \in \Delta^{\mathcal{I}_\nu}, \nu \in \mathbb{I}\}$ is a Pr bisimulation, we obtain the following property for formulae that are $\mathbb{C}$-invariant under Pr bisimulation.

Proposition 1. *If the FOL formula $\phi(x)$ is $\mathbb{C}$-invariant under Pr bisimulation, then it is $\mathbb{C}$-invariant under finite disjoint unions.*

By Theorem 1, this implies that FOL formulae that are equivalent to $\mathcal{ALCSCC}$ concepts are $\mathbb{C}_{\mathsf{fb}}$- and $\mathbb{C}_{\mathsf{fin}}$-invariant under disjoint union. Before we can state the crucial lemma from [24], we must introduce one more notation. We call the class $\mathbb{C}$ of interpretations *localizable* if it is closed under both neighborhoods and finite disjoint unions.[7] Note that our classes $\mathbb{C}_{\mathsf{all}}$, $\mathbb{C}_{\mathsf{fb}}$ and $\mathbb{C}_{\mathsf{fin}}$ are localizable.

Lemma 1 ([24]). *If $\mathbb{C}$ is localizable, then any FOL formula $\phi(x)$ of quantifier depth q that is $\mathbb{C}$-invariant under finite disjoint unions is $(2^q - 1)$-local w.r.t. $\mathbb{C}$.*

Combining this lemma with Proposition 1, we can now link ℓ-locality with invariance under Pr bisimulation.

[7] These conditions on $\mathbb{C}$ are not stated explicitly in [24], but are implicitly assumed.

Corollary 1. *If $\mathbb{C}$ is localizable, then any* FOL *formula $\phi(x)$ of quantifier depth q that is $\mathbb{C}$-invariant under* Pr *bisimulation is ℓ-local w.r.t. $\mathbb{C}$ for $\ell := 2^q - 1$.*

Our next goal is now to show that, for FOL formulae, invariance under Pr bisimulation is equivalent to invariance under Pr (q,ℓ)-bisimulation for some $q, \ell \in \mathbb{N}$. Our first step in this direction is the following result for trees, whose proof can be found in [11].

Theorem 3. *If $\mathcal{I}$, $\mathcal{J}$ are trees of depth at most ℓ with roots d, e that are* Pr *(q,ℓ)-bisimilar, then these roots satisfy the same* FOL *formulae $\phi(x)$ of quantifier depth at most q.*

While not all interpretations in a class $\mathbb{C}$ need to be tree-shaped, we show that, for every interpretation in $\mathbb{C}_{\mathsf{all}}$, $\mathbb{C}_{\mathsf{fb}}$ or $\mathbb{C}_{\mathsf{fin}}$, it is possible to find a Pr bisimilar interpretation in this class where the ℓ-neighborhood of a specific individual d is a tree with root d. Normally, this is achieved by unravelling [13], but this may yield an infinite interpretation, and is thus not suitable for our setting, where we are also interested in the class $\mathbb{C}_{\mathsf{fin}}$. Instead, we introduce *partial unravelling* of $\mathcal{I}$, which preserves finiteness and (like unraveling) finite branching. Intuitively, the *ℓ-unravelling* of an interpretation $\mathcal{I}$ at an element $d \in \Delta^{\mathcal{I}}$ applies unraveling up to length ℓ, and then adds a copy of $\mathcal{I}$ at the end. The exact definition of this operation, which is an adaptation of the unravelling operation described in [13], can be found in [11]. Here, we only state two important properties of it.

Proposition 2. *Let $\mathcal{I}^d_\ell$ be the ℓ-unravelling of the interpretation $\mathcal{I}$ at $d \in \Delta^{\mathcal{I}}$, $\langle d \rangle$ the element corresponding to d in $\mathcal{I}^d_\ell$. Then,*

1. *The elements $d \in \Delta^{\mathcal{I}}$ and $\langle d \rangle \in \Delta^{\mathcal{I}^d_\ell}$ are* Pr *bisimilar.*
2. *The ℓ-neighborhood $\mathcal{N}^{\mathcal{I}^d_\ell}_\ell[\![\langle d \rangle]\!]$ of $\langle d \rangle$ in $\mathcal{I}^d_\ell$ is a tree of depth at most ℓ with root $\langle d \rangle$.*

The class $\mathbb{C}$ of interpretations is *closed under partial unravelling* if $\mathcal{I} \in \mathbb{C}$ implies $\mathcal{I}^d_\ell \in \mathbb{C}$ for all $\ell \in \mathbb{N}$. The following result links invariance under Pr bisimulation with invariance under Pr (q,ℓ)-bisimulation for FOL formulae.

Theorem 4. *Let $\mathbb{C}$ be localizable and closed under partial unravelling. For all* FOL *formulae $\phi(x)$, the following are equivalent:*

1. *$\phi(x)$ is $\mathbb{C}$-invariant under* Pr *bisimulation.*
2. *$\phi(x)$ is $\mathbb{C}$-invariant under* Pr *(q,ℓ)-bisimulation for some $q, \ell \in \mathbb{N}$.*

Proof. The implication "2. ⇒ 1." is an immediate consequence of the fact that every Pr bisimulation is also a Pr (q,ℓ)-bisimulation for all $q, \ell \in \mathbb{N}$.

To prove the other direction, we assume 1. and that $\phi(x)$ has quantifier depth q. By Corollary 1 we deduce that $\phi(x)$ is ℓ-local w.r.t. $\mathbb{C}$ for $\ell := 2^q - 1$. Given $\mathcal{I}, \mathcal{J} \in \mathbb{C}$ and $d \in \Delta^{\mathcal{I}}$, $e \in \Delta^{\mathcal{J}}$, we know that the ℓ-unravellings $\mathcal{I}^d_\ell$ and $\mathcal{J}^e_\ell$

and the ℓ-neighborhoods $\mathcal{N}_d := \mathcal{N}_\ell^{\mathcal{I}_\ell^d}[\![\langle d\rangle]\!]$ and $\mathcal{N}_e := \mathcal{N}_\ell^{\mathcal{J}_\ell^e}[\![\langle e\rangle]\!]$ also belong to $\mathbb{C}$. Since $\phi(x)$ is $\mathbb{C}$-invariant under Pr bisimulation and ℓ-local w.r.t. $\mathbb{C}$ we obtain

$$\begin{array}{l} \mathcal{I} \models \phi(d) \text{ iff } \mathcal{I}_\ell^d \models \phi(\langle d\rangle) \text{ iff } \mathcal{N}_d \models \phi(\langle d\rangle) \text{ and} \\ \mathcal{J} \models \phi(e) \text{ iff } \mathcal{J}_\ell^e \models \phi(\langle e\rangle) \text{ iff } \mathcal{N}_e \models \phi(\langle e\rangle). \end{array} \quad \text{(by Proposition 2)}$$

If ρ is a Pr (q,ℓ)-bisimulation with $(d,e) \in \rho$, then combining this relation with the Pr bisimulations linking d and $\langle d\rangle$ and e and $\langle e\rangle$ shows that there is a Pr (q,ℓ)-bisimulation ρ' between $\mathcal{I}_\ell^d$ and $\mathcal{I}_\ell^e$ with $(\langle d\rangle, \langle e\rangle) \in \rho'$. Since such a bisimulation looks only ℓ steps into the interpretation, the restriction of ρ' to the respective ℓ-neighborhoods $\mathcal{N}_d$ and $\mathcal{N}_e$ is also a Pr (q,ℓ)-bisimulation. Proposition 2 says that these neighborhoods are trees of depth at most ℓ, and thus we can apply Theorem 3 to obtain $\mathcal{N}_d \models \phi(\langle d\rangle)$ iff $\mathcal{N}_e \models \phi(\langle e\rangle)$. □

Together with Theorem 2, this yields the desired non-definability results since the classes $\mathbb{C}_{\mathsf{all}}$, $\mathbb{C}_{\mathsf{fb}}$, and $\mathbb{C}_{\mathsf{fin}}$ are localizable and closed under partial unravelling.

Corollary 2. *Let $\mathbb{C}$ be localizable and closed under partial unravelling. Then there are $\mathcal{ALCSCC}$ concepts that are not* FOL*-definable w.r.t. $\mathbb{C}$.*

The first-order fragment of $\mathcal{ALCSCC}$. In [7], we have established that the FOL-definable subset of $\mathcal{ALCSCC}^\infty$ corresponds to the DL $\mathcal{ALCQ}t$. This DL can be seen both as the extension of $\mathcal{ALCQ}$ where safe role types instead of just role names can be used in qualified number restrictions, and as the restriction of $\mathcal{ALCSCC}$ where only successor restrictions of the form $\mathsf{succ}(|\tau \cap C| \geqslant q)$ are available, where τ is a safe role type, $q \in \mathbb{N}$, and C is an $\mathcal{ALCQ}t$ concept. To make the relationship to qualified number restrictions clear, we write such successor restrictions as $(\geqslant q\, \tau.C)$, and call them qualified number restrictions. Saying that this result was proved in [7] for $\mathcal{ALCSCC}^\infty$ means that it was shown w.r.t. the class $\mathbb{C}_{\mathsf{all}}$. In the following we prove that it also holds for the classes $\mathbb{C}_{\mathsf{fb}}$ and $\mathbb{C}_{\mathsf{fin}}$.

It is easy to see that every $\mathcal{ALCQ}t$ concept can be translated into an equivalent FOL formula with one free variable, and thus $\mathcal{ALCQ}t$ is a FOL-definable fragment of $\mathcal{ALCSCC}$. We will show that all FOL-definable concepts of $\mathcal{ALCSCC}$ are equivalent to one in $\mathcal{ALCQ}t$. We define the *depth* of an $\mathcal{ALCQ}t$ concept to be the maximal nesting of qualified number restrictions and the *breadth* to be the maximal number occurring in a qualified number restriction. With $\mathcal{ALCQ}t_{q,\ell}$ we denote the set of $\mathcal{ALCQ}t$ concepts of depth at most ℓ and breadth at most q. The following results for $\mathcal{ALCQ}t_{q,\ell}$ are established in [11].

Proposition 3. *Let $\mathbb{C}$ be a class of interpretations, $q, \ell \in \mathbb{N}$, and assume that $\mathsf{N_C}$ and $\mathsf{N_R}$ are finite. Then the following holds:*

1. *Every $\mathcal{ALCQ}t_{q,\ell}$ concept is $\mathbb{C}$-invariant under Pr (q,ℓ)-bisimulation.*
2. *Up to $\mathbb{C}$-equivalence, there are only finitely many $\mathcal{ALCQ}t_{q,\ell}$ concepts.*
3. *For every $\mathcal{I} \in \mathbb{C}$ and $d \in \Delta^\mathcal{I}$ there is an $\mathcal{ALCQ}t_{q,\ell}$ concept $\mathsf{Bisim}_\ell^q[d]$ such that $d \in \mathsf{Bisim}_\ell^q[d]^\mathcal{I}$ and $e \in \mathsf{Bisim}_\ell^q[d]^\mathcal{J}$ for an interpretation $\mathcal{J} \in \mathbb{C}$ and $d \in \Delta^\mathcal{J}$ implies that d and e are (q,ℓ)-bisimilar.*

Combining these results with Theorems 1 and 4, we obtain the following characterization of the FOL fragment on $\mathcal{ALCSCC}$.

Theorem 5. *Let $\mathbb{C}$ be localizable and closed under partial unravelling and $\mathsf{N_C}$, $\mathsf{N_R}$ be finite. For all FOL formulae $\phi(x)$, the following are equivalent:*

1. *$\phi(x)$ is $\mathbb{C}$-equivalent to some $\mathcal{ALCSCC}$ concept.*
2. *$\phi(x)$ is $\mathbb{C}$-invariant under Pr bisimulation.*
3. *$\phi(x)$ is $\mathbb{C}$-invariant under Pr (q,ℓ)-bisimulation for some $q, \ell \in \mathbb{N}$.*
4. *$\phi(x)$ is $\mathbb{C}$-equivalent to some $\mathcal{ALCQ}t$ concept.*

Proof. That 1. implies 2. follows from Theorem 1 and the equivalence between 2. and 3. is stated in Theorem 4. In addition, 4. trivially implies 1.

Thus, it is sufficient to show that 3. implies 4. To this purpose, we define $C_\phi := \bigsqcup\{\mathsf{Bisim}^q_\ell[d] \mid \mathcal{I} \in \mathbb{C}, d \in \Delta^\mathcal{I} \text{ and } \mathcal{I} \models \phi(d)\}$. By 2. of Proposition 3, this disjunction is finite (up to equivalence), and thus C_ϕ is a well-formed $\mathcal{ALCQ}t_{q,\ell}$ concept. First, assume that $\mathcal{I} \models \phi(d)$ with $\mathcal{I} \in \mathbb{C}$ and $d \in \Delta^\mathcal{I}$. Then, $d \in C_\phi^\mathcal{I}$ trivially follows from the fact that $\mathsf{Bisim}^q_\ell[d]$ occurs as a disjunct in C_ϕ.

Conversely, if $d \in C_\phi^\mathcal{I}$, then $d \in (\mathsf{Bisim}^q_\ell[e])^\mathcal{I}$ for some $\mathcal{J} \in \mathbb{C}$ and $e \in \Delta^\mathcal{J}$ such that $\mathcal{J} \models \phi(e)$. By 3. of Proposition 3, this implies that d and e are Pr (q,ℓ)-bisimilar. Hence, 3. of the present proposition implies that $\mathcal{I} \models \phi(d)$. Thus, we have shown that $\phi(x)$ and C_ϕ are $\mathbb{C}$-equivalent. □

Recall that the classes $\mathbb{C}_{\mathsf{all}}$, $\mathbb{C}_{\mathsf{fb}}$ or $\mathbb{C}_{\mathsf{fin}}$ satisfy the assumptions of Theorem 5.

4 The Expressive Power of DLs with Concrete Domains

In [9,10] we have investigated the *abstract expressive power* of DLs with concrete domains, which only considers the abstract part of interpretations, i.e., ignores the values assigned to features. This allowed us to compare classical logics like $\mathcal{ALC}$ and FOL with DLs with concrete domains. Here, we want to compare extensions of $\mathcal{ALC}$ with different concrete domains using an appropriate notion of bisimulation, called $\mathfrak{D}$ bisimulation if $\mathfrak{D}$ is the concrete domain under consideration, and characterize $\mathcal{ALC}(\mathfrak{D})$ as the fragment of $\mathsf{FOL}(\mathfrak{D})$ that is invariant under $\mathfrak{D}$ bisimulation. The employed notion of bisimulation is the one for $\mathcal{ALC}$ (see, e.g., [13]) extended with an additional clause that deals with feature values. As in the previous section, we show our results not only for the class of all interpretations, but also for the restrictions to finitely branching and finite ones.

Definition 6. *Let $\mathfrak{D}$ be a concrete domain and $\mathcal{I}$, $\mathcal{J}$ interpretations of $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_F}$ that assign elements of $\mathfrak{D}$ to features from $\mathsf{N_F}$. The relation $\rho \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{J}$ is a $\mathfrak{D}$* bisimulation *between $\mathcal{I}$ and $\mathcal{J}$ if for all $A \in \mathsf{N_C}$, all $r \in \mathsf{N_R}$, all k-ary relations P of $\mathfrak{D}$, and all feature paths $p_1, \ldots, p_k$ over $\mathsf{N_R}$ and $\mathsf{N_F}$:*

atomic *if $(d,e) \in \rho$ then $d \in A^\mathcal{I}$ iff $e \in A^\mathcal{J}$;*
forth *if $(d,e) \in \rho$ and $d' \in r^\mathcal{I}(d)$, then there is $e' \in r^\mathcal{J}(e)$ such that $(d',e') \in \rho$;*
back *if $(d,e) \in \rho$ and $e' \in r^\mathcal{J}(e)$, then there is $d' \in r^\mathcal{I}(d)$ such that $(d',e') \in \rho$.*

features *if* $(d, e) \in \rho$, *then there is* $(v_1, \ldots, v_k) \in P^D$ *with* $v_1 \in p_1^{\mathcal{I}}(d)$, $\ldots$, $v_k \in p_k^{\mathcal{I}}(d)$ *iff there is* $(w_1, \ldots, w_k) \in P^D$ *with* $w_1 \in p_1^{\mathcal{J}}(e)$, $\ldots$, $w_k \in p_k^{\mathcal{J}}(e)$.

Bisimilarity between individuals and $\mathbb{C}$*-invariance w.r.t.* $\mathfrak{D}$ *bisimulation are defined similarly to how it was done in Definition 2 w.r.t.* Pr *bisimulation.*

A result analogous to Theorem 1 holds for $\mathcal{ALC}(\mathfrak{D})$ concepts if the concrete domain $\mathfrak{D}$ is weakly closed under negation.

Theorem 6. *If* $\mathfrak{D}$ *is* WCUN *and* $\mathbb{C}$ *is a class of interpretations of* $\mathsf{N_C}$, $\mathsf{N_R}$ *and* $\mathsf{N_F}$ *that assign elements of* $\mathfrak{D}$ *to features from* $\mathsf{N_F}$, *then every* $\mathcal{ALC}(\mathfrak{D})$ *concept is* $\mathbb{C}$*-invariant under* $\mathfrak{D}$ *bisimulation.*

Proof. The proof by structural induction on the concept C proceeds like the one for $\mathcal{ALC}$ in [13], except for the cases where C is a CD-restriction. We only consider these cases explicitly here. Thus, let ρ be a $\mathfrak{D}$ bisimulation between $\mathcal{I}$ and $\mathcal{J}$ with $(d, e) \in \rho$. We show that d and e satisfy the same CD-restrictions.

If $C := \exists p_1, \ldots, p_k.P$ then $d \in C^{\mathcal{I}}$ implies the existence of $v_1 \in p_1^{\mathcal{I}}(d), \ldots, v_k \in p_k^{\mathcal{I}}(d)$ such that $(v_1, \ldots, v_k) \in P^D$. Since ρ satisfies features, there must be $w_1 \in p_1^{\mathcal{J}}(e)$, $\ldots$, $w_k \in p_k^{\mathcal{J}}(e)$ such that $(w_1, \ldots, w_k) \in P^D$, hence $e \in C^{\mathcal{J}}$. Similarly, we can show that $e \in C^{\mathcal{J}}$ implies $d \in C^{\mathcal{I}}$.

If $C := \forall p_1, \ldots, p_k.P$, then $d \in C^{\mathcal{I}}$ implies that $(v_1, \ldots, v_k) \in P^D$ for all values $v_1 \in p_1^{\mathcal{I}}(d)$, $\ldots$, $v_k \in p_k^{\mathcal{I}}(d)$. Since $\mathfrak{D}$ is WCUN, this is the case iff there are relations $P_1, \ldots, P_{n_P}$ of $\mathfrak{D}$ such that $(v_1, \ldots, v_k) \notin P_i^D$ for $i = 1, \ldots, n_P$. Using the features condition of ρ, we deduce that $(w_1, \ldots, w_k) \notin P_i^D$ for all $w_1 \in p_1^{\mathcal{J}}(e)$, $\ldots$, $w_k \in p_k^{\mathcal{J}}(e)$ and $i = 1, \ldots, n_P$. By WCUN it follows that $(w_1, \ldots, w_k) \in P^D$, and we conclude that $e \in C^{\mathcal{J}}$. The proof of the other direction is symmetric. □

A non-expressivity result. We can use the notion of $\mathfrak{D}$ bisimulation to show that $\mathcal{ALC}(\mathfrak{D})$ cannot express certain concepts of the DL $\mathcal{ALC}(\mathfrak{D}')$, where $\mathfrak{D}'$ has the same domain set as $\mathfrak{D}$, but different relations. Coming back to the example in the introduction, we compare the expressive power of $\mathfrak{Q}_{+1}$ and $\mathfrak{Q}_{+2}$, both having domain set $\mathbb{Q}$, where the former has a binary relation $+_1$ relating $q \in \mathbb{Q}$ and $q + 1$ (and the complementary relation $\neq_{+_1}$) and the latter has a binary relation $+_2$ relating q and $q + 2$ (and the complementary relation $\neq_{+_2}$).

These two DLs have the same *abstract expressive power*. In fact, we can interchange CD-restrictions using relations $+_1$ and $\neq_{+_1}$ with restrictions of the same kind (existential or universal) using relations $+_2$ and $\neq_{+_2}$. Abstract models of a concept in one of these DLs are then the same as of the corresponding concept in the other DL: in one direction, we just double the feature values, and in the other we halve them. Nevertheless, we can show that their *concrete expressive power*, which takes the feature values into account, is incomparable.

Proposition 4. *Let* $\mathbb{C}$ *be* $\mathbb{C}_{\mathsf{all}}$, $\mathbb{C}_{\mathsf{fb}}$, *or* $\mathbb{C}_{\mathsf{fin}}$. *There are* $\mathcal{ALC}(\mathfrak{Q}_{+_1})$ *concepts that are not* $\mathbb{C}$*-equivalent to any* $\mathcal{ALC}(\mathfrak{Q}_{+_2})$ *concept (and vice versa).*

Proof. First, consider the $\mathcal{ALC}(\mathfrak{Q}_{+_1})$ concept $C := \exists rf, rf.{+_1}$ and assume by contradiction that it is $\mathbb{C}_{\mathsf{all}}$-equivalent to some $\mathcal{ALC}(\mathfrak{Q}_{+_2})$ concept D. Let us

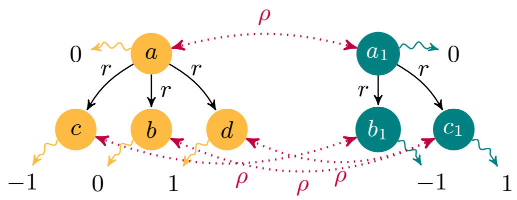

Fig. 1. A $\mathfrak{Q}_{+_2}$ bisimulation ρ between $\mathcal{I}$ (left) and $\mathcal{J}$ (right).

consider the interpretations $\mathcal{I}$ and $\mathcal{J}$ depicted in Figure 1. Then, $a \in C^{\mathcal{I}}$ and by equivalence $a \in D^{\mathcal{I}}$, while $a_1 \notin C^{\mathcal{J}}$ and so $a_1 \notin D^{\mathcal{J}}$ by equivalence. This leads to a contradiction, since the relation ρ between $\mathcal{I}$ and $\mathcal{J}$ is a $\mathfrak{Q}_{+_2}$ bisimulation relating a and a_1, and by Theorem 6 this means that $a \in D^{\mathcal{I}}$ iff $a_1 \in D^{\mathcal{J}}$. Therefore, we conclude that C and D cannot be equivalent w.r.t. any class of interpretations that contains the two interpretations of Figure 1. Vice versa, we can show with a similar argument that $\exists rf, rf.+_2$ cannot be expressed in $\mathcal{ALC}(\mathfrak{Q}_{+_1})$, but this requires slightly different interpretations. □

FOL with concrete domains and $\mathcal{ALC}(\mathfrak{D})$. Since we are interested in characterizing the concrete expressive power of $\mathcal{ALC}(\mathfrak{D})$, which takes the feature values into account, we cannot compare $\mathcal{ALC}(\mathfrak{D})$ with FOL, where no such values are available. Instead, we consider the extension $\mathsf{FOL}(\mathfrak{D})$ of FOL with the concrete domain $\mathfrak{D}$ as introduced in [9,10]. The logic $\mathsf{FOL}(\mathfrak{D})$ is obtained from FOL by adding *definedness predicates* $\mathrm{Def}(f)(t)$ with $f \in \mathsf{N_F}$ and t a first-order term, and *concrete domain predicates* $P(f_1, \ldots, f_k)(t_1, \ldots, t_k)$ where P is a k-ary relation of $\mathfrak{D}$, each t_i is a first-order term and $f_i \in \mathsf{N_F}$ for $i = 1, \ldots, k$.

The semantics of $\mathsf{FOL}(\mathfrak{D})$ formulae is defined in terms of first-order interpretations $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ that additionally assign partial functions $f^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \rightharpoonup D$ to $f \in \mathsf{N_F}$. The semantics of terms, Boolean connectives and first-order quantifiers is defined as usual. Denoting the interpretation of a first-order term t w.r.t $\mathcal{I}$ and a variable assignment w as $t^{\mathcal{I},w}$, the new predicates are interpreted as follows:

- $\mathcal{I} \models \mathrm{Def}(f)(t^{\mathcal{I},w})$ if $f^{\mathcal{I}}(t^{\mathcal{I},w})$ is defined, and
- $\mathcal{I} \models P(f_1, \ldots, f_k)(t_1^{\mathcal{I},w}, \ldots, t_k^{\mathcal{I},w})$ if $(f_1^{\mathcal{I}}(t_1^{\mathcal{I},w}), \ldots, f_k^{\mathcal{I}}(t_k^{\mathcal{I},w})) \in P^D$.

Note that if $(f_1^{\mathcal{I}}(t_1^{\mathcal{I},w}), \ldots, f_k^{\mathcal{I}}(t_k^{\mathcal{I},w})) \in P^D$ then each $f_i^{\mathcal{I}}(t_i^{\mathcal{I},w})$ must be defined.

It is easy to see (and explicitly shown in [9,10]) that $\mathcal{ALC}(\mathfrak{D})$ is a fragment of $\mathsf{FOL}(\mathfrak{D})$. Our goal is to prove that it is the fragment of $\mathsf{FOL}(\mathfrak{D})$ that is invariant under $\mathfrak{D}$ bisimulation, not just for the class of all interpretations, but also for finite and finitely branching interpretations. For this, we use an approach that is very similar to the one employed in Section 3. Recall that Lemma 1 turned out to be an important model-theoretic tool in that approach since it provided us with locality results for FOL formulae expressing $\mathcal{ALCSCC}$ concepts. The corresponding result also holds for $\mathsf{FOL}(\mathfrak{D})$. Note that the notions of *finite disjoint union* and the corresponding $\mathbb{C}$-invariance w.r.t. classes $\mathbb{C}$ of interpretations of $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_F}$ are obtained by extending Definition 5 to account for feature names in the obvious way. For interpretations of $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_F}$ we define ℓ-neighborhoods by using the same notion of distance employed in Definition 4. This means that the distance of two individuals is not determined by concrete domain predicates, but only by role names. The notions of ℓ-locality of a $\mathsf{FOL}(\mathfrak{D})$ formula and of $\mathbb{C}$-invariance w.r.t. classes $\mathbb{C}$ of interpretations of $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_F}$ are obtained by extending Definition 4 using this notion of neighborhood. In particular, the extension of the classes $\mathbb{C}_{\mathsf{all}}$, $\mathbb{C}_{\mathsf{fb}}$, and $\mathbb{C}_{\mathsf{fin}}$ to interpretations taking feature names into account are defined in the obvious way, and these classes are localizable.

Lemma 2. *If $\mathbb{C}$ is localizable, then a $\mathsf{FOL}(\mathfrak{D})$ formula $\phi(x)$ of quantifier depth q that is $\mathbb{C}$-invariant under disjoint unions is ℓ-local w.r.t. $\mathbb{C}$ for $\ell := 2^q - 1$.*

This result can be proved similarly to Lemma 1, by employing the translation of $\mathsf{FOL}(\mathfrak{D})$ formulae and $\mathsf{FOL}(\mathfrak{D})$ interpretations into FOL formulae and FOL interpretations introduced in [9,10] (see [11] for details).

In the following, we assume that the concrete domain $\mathfrak{D}$ is WCUN and has finitely many relations; both conditions are always satisfied by ω-admissible concrete domains [21,14]. Following the approach employed in the previous section, we introduce a bounded version of $\mathfrak{D}$ bisimulation, where now only the depth is bounded since there are no cardinality constraints.

Definition 7. *Let $\mathcal{I}$, $\mathcal{J}$ be interpretations of $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_F}$ and $\ell \in \mathbb{N}$. The relation $\rho \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{J}}$ is a $\mathfrak{D}$* 0-bisimulation *if ρ satisfies the* **atomic** *condition of Definition 6 and for all k-ary relations P of $\mathfrak{D}$ and $f_1, \dots, f_k \in \mathsf{N_F}$:*

values *if $(d, e) \in \rho$ then $(f_1^{\mathcal{I}}(d), \dots, f_k^{\mathcal{I}}(d)) \in P^D$ iff $(f_1^{\mathcal{J}}(e), \dots, f_k^{\mathcal{J}}(e)) \in P^D$.*

The relation ρ is a $\mathfrak{D}$ $(\ell + 1)$-bisimulation *if it is a $\mathfrak{D}$ ℓ-bisimulation that additionally satisfies the* **features** *conditions of Definition 6, and for all $r \in \mathsf{N_R}$ the following are satisfied:*

ℓ-**forth** *if $(d, e) \in \rho$ and d' is an r-successor of d, then there exist an r-successor e' of e and a $\mathfrak{D}$ ℓ-bisimulation ρ' such that $(d', e') \in \rho'$;*
ℓ-**back** *if $(d, e) \in \rho$ and e' is an r-successor of e, then there exist an r-successor d' of d and a $\mathfrak{D}$ ℓ-bisimulation ρ' such that $(d', e') \in \rho'$.*

The notions of bisimilarity and $\mathbb{C}$-invariance w.r.t. $\mathfrak{D}$ ℓ-bisimulation are defined similarly to how it was done in Definition 2.

In [11] we show that, under the assumption that the concrete domain $\mathfrak{D}$ is WCUN and has finitely many relations, results analogous to Proposition 1, Corollary 1, Theorem 3, Proposition 2, Theorem 4, and Proposition 3 also hold for $\mathsf{FOL}(\mathfrak{D})$ and $\mathcal{ALC}(\mathfrak{D})$, where $\mathcal{ALC}(\mathfrak{D})$ plays both the role of $\mathcal{ALCSCC}$ and of $\mathcal{ALCQ}t$. Similarly to the proof of Theorem 4, these results can be combined to show the following characterization of $\mathcal{ALC}(\mathfrak{D})$ as the fragment of $\mathsf{FOL}(\mathfrak{D})$ that is invariant under $\mathfrak{D}$ bisimulation.

Theorem 7. *Let $\mathbb{C}$ be localizable and closed under partial unravelling, $\mathfrak{D}$ be WCUN and have finitely many relations, and $\mathsf{N_C}$, $\mathsf{N_R}$, $\mathsf{N_F}$ be finite. Then the following are equivalent for all $\mathsf{FOL}(\mathfrak{D})$ formulae $\phi(x)$:*

1. *$\phi(x)$ is $\mathbb{C}$-invariant under $\mathfrak{D}$ bisimulation.*
2. *$\phi(x)$ is $\mathbb{C}$-invariant under $\mathfrak{D}$ ℓ-bisimulation for some $\ell \in \mathbb{N}$.*
3. *$\phi(x)$ is equivalent to an $\mathcal{ALC}(\mathfrak{D})$ concept.*

Recall that the classes $\mathbb{C}_{\mathsf{all}}$, $\mathbb{C}_{\mathsf{fb}}$ and $\mathbb{C}_{\mathsf{fin}}$ satisfy the assumptions of Theorem 7. We further remark that, in contrast to the case of $\mathcal{ALCSCC}$, where there are concepts that are not FOL-definable, every $\mathcal{ALC}(\mathfrak{D})$ concept is $\mathsf{FOL}(\mathfrak{D})$-definable.

5 Conclusion

We have investigated the expressive power of concept description languages that allow their users to employ numerical constraints when defining concepts in two orthogonal ways. In contrast to our previous results on the expressive power of such languages [7,8,9,10], the approach employed here also works for restricted classes of interpretations such as finitely branching or finite ones. In [8], we have characterized the expressive power of TBoxes and cardinality boxes of $\mathcal{ALCSCC}^{\infty}$ (where arbitrary interpretations are considered) using global Pr bisimulations. It is at the moment not clear to us whether the results obtained there can be extended to the restricted classes of interpretations considered in the present paper. Another interesting topic for future research is to study the expressive power of $\mathcal{ALCOSCC}(\mathfrak{D})$, a joint extension of both $\mathcal{ALCSCC}$ and $\mathcal{ALC}(\mathfrak{D})$, whose complexity has recently been analyzed in [5]. The DLs $\mathcal{ALCSCC}$ and $\mathcal{ALC}(\mathfrak{D})$ are closed under all Boolean operations, whereas Kurtonina and de Rijke [20] characterize the expressive power of sub-Boolean fragments of $\mathcal{ALC}$. It would be interesting to see whether their results can be extended to the corresponding fragments of $\mathcal{ALCSCC}$ and $\mathcal{ALC}(\mathfrak{D})$. Like most bisimulation-based characterizations of the expressive power of logics, we assume here that the concept D in the DL $\mathcal{L}_2$ expressing the concept C in the DL $\mathcal{L}_1$ must be built over the same signature as C, i.e., no auxiliary symbols may be used. It would again be interesting to see whether inexpressivity results such as the one in Proposition 4 still hold if the use of auxiliary symbols is allowed, as for instance in [1,2]. In this context, work on conservative extensions could become relevant [16].

Acknowledgments. This work was partially supported by DFG grant 389792660 as part of TRR 248 – CPEC, by the German Federal Ministry of Education and Research (BMBF, SCADS22B) and the Saxon State Ministry for Science, Culture and Tourism (SMWK) by funding the competence center for Big Data and AI "ScaDS.AI Dresden/Leipzig".

References

1. Baader, F.: A formal definition for the expressive power of knowledge representation languages. In: 9th European Conference on Artificial Intelligence, ECAI 1990, Proceedings. pp. 53–58 (1990)

2. Baader, F.: A formal definition for the expressive power of terminological knowledge representation languages. J. Log. Comput. **6**(1), 33–54 (1996). https://doi.org/10.1093/LOGCOM/6.1.33
3. Baader, F.: A new description logic with set constraints and cardinality constraints on role successors. In: Dixon, C., Finger, M. (eds.) Proc. of the 11th Int. Symposium on Frontiers of Combining Systems (FroCoS'17). Lecture Notes in Computer Science, vol. 10483, pp. 43–59. Springer-Verlag, Brasília, Brazil (2017). https://doi.org/10.1007/978-3-319-66167-4_3
4. Baader, F., Bednarczyk, B., Rudolph, S.: Satisfiability and query answering in description logics with global and local cardinality constraints. In: ECAI 2020 – 24th European Conference on Artificial Intelligence. pp. 616–623. IOS Press (2020). https://doi.org/10.3233/FAIA200146
5. Baader, F., Borgwardt, S., De Bortoli, F., Koopmann, P.: Concrete domains meet expressive cardinality restrictions in description logics. In: Barrett, C., Waldmann, U. (eds.) CADE-30: 30th International Conference on Automated Deduction, Proceedings. Lecture Notes in Computer Science, Springer (2025), to appear.
6. Baader, F., Calvanese, D., McGuinness, D.L., Nardi, D., Patel-Schneider, P.F. (eds.): The Description Logic Handbook: Theory, Implementation, and Applications. Cambridge University Press (2003). https://doi.org/10.1017/CBO9780511711787
7. Baader, F., De Bortoli, F.: On the expressive power of description logics with cardinality constraints on finite and infinite sets. In: Herzig, A., Popescu, A. (eds.) Frontiers of Combining Systems – 12th International Symposium, FroCoS 2019, Proceedings. Lecture Notes in Computer Science, vol. 11715, pp. 203–219. Springer (2019). https://doi.org/10.1007/978-3-030-29007-8_12
8. Baader, F., De Bortoli, F.: Description logics that count, and what they can and cannot count. In: Kovacs, L., Korovin, K., Reger, G. (eds.) ANDREI-60. Automated New-era Deductive Reasoning Event in Iberia. EPiC Series in Computing, vol. 68, pp. 1–25. EasyChair (2020). https://doi.org/https://doi.org/10.29007/ltzn
9. Baader, F., De Bortoli, F.: The abstract expressive power of first-order and description logics with concrete domains. In: Proceedings of the 39th ACM/SIGAPP Symposium on Applied Computing, SAC '24. pp. 754–761. Association for Computing Machinery, New York, NY, USA (2024). https://doi.org/10.1145/3605098.3635984
10. Baader, F., De Bortoli, F.: Logics with concrete domains: First-order properties, abstract expressive power, and (un)decidability. SIGAPP Appl. Comput. Rev. **24**(3), 5–17 (Oct 2024). https://doi.org/10.1145/3699839.3699840
11. Baader, F., De Bortoli, F.: The Expressive Power of Description Logics with Numerical Constraints over Restricted Classes of Models (Extended Version). LTCS-Report 25-03, Chair of Automata Theory, Institute of Theoretical Computer Science, Technische Universität Dresden, Dresden, Germany (2025), https://doi.org/10.25368/2025.126
12. Baader, F., Hanschke, P.: A scheme for integrating concrete domains into concept languages. In: Mylopoulos, J., Reiter, R. (eds.) Proceedings of the 12th International Joint Conference on Artificial Intelligence, IJCAI 1991. pp. 452–457. Morgan Kaufmann (1991), http://ijcai.org/Proceedings/91-1/Papers/070.pdf
13. Baader, F., Horrocks, I., Lutz, C., Sattler, U.: An Introduction to Description Logic. Cambridge University Press (2017). https://doi.org/10.1017/9781139025355
14. Baader, F., Rydval, J.: Using model theory to find decidable and tractable description logics with concrete domains. Journal of Automated Reasoning **66**(3), 357–407 (2022). https://doi.org/10.1007/s10817-022-09626-2

15. Borgida, A.: On the relative expressiveness of description logics and predicate logics. Artif. Intell. **82**(1-2), 353–367 (1996). https://doi.org/10.1016/0004-3702(96)00004-5
16. Botoeva, E., Konev, B., Lutz, C., Ryzhikov, V., Wolter, F., Zakharyaschev, M.: Inseparability and conservative extensions of description logic ontologies: A survey. In: Pan, J.Z., Calvanese, D., Eiter, T., Horrocks, I., Kifer, M., Lin, F., Zhao, Y. (eds.) Reasoning Web: Logical Foundation of Knowledge Graph Construction and Query Answering – 12th International Summer School 2016, Tutorial Lectures. Lecture Notes in Computer Science, vol. 9885, pp. 27–89. Springer (2016). https://doi.org/10.1007/978-3-319-49493-7_2
17. Calvanese, D.: Finite model reasoning in description logics. In: Aiello, L.C., Doyle, J., Shapiro, S.C. (eds.) Proceedings of the Fifth International Conference on Principles of Knowledge Representation and Reasoning (KR'96). pp. 292–303. Morgan Kaufmann (1996)
18. Hollunder, B., Baader, F.: Qualifying number restrictions in concept languages. In: Allen, J.F., Fikes, R., Sandewall, E. (eds.) Proceedings of the 2nd International Conference on Principles of Knowledge Representation and Reasoning (KR'91). pp. 335–346. Morgan Kaufmann (1991)
19. Kuncak, V., Rinard, M.: Towards efficient satisfiability checking for Boolean algebra with Presburger arithmetic. In: Pfenning, F. (ed.) Automated Deduction – 28th International Conference on Automated Deduction – CADE-21, Proceedings. Lecture Notes in Computer Science, vol. 4603, pp. 215–230. Springer Berlin Heidelberg, Berlin, Heidelberg (2007). https://doi.org/10.1007/978-3-540-73595-3_15
20. Kurtonina, N., de Rijke, M.: Expressiveness of concept expressions in first-order description logics. Artif. Intell. **107**(2), 303–333 (1999). https://doi.org/10.1016/S0004-3702(98)00109-X
21. Lutz, C., Miličić, M.: A tableau algorithm for description logics with concrete domains and general TBoxes. Journal of Automated Reasoning **38**(1), 227–259 (2007). https://doi.org/10.1007/s10817-006-9049-7
22. Lutz, C., Piro, R., Wolter, F.: Description logic TBoxes: Model-theoretic characterizations and rewritability. In: Proc. of the 22nd Int. Joint Conf. on Artificial Intelligence (IJCAI'11). pp. 983–988. IJCAI/AAAI (2011). https://doi.org/10.5591/978-1-57735-516-8/IJCAI11-169
23. Lutz, C., Sattler, U., Tendera, L.: The complexity of finite model reasoning in description logics. Inf. Comput. **199**(1-2), 132–171 (2005). https://doi.org/10.1016/J.IC.2004.11.002
24. Otto, M.: Model theoretic methods for fragments of FO and special classes of (finite) structures. In: Esparza, J., Michaux, C., Steinhorn, C. (eds.) Finite and Algorithmic Model Theory, pp. 271–338. Cambridge University Press, Cambridge (2011). https://doi.org/10.1017/CBO9780511974960.007
25. Otto, M.: Graded modal logic and counting bisimulation (Jul 2023), http://arxiv.org/abs/1910.00039
26. de Rijke, M.: A note on graded modal logic. Stud Logica **64**(2), 271–283 (2000). https://doi.org/10.1023/A:1005245900406
27. Rosen, E.: Modal logic over finite structures. Journal of Logic, Language, and Information **6**(4), 427–439 (1997), https://www.jstor.org/stable/40181601
28. van Benthem, J.: Modal Correspondence Theory. Ph.D. thesis, Mathematical Institute, University of Amsterdam, The Netherlands (1976)