Complexity Theory

Space Complexity

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Computational Logic

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Review

Space Complexity

Review: Space Complexity Classes

Recall our earlier definition of space complexities:

Definition 10.1

Let $f : \mathbb{N} \to \mathbb{R}^+$ be a function.

- $\mathrm{DS{\scriptsize PACE}}(f(n))$ is the class of all languages $\mathcal{L}$ for which there is an $O(f(n))$-space bounded Turing machine deciding $\mathcal{L}$.
- $\mathrm{NS{\scriptsize PACE}}(f(n))$ is the class of all languages $\mathcal{L}$ for which there is an $O(f(n))$-space bounded nondeterministic Turing machine deciding $\mathcal{L}$.

Being $O(f(n))$-space bounded requires a (nondeterministic) TM

- to halt on every input and
- to use $\leq f(|w|)$ tape cells on every computation path.

Space Complexity Classes

Some important space complexity classes:

$$\mathrm{L} = \textsc{LogSpace} = \textsc{DSpace}(\log n)$$ logarithmic space

$$\textsc{PSpace} = \bigcup_{d\geq 1} \textsc{DSpace}(n^d)$$ polynomial space

$$\textsc{ExpSpace} = \bigcup_{d\geq 1} \textsc{DSpace}(2^{n^d})$$ exponential space

$$\mathrm{NL} = \textsc{NLogSpace} = \textsc{NSpace}(\log n)$$ nondet. logarithmic space

$$\textsc{NPSpace} = \bigcup_{d\geq 1} \textsc{NSpace}(n^d)$$ nondet. polynomial space

$$\textsc{NExpSpace} = \bigcup_{d\geq 1} \textsc{NSpace}(2^{n^d})$$ nondet. exponential space

The Power Of Space

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Example 10.2

SAT can be solved in linear space:
Just iterate over all possible truth assignments (each linear in size) and check if one satisfies the formula.

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Example 10.2

SAT can be solved in linear space:
Just iterate over all possible truth assignments (each linear in size) and check if one satisfies the formula.

Example 10.3

TAUTOLOGY can be solved in linear space:
Just iterate over all possible truth assignments (each linear in size) and check if all satisfy the formula.

More generally: $\mathrm{NP} \subseteq \mathrm{PSPACE}$ and $\mathrm{coNP} \subseteq \mathrm{PSPACE}$

Linear Compression

Theorem 10.4

For every function $f : \mathbb{N} \to \mathbb{R}^{+}$, for all $c \in \mathbb{N}$, and for every f-space bounded (deterministic/nondeterminsitic) Turing machine $\mathcal{M}$:

there is a $\max\{1, \frac{1}{c}f(n)\}$-space bounded (deterministic/nondeterminsitic) Turing machine $\mathcal{M}'$ that accepts the same language as $\mathcal{M}$.

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Proof idea.

Similar to (but much simpler than) linear speed-up. □

This justifies using O-notation for defining space classes.

Tape Reduction

Theorem 10.5

For every function $f : \mathbb{N} \to \mathbb{R}^+$ *all* $k \geq 1$ *and* $\mathcal{L} \subseteq \Sigma^*$*:*

If $\mathcal{L}$ *can be decided by an* f*-space bounded* k*-tape Turing-machine,*

it can also be decided by an f*-space bounded* 1*-tape Turing-machine*

Tape Reduction

Theorem 10.5

For every function $f : \mathbb{N} \to \mathbb{R}^+$ all $k \geq 1$ and $\mathcal{L} \subseteq \Sigma^*$:

If $\mathcal{L}$ can be decided by an f-space bounded k-tape Turing-machine,

it can also be decided by an f-space bounded 1-tape Turing-machine

Proof idea.

Combine tapes with a similar reduction as for time. Compress space to avoid linear increase. □

Recall that we still use a separate read-only input tape to define some space complexities, such as $\textsc{LogSpace}$.

Time vs. Space

Theorem 10.6

For all functions $f : \mathbb{N} \to \mathbb{R}^+$:

$$\mathrm{DT{\small IME}}(f) \subseteq \mathrm{DS{\small PACE}}(f) \qquad \textit{and} \qquad \mathrm{NT{\small IME}}(f) \subseteq \mathrm{NS{\small PACE}}(f)$$

Proof.

Visiting a cell takes at least one time step. □

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Proof.

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Theorem 10.7

For all functions $f : \mathbb{N} \to \mathbb{R}^+$ *with* $f(n) \geq \log n$*:*

$$\text{DSPACE}(f) \subseteq \text{DTIME}(2^{O(f)}) \qquad \textit{and} \qquad \text{NSPACE}(f) \subseteq \text{DTIME}(2^{O(f)})$$

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For all functions $f : \mathbb{N} \to \mathbb{R}^+$ with $f(n) \geq \log n$:

$$\mathrm{DS{\small PACE}}(f) \subseteq \mathrm{DT{\small IME}}(2^{O(f)}) \qquad \textit{and} \qquad \mathrm{NS{\small PACE}}(f) \subseteq \mathrm{DT{\small IME}}(2^{O(f)})$$

Proof.

Based on configuration graphs and a bound on the number of possible configurations.

Number of Possible Configurations

Let $\mathcal{M} := (Q, \Sigma, \Gamma, q_0, \delta, q_{\text{start}})$ be a 2-tape Turing machine (1 read-only input tape + 1 work tape)

Recall: A configuration of $\mathcal{M}$ is a quadruple (q, p_1, p_2, x) where

- $q \in Q$ is the current state,
- $p_i \in \mathbb{N}$ is the head position on tape i, and
- $x \in \Gamma^*$ is the tape content.

Let $w \in \Sigma^*$ be an input to $\mathcal{M}$ and $n := |w|$. Then also $p_1 \le n$.

If $\mathcal{M}$ is $f(n)$-space bounded we can assume $p_2 \le f(n)$ and $|x| \le f(n)$

Hence, there are at most

$$|Q| \cdot n \cdot f(n) \cdot |\Gamma|^{f(n)} \quad = \quad n \cdot 2^{O(f(n))} \quad = \quad 2^{O(f(n))}$$

different configurations on inputs of length n
(the last equality requires $f(n) \ge \log n$).

Configuration Graphs

The possible computations of a TM $\mathcal{M}$ (on input w) form a directed graph:

- Vertices: configurations that $\mathcal{M}$ can reach (on input w)
- Edges: there is an edge from C_1 to C_2 if $C_1 \vdash_{\mathcal{M}} C_2$ (C_2 reachable from C_1 in a single step)

This yields the configuration graph

- Could be infinite in general.
- For $f(n)$-space bounded 2-tape TMs, there can be at most $2^{O(f(n))}$ vertices and $2 \cdot (2^{O(f(n))})^2 = 2^{O(f(n))}$ edges

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A computation of $\mathcal{M}$ on input w corresponds to a path in the configuration graph from the start configuration to a stop configuration.

Hence, to test if $\mathcal{M}$ accepts input w,

- construct the configuration graph and
- find a path from the start to an accepting stop configuration.

Time vs. Space

Theorem 10.6

For all functions $f : \mathbb{N} \to \mathbb{R}^+$:

$$\textsc{DTime}(f) \subseteq \textsc{DSpace}(f) \qquad \textit{and} \qquad \textsc{NTime}(f) \subseteq \textsc{NSpace}(f)$$

Proof.

Visiting a cell takes at least one time step. □

Theorem 10.7

For all functions $f : \mathbb{N} \to \mathbb{R}^+$ with $f(n) \geq \log n$:

$$\textsc{DSpace}(f) \subseteq \textsc{DTime}(2^{O(f)}) \qquad \textit{and} \qquad \textsc{NSpace}(f) \subseteq \textsc{DTime}(2^{O(f)})$$

Proof.

Build the configuration graph (time $2^{O(f(n))}$) and find a path from the start to an accepting stop configuration (time $2^{O(f(n))}$). □

Basic Space/Time Relationships

Applying the results of the previous slides, we get the following relations:

$$\mathrm{L} \subseteq \mathrm{NL} \subseteq \mathrm{P} \subseteq \mathrm{NP} \subseteq \textsc{PSpace} \subseteq \textsc{NPSpace} \subseteq \textsc{ExpTime} \subseteq \textsc{NExpTime}$$

We also noted $\mathrm{P} \subseteq \textsc{co}\mathrm{NP} \subseteq \textsc{PSpace}$.

Open questions:

- What is the relationship between space classes and their co-classes?
- What is the relationship between deterministic and non-deterministic space classes?

Nondeterminism in Space

Most experts think that nondeterministic TMs can solve strictly more problems when given the same amount of time than a deterministic TM:
Most believe that $\mathrm{P} \subsetneq \mathrm{NP}$

How about nondeterminism in space-bounded TMs?

Nondeterminism in Space

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How about nondeterminism in space-bounded TMs?

Theorem 10.8 (Savitch's Theorem, 1970)

For any function $f : \mathbb{N} \to \mathbb{R}^+$ with $f(n) \geq \log n$:

$$\mathrm{NS{\small PACE}}(f(n)) \subseteq \mathrm{DS{\small PACE}}(f^2(n)).$$

That is: nondeterminism adds almost no power to space-bounded TMs!

Consequences of Savitch's Theorem

Savitch's Theorem: For any function $f : \mathbb{N} \to \mathbb{R}^+$ with $f(n) \geq \log n$:

$$\textsc{NSpace}(f(n)) \subseteq \textsc{DSpace}(f^2(n)).$$

Corollary 10.9

$\textsc{PSpace} = \textsc{NPSpace}$.

Proof.

$\textsc{PSpace} \subseteq \textsc{NPSpace}$ is clear. The converse follows since the square of a polynomial is still a polynomial. $\square$

Similarly for "bigger" classes, e.g., $\textsc{ExpSpace} = \textsc{NExpSpace}$.

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Proof.

$\mathrm{PS{\scriptsize PACE}} \subseteq \mathrm{NPS{\scriptsize PACE}}$ is clear. The converse follows since the square of a polynomial is still a polynomial. □

Similarly for "bigger" classes, e.g., $\mathrm{E{\scriptsize XP}S{\scriptsize PACE}} = \mathrm{NE{\scriptsize XP}S{\scriptsize PACE}}$.

Corollary 10.10

$\mathrm{NL} \subseteq \mathrm{DS{\scriptsize PACE}}(O(\log^2 n))$.

Note that $\log^2(n) \notin O(\log n)$, so we do not obtain $\mathrm{NL} = \mathrm{L}$ from this.

Proving Savitch's Theorem

Simulating nondeterminism with more space:

- Use configuration graph of nondeterministic space-bounded TM
- Check if an accepting configuration can be reached
- Store only one computation path at a time (depth-first search)

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What to do?

Things we can do:

- Store one configuration:
  - one configuration requires $\log n + O(f(n))$ space
  - if $f(n) \geq \log n$, then this is $O(f(n))$ space
- Store $\log n$ configurations (remember we have $\log^2 n$ space)
- Iterate over all configurations (one by one)

Proving Savitch's Theorem: Key Idea

To find out if we can reach an accepting configuration, we solve a slighly more general question:

> **YIELDABILITY**
>
> *Input:* TM configurations C_1 and C_2, integer k
>
> *Problem:* Can TM get from C_1 to C_2 in at most k steps?

Approach: check if there is an intermediate configuration C' such that

(1) C_1 can reach C' in $k/2$ steps and

(2) C' can reach C_2 in $k/2$ steps

$\leadsto$ Deterministic: we can try all C' (iteration)

$\leadsto$ Space-efficient: we can reuse the same space for both steps

An Algorithm for Yieldability

```
CanYield(C1,C2,k) {
  if k = 1 :
    return (C1 = C2) or (C1 ⊢_M C2)
  else if k > 1 :
    for each configuration C of M for input size n :
      if CanYield(C1,C,k/2) and
         CanYield(C,C2,k/2) :
        return true
  // eventually, if no success:
  return false
}
```

- We only call CanYield only with k a power of 2, so $k/2 \in \mathbb{N}$

Space Requirement for the Algorithm

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 Space Complexity

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- During iteration (line 05), we store one C in $O(f(n))$
- Calls in lines 06 and 07 can reuse the same space
- Maximum depth of recursive call stack: $\log_2 k$

Overall space usage: $O(f(n) \cdot \log k)$

Simulating Nondeterministic Space-Bounded TMs

Input: TM $\mathcal{M}$ that runs in $\mathrm{NS{\small PACE}}(f(n))$; input word w of length n

Algorithm:

- Modify $\mathcal{M}$ to have a unique accepting configuration C_{accept}
  when accepting, erase tape and move head to the very left
- Select d such that $2^{df(n)} \geq |Q| \cdot n \cdot f(n) \cdot |\Gamma|^{f(n)}$
- Return $\textsc{CanYield}(C_{\text{start}}, C_{\text{accept}}, k)$ with $k = 2^{df(n)}$

Space requirements:
CanYield runs in

$$O\left(f(n) \cdot \log k\right) = O\left(f(n) \cdot \log 2^{df(n)}\right) = O(f(n) \cdot df(n)) = O(f^2(n))$$

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- Even if we knew f, it might not be easy to compute!

Solution: replace $f(n)$ by a parameter ℓ and probe its value

(1) Start with $\ell = 1$

(2) Check if $\mathcal{M}$ can reach any configuration with more than ℓ tape cells (iterate over all configurations of size $\ell + 1$; use CanYield on each)

(3) If yes, increase ℓ by 1; goto (2)

(4) Run algorithm as before, with $f(n)$ replaced by ℓ

Therefore: we don't need to know f at all. This finishes the proof. □

Relationships of Space and Time

Summing up, we get the following relations:

$$\mathrm{L} \subseteq \mathrm{NL} \subseteq \mathrm{P} \subseteq \mathrm{NP} \subseteq \mathrm{PSPACE} = \mathrm{NPSPACE} \subseteq \mathrm{EXPTIME} \subseteq \mathrm{NEXPTIME}$$

We also noted $\mathrm{P} \subseteq \mathrm{coNP} \subseteq \mathrm{PSPACE}$.

Open questions:

- Is Savitch's Theorem tight?
- Are there any interesting problems in these space classes?
- We have $\mathrm{PSPACE} = \mathrm{NPSPACE} = \mathrm{coNPSPACE}$.
  But what about L, NL, and coNL?

$\leadsto$ the first: nobody knows; the others: see upcoming lectures